Eigenfunctions of the Vector-Potential Operator of the Electromagnetic Field and Semiclassical Electrodynamics

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(Received 18 April 1972)

A complete orthonormal set of eigenfunctions of the vector-potential operator is introduced. It is shown that the semiclassical approximation is naturally determined in the basis of the functions introduced. The extraction of the semiclassical approximation from nonrelativistic quantum electrodynamics allows the determination of the limits of applicability of the semiclassical approach. It is shown that the semiclassical approximation is inapplicable at sufficiently large photon densities. The advantages of using our set of eigenfunctions for the description of the interaction of a strong quantized field with a quantum system are discussed.

I. INTRODUCTION

The coherent states, introduced into optics by Glauber,[1] are used with success in problems on light radiation, in considering theoretical schemes of radiation receivers, and in analyzing interference phenomena.[2,3] These functions form a "natural" system for problems of these types because either the normally ordered operators or the oscillator shift operator arises automatically there. However, many advantages of coherent states are lost in considering the Schrödinger equation for the evolution operator describing the interaction of a sufficiently strong electromagnetic field with a quantum system.

This paper presents a complete orthonormal set of eigenfunctions of the vector-potential operator as being more appropriate for the solution of such problems. A study of the properties of the field operators in the new representation shows that these functions allow the semiclassical theory of radiation to be derived properly from the description of the interaction of a strong quantized field and a quantum system.

In Sec. II the eigenfunctions are introduced and their analytical properties and the connection with other representations are considered. In Sec. III the matrix elements of the field operators are determined and the characteristics of operators are studied. It is shown in Sec. IV that there is an analog of the optical equivalence theorem[3] in the basis of the eigenfunctions for the field density matrix. In Sec. V, by using properties of the functions introduced and of the operators in their basis, we analyze the question of the extraction of the semiclassical approximation from the description of the interaction of a quantized field and a quantum system. In particular, the separation of the semiclassical evolution operator and the semiclassical nonlinear Maxwell equations is considered. The limits of applicability of the semiclassical approximation are discussed. In Sec. VI the quantum corrections for the Maxwell equations are found; they occur at sufficiently large intensity of the incident field. In Sec. VII an expression for the field density matrix in the semiclassical approximation is obtained which allows the determination of the density matrix according to the known solution of the Maxwell equations.

II. EIGENFUNCTIONS OF THE VECTOR-POTENTIAL OPERATOR

The eigenfunctions ψ_a of the vector-potential operator of the electromagnetic field $\vec{A}_{op}(\vec{r})$ satisfy the equation

$$\vec{A}_{op}(\vec{r})\psi_a = \vec{A}(\vec{r})\psi_a , \tag{1}$$

where

$$\vec{A}_{op}(\vec{r}) = \sum_{\vec{k},\lambda} \left(\frac{2\pi\hbar c}{kL^3}\right)^{1/2} \vec{e}_\lambda(\vec{k})[a_\lambda^\dagger(\vec{k})e^{i\vec{k}\cdot\vec{r}} + a_\lambda(\vec{k})e^{-i\vec{k}\cdot\vec{r}}] . \tag{2}$$

L^3 is the quantization volume; $\hbar$ is Planck's constant; c is the velocity of light; $\vec{r}$ denotes the point coordinates; $a_\lambda(\vec{k})$ and $a_\lambda^\dagger(\vec{k})$ are the photon annihilation and creation operators with polarization $\vec{e}_\lambda(\vec{k})$ $(\lambda = 1,2)$, and the wave vector $\vec{k} = (2\pi/L)\vec{n}$, with $\vec{n} = (n_1, n_2, n_3)$, where the n_j are integers. The eigenvalues are

$$\vec{A}(\vec{r}) = \sum_{\vec{k},\lambda} \left(\frac{2\pi\hbar c}{kL^3}\right)^{1/2} \vec{e}_\lambda(\vec{k}) a_{\vec{k},\lambda} e^{i\vec{k}\cdot\vec{r}} .$$

The set of numbers $a_{\vec{k},\lambda}$ will sometimes be denoted by a. From the Hermitian character of $\vec{A}_{op}(\vec{r})$, $a_{\vec{k},\lambda} = a^*_{-\vec{k},\lambda}$.

The function ψ_a may be described as the expansion

$$\psi_a = \int Q(\cdots \alpha_{\vec{k},\lambda} \cdots) \prod_{\vec{k},\lambda} \phi_{\alpha_{\vec{k},\lambda}}(\zeta_{\vec{k},\lambda}) \times e^{-|\alpha_{\vec{k},\lambda}|^2/2} \pi^{-1} d^2\alpha_{\vec{k},\lambda}, \tag{3}$$

where $\phi_{\alpha_{\vec{k},\lambda}}(\zeta)$ is the coherent state of a one-dimensional oscillator with the coordinate ζ and $d^2\alpha = d\mathrm{Re}\,\alpha\, d\mathrm{Im}\,\alpha$. The substitution of (3) into (1) and the use of standard calculations, based on the characteristics of coherent states[1-3] and the definition (2), allow one to obtain a set of equations which can be separated into pairs:

$$\left(\frac{\partial}{\partial \alpha^*_{\vec{k},\mu}} + \alpha^*_{-\vec{k},\mu} - a_{\vec{k},\mu}\right) Q = \left(\frac{\partial}{\partial \alpha^*_{-\vec{k},\mu}} + \alpha^*_{\vec{k},\mu} - a^*_{\vec{k},\mu}\right) Q = 0. \tag{4}$$

After solution of this set of equations and calculation of a normalizing factor one can write

$$\psi_a = \left(\prod_{\lambda} \prod_{\vec{k}}{}^{\dagger} \frac{1}{\sqrt{\pi}} e^{-|a_{\vec{k},\lambda}|^2/2}\right) \times \int \exp\left(\sum_{\lambda} \sum_{\vec{k}}{}^{\dagger} \alpha^*_{\vec{k},\lambda} a_{\vec{k},\lambda} + \alpha^*_{-\vec{k},\lambda} a^*_{\vec{k},\lambda} - \alpha^*_{-\vec{k},\lambda} \alpha^*_{\vec{k},\lambda}\right) \times \prod_{\vec{k},\lambda} \phi_{\alpha_{\vec{k},\lambda}}(\zeta_{\vec{k},\lambda}) e^{-|\alpha_{\vec{k},\lambda}|^2/2} \pi^{-1} d^2\alpha_{\vec{k},\lambda}.$$

The symbols $\sum^{\dagger}$ and $\prod^{\dagger}$ are defined by the following rule: For each $k_3 > 0$ all the vectors with any k_1 and k_2 are considered; at $k_3 = 0$ all the vectors with $k_2 > 0$ and any k_1 are considered, etc.; the case $k_1 = k_2 = k_3 = 0$ is excluded.

The only limitation set on Q amounts to the fact that it should be an integral function of its arguments. This condition is satisfied at any complex $a_{\vec{k},\lambda}$, which implies that the operator $\vec{A}_{\mathrm{op}}(\vec{r})$ has a continuous spectrum.

The functions considered are continuous at $a_{\vec{k},\lambda}$ and $a^*_{\vec{k},\lambda}$, but not regular at $a_{\vec{k},\lambda}$. The nonregular behavior can be seen by means of the Komi-Ritan conditions.

The functions ψ_a have all the properties of eigenfunctions of a Hermitian operator possessing a continuous spectrum. In particular,

$$\langle \psi_a | \psi_b \rangle = \delta(a-b) \equiv \prod_{\lambda} \prod_{\vec{k}}{}^{\dagger} \delta(\mathrm{Re}\,a_{\vec{k},\lambda} - \mathrm{Re}\,b_{\vec{k},\lambda}) \times \delta(\mathrm{Im}\,a_{\vec{k},\lambda} - \mathrm{Im}\,b_{\vec{k},\lambda}),$$

$$\int \psi_a(\zeta)\psi_a(\zeta')da = \prod_{\vec{k},\lambda} \delta(\zeta_{\vec{k},\lambda} - \zeta'_{\vec{k},\lambda}),$$

$$da = \prod_{\lambda} \prod_{\vec{k}}{}^{\dagger} d\mathrm{Re}\,a_{\vec{k},\lambda} d\mathrm{Im}\,a_{\vec{k},\lambda}.$$

The connection of the representation of these eigenfunctions with other representations is defined by the value $c_F(a) = \langle \psi_a | F \rangle$.

The connection with coherent states is defined by the expansion (3):

$$c_\beta(a) = \langle \psi_a | \phi_\beta \rangle = e^{-|\beta_{\vec{k},\lambda}|^2/2} Q_a(\cdots \beta^*_{\vec{k},\lambda} \cdots).$$

The functions ψ_a are related to the eigenfunctions of the Hamiltonian of the free field as follows ($H_R \kappa_n = E_n \kappa_n$, $\kappa_n = \prod_{\vec{k},\lambda} \kappa_{n_{\vec{k},\lambda}}$):

$$c_n(a) = \langle \psi_a | \kappa_n \rangle = \prod_{\lambda} \prod_{\vec{k}}{}^{\dagger} \frac{1}{\sqrt{\pi}} e^{-|a_{\vec{k},\lambda}|^2/2} (n_{\vec{k},\lambda}!\, n_{-\vec{k},\lambda}!)^{1/2} \times a^{n_{-\vec{k},\lambda}} a^{n_{\vec{k},\lambda}} \sigma_{n_{-\vec{k},\lambda} n_{\vec{k},\lambda}}(-1/|a_{\vec{k},\lambda}|^2).$$

Here

$$\sigma_{nm}(x) = \sum_{l=0}^{\infty} \frac{x^l}{l!\,(n-l)!\,(m-l)!} = \frac{x^{n_<}}{n_>!} \mathcal{L}^{n_> - n_<}_{n_<}\left(-\frac{1}{x}\right).$$

$\mathcal{L}^n_m$ is the Laguerre polynomial and

$$n_> \equiv \max(n,m),$$

$$n_< \equiv \min(n,m).$$

The connection of the functions $\kappa_{n_{\vec{k},\lambda}}$ with coherent states will be used in later calculations.

III. MATRIX ELEMENTS

A matrix element of any field operator F in the representation of the functions ψ_a is of the form

$$\langle \psi_a | F | \psi_b \rangle = \int Q^*_a(\cdots \alpha^* \cdots) Q_b(\cdots \beta^* \cdots) \exp\left[-\tfrac{1}{2}\sum_{\vec{k},\lambda} (|\alpha_{\vec{k},\lambda}|^2 + |\beta_{\vec{k},\lambda}|^2)\right] \times \langle \prod \phi_{\alpha_{\vec{k},\lambda}} | F | \prod \phi_{\beta_{\vec{k},\lambda}} \rangle \prod_{\vec{k},\lambda} \pi^{-2} d^2\alpha_{\vec{k},\lambda}\, d^2\beta_{\vec{k},\lambda}. \tag{5}$$

If the integral (5) exists in the usual sense, the corresponding operator may be called a nonsingular one. When the relation (5) results in a singular function, one may call the operator F a singular one. The operator F will be weakly singular when (5) has a singularity of the type of Dirac's δ function and strongly singular if the calculation (5) results in a relation of the kind $D(a)\delta(a-b)$, where $D(a)$ is a certain differential operator.

The generalized functions, arising in the case of singular operators, are defined for a certain class of functions $f(a, b)$, which may be introduced as follows.

We shall consider the expression

$$\int\langle\psi_a|F|\psi_b\rangle f(a,b)\,da\,db=\int Q_a^*(\alpha^*)Q_b(\beta^*)f(a,b)F(\cdots\alpha^*\cdots,\cdots\beta\cdots)\exp\Big[-\tfrac{1}{2}\sum_{\vec{k},\lambda}(|\alpha_{\vec{k},\lambda}|^2+|\beta_{\vec{k},\lambda}|^2)\Big]\times\prod_{\vec{k},\lambda}\pi^{-2}d^2\alpha_{\vec{k},\lambda}\,d^2\beta_{\vec{k},\lambda}\,da\,db\,.$$

It may be shown that

$$\exp\Big[-\tfrac{1}{2}\sum_{\vec{k},\lambda}(|\mu_{\vec{k},\lambda}|^2+|\nu_{\vec{k},\lambda}|^2)\Big]\int Q_a^*(\nu^*_{\vec{k},\lambda})Q_b(\mu^*_{\vec{k},\lambda})\,da\,db=\langle\mu|\nu\rangle\;,$$

where $|\mu\rangle$ and $|\nu\rangle$ are coherent states. It follows that for a certain Λ operator the following relation holds:

$$\exp\Big[-\tfrac{1}{2}\sum_{\vec{k},\lambda}(|\mu_{\vec{k},\lambda}|^2+|\nu_{\vec{k},\lambda}|^2)\Big]\int\langle\psi_a|\Lambda|\psi_b\rangle\,Q_a^*(\nu^*)Q_b(\mu^*)\,da\,db=\langle\mu|\Lambda|\nu\rangle\,.$$

Now assuming $f(a, b)$ to be a matrix element of a nonsingular operator Λ we obtain

$$\int\langle\psi_a|F|\psi_b\rangle f(a,b)\,da\,db=\mathrm{Tr}\,F\Lambda<\infty\,. \tag{6}$$

The condition (6) determines the unknown class of functions: One should sort all the operators Λ for which the last trace is finite.

The creation and annihilation operators are singular ones. Their matrix elements in the ψ_a representation are calculated by means of standard relations, connected with functions of coherent states and with the representations of derivatives of δ functions.

The result can be described as

$$\begin{aligned}\langle\psi_a|a_\lambda(\vec{k})|\psi_b\rangle&=\Big(\tfrac{1}{2}a_{\vec{k},\lambda}+\frac{\partial}{\partial a^*_{\vec{k},\lambda}}\Big)\delta(a-b)\,,\\ \langle\psi_a|a^\dagger_\lambda(\vec{k})|\psi_b\rangle&=\Big(\tfrac{1}{2}a^*_{\vec{k},\lambda}-\frac{\partial}{\partial a_{\vec{k},\lambda}}\Big)\delta(a-b)\,.\end{aligned} \tag{7}$$

The relations (7) allow the matrix elements of field operators in the interaction representation to be calculated:

$$F_{\rm op}(t)=e^{iH_Rt/\hbar}F_{\rm op}e^{-iH_Rt/\hbar}\;,$$

where H_R is the Hamiltonian of the free quantized field and t is the time. Then

$$\langle\psi_a|\vec{A}_{\rm op}(\vec{r},t)|\psi_b\rangle=\vec{A}_a(\vec{r},t)\delta(a-b)\,,$$

where

$$\vec{A}_a(\vec{r},t)\equiv\vec{A}_1(\vec{r},t)+\vec{A}_2(\vec{r},t)\,,$$

$$\begin{aligned}\vec{A}_1(\vec{r},t)&=\sum_{\vec{k},\lambda}\Big(\frac{2\pi\hbar c}{kL^3}\Big)^{1/2}\vec{e}_\lambda(\vec{k})\,e^{i\vec{k}\cdot\vec{r}}\,a_{\vec{k},\lambda}\cos kct\\ \vec{A}_2(\vec{r},t)&=\sum_{\vec{k},\lambda}\Big(\frac{2\pi\hbar c}{kL^3}\Big)^{1/2}\vec{e}_\lambda(\vec{k})e^{-i\vec{k}\cdot\vec{r}}\frac{\partial}{\partial a_{\vec{k},\lambda}}\sin kct\;.\end{aligned} \tag{8}$$

Similar expressions for the magnetic and electric field intensity operators follow from the formulas

$$\vec{H}_{\rm op}=\mathrm{curl}\,\vec{A}_{\rm op}$$

and

$$\vec{E}_{\rm op}=-\frac{1}{c}\frac{\partial\vec{A}_{\rm op}}{\partial t}\,.$$

It should be noted that the above-determined weakly singular operators have the desired properties, making them convenient in calculations. For example, the matrix element of a product of operators equals the product of the matrix elements of the operators, i.e., its factors. This implies, in particular, that the weakly singular operators commute. Further, if $\langle\psi_a|F|\psi_b\rangle=F_1\delta(a-b)$, then $\langle\psi_a|f(F)|\psi_b\rangle=f(F_1)\delta(a-b)$, where f is a whole function of its argument. The enumerated properties denote in fact the substitution of field operators by c numbers.

IV. DENSITY MATRIX

It follows from the properties of the density matrix[4] that it is nonsingular.

One can show that in the basis ψ_a the density matrix cannot be a diagonal one.

Let $\langle\psi_a|\rho|\psi_b\rangle=\rho_a\delta(a-b)$. It follows from the orthogonality of the ψ_a functions that $\rho=\rho(\vec{A}_{\rm op})$. But even for stationary fields it follows from $[H,\rho]$ that $\rho\neq\rho(\vec{A}_{\rm op})$. The case $\rho_a=1$ is excluded by the requirement of finiteness of the trace ρ. Therefore, the off-diagonal matrix elements of the density matrix are not equal to zero. From here it also follows that a physical (real) field cannot exist in a pure state ψ_a. In fact, we could write an average value of the F operator as $\langle\psi_a|F|\psi_a\rangle$, that is, a special case of averaging according to a diagonal density matrix of the type

$$\int P(a)|\psi_a\rangle\langle\psi_a|\,da\,,\quad P(a)=1\,.$$

It is mentioned above that such a density matrix does not exist. Formally, the unphysical character of states ψ_a is connected with their irregularity according to $a_{\vec{k},\lambda}$. In the case of regular functions the off-diagonal matrix elements are determined by the diagonal ones and the relation $\int P(a)|a\rangle\langle a|da$ gives a proper formula of analytic continuation.[3]

From the nonsingularity of ρ and relations (6) and (8) it follows that an average value of any operator $F(\vec{A}_{op}(\vec{r},t),\vec{H}_{op}(\vec{r},t),\vec{E}_{op}(\vec{r},t))$ is defined by the diagonal matrix elements of the density matrix $\langle\psi_a|\rho|\psi_a\rangle\equiv\sigma(a)$,

$$\mathrm{Tr}\rho F=\int F(\vec{A}_a(\vec{r},t),\vec{H}_a(\vec{r},t),\vec{E}_a(\vec{r},t))\,\sigma(a)da\,. \tag{9}$$

This relation is similar to the optical theorem of equivalence,[3] but it differs from the latter by the presence in (9) of the derivatives $\sigma(a)$. All the derivatives of $\sigma(a)$ over $a_{\vec{k},\lambda}$ and $a^*_{\vec{k},\lambda}$ exist, as the functions of ψ_a are separately regular over $a_{\vec{k},\lambda}$ and $a^*_{\vec{k},\lambda}$.

It follows from (6)–(8) that the field operators $\vec{A}_{op}(\vec{r},t)$, $\vec{H}_{op}(\vec{r},t)$, and $\vec{E}_{op}(\vec{r},t)$ and the photon number operator $n_{\vec{k},\lambda}$ may be represented as a sum of weakly singular and strongly singular operators. On the basis of (9) it may be shown that at $\langle n_{\vec{k},\lambda}\rangle \gg 1$ one may confine oneself only to the weakly singular operator. It is known[5] that in this case the free field is described as a classical one. From this and from the properties of weakly singular operators follows their connection with the semiclassical description.

V. EXTRACTION OF SEMICLASSICAL ELECTRODYNAMICS FROM THE TOTAL QUANTUM DESCRIPTION

The Schrödinger equation for the field interacting with a sufficiently small quantum system (such that the long-wave approximation will take place), registered in the interaction representation for the evolution operator, $g(t)$, is of the form

$$i\hbar\frac{\partial}{\partial t}g(t)=H_{0R}(t)\,g(t)\,,\quad g(0)=1 \tag{10}$$

$$H_{0R}(t)=-\sum_j\frac{e_j}{m_jc}\vec{p}_j\cdot\vec{A}_{op}(\vec{r}_j,t)+\sum_j\frac{e_j^{\,2}}{2m_jc^2}\vec{A}_{op}^{\,2}(\vec{r}_j,t)\,, \tag{11}$$

where e_j, m_j, and $\vec{p}_j(t)$ are the charge, the mass, and the momentum operator of the j particle, respectively. We confine ourselves to a linear term according to the field in the Hamiltonian (11). A square term in (11) may be removed, making the proper unitary transformation. It is not difficult to check that after the unitary transformation the first term in the interaction Hamiltonian remains linear.

The solution of Eq. (10) may be symbolized as

$$g=G(1+N)\,, \tag{12}$$

$$G=\lim_{n\to\infty} g_0g_1\cdots g_n\,, \tag{13}$$

$$i\hbar\frac{\partial}{\partial t}g_n=M_{n-1,1}g_n\,, \tag{14}$$

$$N=Q-1\,,\quad Q=\lim_{n\to\infty}Q_n\,, \tag{15}$$

$$i\hbar\frac{\partial}{\partial t}Q_n=M_nQ_n\,, \tag{16}$$

$$i\hbar\frac{\partial}{\partial t}Q=(G^{-1}M_{-1,2}G)_2Q\,. \tag{17}$$

Here and in what follows the following designations are taken: Index 1 corresponds to the weakly singular operator; 2 corresponds to the strongly singular one; for example,

$$\begin{aligned}
&\vec{A}_{op}(t)=\vec{A}_1(t)+\vec{A}_2(t)\,,\\
&H_{0R}(t)=H_{0R,1}(t)+H_{0R,2}(t)\,,\\
&M_{-1,1}=H_{0R,1}\,,\\
&M_{-1,2}=H_{0R,2}\,,\\
&M_{0,1}=(g_0^{\,-1}H_{0R,2}g_0)_i\,,\\
&M_n=g_n^{\,-1}M_{n-1,2}g_n\\
&\quad=M_{n,1}+M_{n,2}\,.
\end{aligned} \tag{18}$$

The proof of relations (12)–(17) is based on the known expression for the Heisenberg operator[6] $\vec{F}_H(t)$,

$$\vec{F}_H(t)=g^{-1}(t)\vec{F}_{op}(t)\,g(t)=\vec{F}_{op}(t)+\frac{i}{\hbar}\int_0^t g^{-1}(\tau)[H_{0R}(\tau),\vec{F}_{op}(t)]\,g(\tau)d\tau\,, \tag{19}$$

which is used to separate the weakly singular operator out of M_n.

It may be shown that $\lim_{n\to\infty}g_n=1$, from which follows the existence of the limits (13), (15).

From (13), (14) it follows that G is a weakly singular operator. A strong singularity N results from the relations (15)–(17).

It is seen from (13), (14) that the weakly singular operator G is a nonunitary one. To single out the unitary operator we shall represent G as

$$G=\lim_{n\to\infty} g_0\theta_1 B_1\theta_2 B_2\cdots\theta_r B_n\,,$$

$$i\hbar\frac{\partial}{\partial t}\theta_n=\vec{\mathrm{P}}_{n-1}\cdot(\vec{\mathrm{A}}_{n-1,1}-\langle\vec{\mathrm{A}}_{n-1,1}\rangle_x)\theta_n\,, \tag{20}$$

$$i\hbar\frac{\partial}{\partial t}B_n=\theta_n^{-1}\vec{\mathrm{P}}_{n-1}\theta_n\cdot\langle\vec{\mathrm{A}}_{n-1,1}\rangle_x B_n\,,$$

where $\vec{\mathrm{P}}\cdot\vec{\mathrm{A}}_1=H_{0R,1}$, $\vec{\mathrm{A}}_{n,1}=(g_n^{-1}\vec{\mathrm{A}}_{n-1,2}g_n)_1$, and $\langle\cdots\rangle_x$ denotes the averaging according to the initial density matrix of the medium R.

The operators θ_n are defined by the value $\vec{\mathrm{A}}_{\mathrm{op}}-\langle\vec{\mathrm{A}}_{\mathrm{op}}\rangle$, i.e., the linear deviation of the average value from the nonaveraged one. At sufficiently large number of particles in the medium one may consider $\vec{\mathrm{A}}_{\mathrm{op}}-\langle\vec{\mathrm{A}}_{\mathrm{op}}\rangle=0$; then G will be of the form

$$G=\lim_{n\to\infty} g_0 B_1 B_2\cdots B_n\,, \tag{21}$$

where

$$i\hbar\frac{\partial}{\partial t}B_n=\vec{\mathrm{P}}_{n-1}\cdot\langle\vec{\mathrm{A}}_{n-1,1}\rangle_x B_n\,. \tag{22}$$

Equation (14) written for g_0 is essentially the Schrödinger equation for the quantum system interacting with a classical field.

One can write $g=g_0Q_0$, where the operator Q_0 satisfies Eq. (16) and determines the boundaries of the semiclassical approximation. From (19), (12), and (15) it follows that the field operator $\vec{\mathrm{F}}_H(t)$ can be written as

$$\vec{\mathrm{F}}_H(t)=\vec{\mathrm{F}}_1(t)+\vec{\mathrm{F}}_2(t)+\frac{i}{\hbar}\int_0^t G^{-1}(\tau)[H_{0R}(\tau),\vec{\mathrm{F}}_{\mathrm{op}}(t)]G(\tau)d\tau$$
$$+\left(\frac{i}{\hbar}\right)^2\int_0^t\int_0^\tau Q^{-1}(\tau_1)[(G^{-1}(\tau_1)M_{-1,2}G(\tau_1))_2,G^{-1}(\tau)[H_{0R}(\tau),\vec{\mathrm{F}}_{\mathrm{op}}(t)]G(\tau)]Q(\tau_1)d\tau d\tau_1\,.$$

Now the linear weakly singular field operator $\vec{\mathrm{F}}_{H,1}(t)$ will be defined as

$$\vec{\mathrm{F}}_{H,1}(t)=\vec{\mathrm{F}}_1(t)+\frac{i}{\hbar}\int_0^t G^{-1}(\tau)[H_{0R}(\tau),\vec{\mathrm{F}}_{\mathrm{op}}(t)]G(\tau)d\tau\,. \tag{23}$$

It is known[5] that the field operators $\vec{\mathrm{A}}_H(t)$, $\vec{\mathrm{E}}_H(t)$, and $\vec{\mathrm{H}}_H(t)$ satisfy the operator Maxwell equations. It is not difficult to make sure that the weakly singular operator (23) satisfies similar equations.

We shall introduce the average field operator

$$\langle\vec{\mathrm{F}}_H(t)\rangle_x=\mathrm{Tr}_x R\,g^{-1}(t)\vec{\mathrm{F}}_{\mathrm{op}}(t)g(t)\,. \tag{24}$$

Here the index x denotes that the summation is taken over the medium states. Turning from the singular operator $\vec{\mathrm{F}}_H(t)$ to the weakly singular operator $\vec{\mathrm{F}}_{H,1}(t)$, we obtain a c function corresponding to the average field operator (24):

$$\langle\vec{\mathrm{F}}_{H,1}(t)\rangle_x=\vec{\mathrm{F}}_1(t)+\frac{i}{\hbar}\mathrm{Tr}_x R\int_0^t G^{-1}(\tau)[H_{0R}(\tau),\vec{\mathrm{F}}_{\mathrm{op}}(t)]G(\tau)d\tau\,. \tag{25}$$

The field, satisfying the nonlinear self-consistent Maxwell equations, should be of the form

$$\vec{\mathrm{F}}_M(\vec{\mathrm{r}},t)=\vec{\mathrm{F}}_1(a)+\frac{i}{\hbar}\mathrm{Tr}_x R\int_0^t C^{-1}(\tau)[H_{0R}(\tau),\vec{\mathrm{F}}_{\mathrm{op}}(t)]C(\tau)d\tau\,, \tag{26}$$

where

$$i\hbar\frac{\partial}{\partial t}C(t)=\vec{\mathrm{P}}\cdot\vec{\mathrm{A}}_M C(t)\,. \tag{27}$$

The equivalence of (26) and (27) to the Maxwell equations is shown by differentiation of (26) with respect to time.

One can show that the weakly singular operator (25) satisfies Eqs. (26) and (27) at equality of G and C. As the form of the equations for G and C does not allow us to say anything about a complete equality, we shall reformulate this requirement as follows. Let $G=C\theta$. One may show that the field (25) satisfies the Maxwell equations (26) and (27) if and only if $\theta=1$.

To determine the conditions under which $\theta=1$ we consider the solution of the system (26), (27) by the method of successive approximations. Let us assume $\vec{\mathrm{A}}_M=\vec{\mathrm{A}}_1(a)$; then

$$i\hbar\frac{\partial}{\partial t}C_0=H_{0R,1}C,\qquad C_0=g_0(a)\,.$$

Substituting the operator C_0 into the field (26) we obtain

$$C^{(1)}=C_0C_1\,,$$

$$i\hbar\frac{\partial}{\partial t}C^{(1)}=\vec{\mathrm{P}}\cdot[\vec{\mathrm{A}}_1(a)+2\langle A_{0,1}(a)\rangle_x]C^{(1)}\,, \tag{28}$$

$$i\hbar\frac{\partial}{\partial t}C_1=2\vec{\mathrm{P}}_0\cdot\langle\vec{\mathrm{A}}_{0,1}(a)\rangle_x C_1\,.$$

Continuing this procedure we obtain that the operator C may be represented as $C=\lim C_0C_1\cdots C_n$ and satisfies the equation

$$i\hbar\frac{\partial}{\partial t}C=\vec{\mathrm{P}}\cdot[\vec{\mathrm{A}}_1(a)+2\langle\vec{\mathrm{A}}_{0,1}(a)\rangle_x+\cdots]C. \quad (29)$$

From Eqs. (21), (22), and (29) and the connection between G and C we obtain the equation for θ,

$$i\hbar\frac{\partial}{\partial t}\theta=C^{-1}\vec{\mathrm{P}}C\cdot[\langle\vec{\mathrm{A}}_{0,1}(a)\rangle_x+\cdots]\theta. \quad (30)$$

From solution of Eq. (30) it follows that $\theta=1$ under the conditions

$$\begin{aligned}&B_1(a)\simeq C_1\simeq 1,\\&G(a)\simeq C\simeq C_0.\end{aligned} \quad (31)$$

The field (25) in this case coincides with the field (26) and is of the form

$$\langle\vec{\mathrm{F}}_{H,1}(t)\rangle_x=\vec{\mathrm{F}}_1(t)+\frac{i}{\hbar}\mathrm{Tr}_x R\int_0^t g_0^{-1}(\tau)[H_{0R}(\tau),\vec{\mathrm{F}}_{\mathrm{op}}(t)]g_0(\tau)d\tau. \quad (32)$$

The formula (32) satisfies formally the non-self-consistent equations. For example [the term with $(\partial/\partial t)\nabla\phi$ is introduced in the current operator],

$$\mathrm{curl}\langle\vec{\mathrm{H}}_{H,1}(t)\rangle_x=\frac{1}{c}\frac{\partial}{\partial t}\langle\vec{\mathrm{E}}_{H,1}(t)\rangle_x+\frac{4\pi}{c}\mathrm{Tr}_x R g_0^{-1}\vec{\mathrm{j}}g_0.$$

However, taking (31) into account we may write

$$g_0\simeq g_0C_1\simeq C(A_M),$$

from which

$$\mathrm{curl}\langle\vec{\mathrm{H}}_{H,1}(t)\rangle_x=\frac{1}{c}\frac{\partial}{\partial t}\langle\vec{\mathrm{E}}_{H,1}(t)\rangle_x+\frac{4\pi}{c}\mathrm{Tr}_x C^{-1}\vec{\mathrm{j}}C. \quad (33)$$

From the relations (30) and (31) defining the applicability limits of the Maxwell semiclassical equations, it follows that the conditions of applicability may be violated at sufficiently large photon occupation numbers.

Hence, for interaction of a strong field with the medium it is impossible to be sure beforehand that one can use the semiclassical approximation. To answer this question the value of the θ [in (30)] or B [in (22)] operators must be estimated for a given situation. The same may be said concerning the applicability limits of the semiclassical evolution operator g_0. It should be noted that in analyzing these values the condition $\langle n_k\rangle\gg 1$ appears at once; the second condition gives the limitation on the density of occupation numbers, depending on the medium with which the field interacts.

VI. QUANTUM CORRECTIONS FOR THE MAXWELL EQUATIONS

To evaluate the quantum corrections we assume

$$G\simeq g_0B_1,\quad B_2\simeq C_2\simeq 1. \quad (34)$$

Now the semiclassical field may be written as

$$\langle\vec{\mathrm{F}}_{H,1}(t)\rangle_x=\vec{\mathrm{F}}_1(t)+\frac{i}{\hbar}\mathrm{Tr}_x R\int_0^t B_1^{-1}g_0[H_{0R}(\tau),\vec{\mathrm{F}}(t)]g_0B_1d\tau. \quad (35)$$

We may write out the current, appearing upon differentiation of the field $\vec{\mathrm{E}}$ written as Eq. (35),

$$\mathrm{Tr}_x RG^{-1}\vec{\mathrm{j}}G=\mathrm{Tr}_x Rg_0^{-1}\vec{\mathrm{j}}g_0+\frac{i}{\hbar}\mathrm{Tr}_x R\int_0^t B_1^{-1}(\tau)\sum_\alpha[g_0^{-1}(\tau)P_\alpha g_0(\tau),g_0^{-1}\vec{\mathrm{j}}g_0]B_1(\tau)$$
$$\times\mathrm{Tr}_x R\int_0^\tau g_0^{-1}(\tau_1)[H_{0R}(\tau_1),A_{2\alpha}(\tau)]g_0(\tau_1)d\tau d\tau_1, \quad (36)$$

where $\alpha=1,2,3$ numbers the unit vectors of the coordinate system. In calculating (36) we used Eq. (22) and the relation (19).

In the self-consistent equation a current appears,

$$\mathrm{Tr}_x RC^{-1}\vec{\mathrm{j}}C, \quad (37)$$

where the C operator satisfies Eq. (28). The self-consistent current (37) can be converted into a form similar to (36). We have

$$\mathrm{Tr}_x RC^{-1}\vec{\mathrm{j}}C=\mathrm{Tr}_x RC_0^{-1}\vec{\mathrm{j}}C_0+2\frac{i}{\hbar}\mathrm{Tr}_x R\int_0^t C_1^{-1}(\tau)\sum_\alpha[C_0^{-1}(\tau)P_\alpha C_0(\tau),g_0^{-1}(t)\vec{\mathrm{j}}g_0(t)]$$
$$\times C_1(\tau)\mathrm{Tr}_x R\int_0^\tau C_0^{-1}(\tau_1)[H_{0R}(\tau_1),A_{2\alpha}(\tau)]C_0(\tau_1)d\tau d\tau_1. \quad (38)$$

Comparing (37) with (35) we obtain

$$\mathrm{Tr}_x RG^{-1}(a)\vec{j}G(a) = \mathrm{Tr}_x RC^{-1}\vec{j}C + \mathrm{Tr}_x R\int_0^t \{B_1^{-1}(\tau)\sum_\alpha [g_0^{-1}(\tau)P_\alpha g(\tau), g_0^{-1}(t)\vec{j}g_0(t)]B_1(\tau) - 2C_1^{-1}(\tau)[C_0^{-1}(\tau)P_\alpha C_0(\tau), g_0^{-1}(t)\vec{j}g_0(t)]C_1(\tau)\} \times \mathrm{Tr}_x R\int_0^\tau g_0^{-1}(\tau_1)[H_{0R}(\tau_1), A_{2\alpha}(\tau)]\,g_0(\tau_1)d\tau d\tau_1\,. \quad (39)$$

We now transform the last term in (39). For this purpose one may represent the operator B_1 as $B_1 = C_1 D$. From Eqs. (22) and (28) it follows that

$$i\hbar\frac{\partial}{\partial t}D = C_1^{-1}C_0^{-1}\vec{P}\cdot C_0 C_1 \mathrm{Tr}_x R\int_0^t C_0^{-1}(\tau)[H_{0R}(\tau), \vec{A}_2(t)]\,C_0(\tau)d\tau\,. \quad (40)$$

Having substituted the definition B_1 in the relation (39) and used Eq. (40), one obtains

$$\mathrm{Tr}_x RG^{-1}(a)\vec{j}G(a) = \mathrm{Tr}_x RC^{-1}\vec{j}C - \mathrm{Tr}_x R\int_0^t C_1^{-1}(\tau)\sum_\alpha [C_0^{-1}(\tau)P_\alpha C_0(\tau)C_0^{-1}(t)\vec{j}C_0(t)]\,C_1(\tau) \times \mathrm{Tr}_x R\int_0^\tau C_0^{-1}(\tau_1)[H_{0R}(\tau_1), A_{2\alpha}(\tau)]\,C_0(\tau_1)d\tau d\tau_1\,.$$

The latter relation can be converted to the form

$$\mathrm{Tr}_x RG^{-1}(a)\vec{j}G(a) = \tfrac{1}{2}\mathrm{Tr}_x RC_0^{-1}\vec{j}C_0 + \tfrac{1}{2}\mathrm{Tr}_x RC^{-1}\vec{j}C\,. \quad (41)$$

Now the equation for the fields $\vec{E}$ and $\vec{H}$ under the limitation (34) will be written as

$$\mathrm{curl}\langle \vec{H}_{H,1}(t)\rangle_x = \frac{1}{c}\frac{\partial}{\partial t}\langle \vec{E}_{H,1}(t)\rangle_x + \frac{2\pi}{c}\mathrm{Tr}_x Rg_0^{-1}\vec{j}g_0 + \frac{2\pi}{c}\mathrm{Tr}_x RC^{-1}\vec{j}C\,, \quad (42)$$

where

$$i\hbar\frac{\partial}{\partial t}C = \vec{P}\cdot\langle \vec{A}_{H,1}(t)\rangle_x C\,. \quad (43)$$

Equation (42) differs from Eq. (33) by the current. When the conditions of applicability of the Maxwell equations (31) are satisfied Eq. (42) converts to the Maxwell equation (33). Equations (42) and (43) describe the change of a strong field in interaction with a quantum medium at larger photon densities than that given by Eqs. (33).

VII. EQUATION FOR THE DENSITY MATRIX

To solve the problems on the change of radiation statistics upon interaction with the substance the equation of motion for the field density matrix is required.

We shall obtain this equation for semiclassic electrodynamics. For this purpose we shall average the semiclassic field, satisfying the Maxwell equations (26), over the initial density matrix ρ of the field:

$$\mathrm{Tr}_\gamma \rho\vec{F}_M(t) = \mathrm{Tr}_\gamma \rho\vec{F}_1(t) + \frac{i}{\hbar}\mathrm{Tr}_{\gamma x}\rho R\int_0^t C^{-1}(\tau)[H_{0R}(\tau), \vec{F}_{\mathrm{op}}(t)]\,C(\tau)\,d\tau = \mathrm{Tr}_\gamma \rho(t)\vec{F}_{\mathrm{op}}(t)\,.$$

The index γ denotes that the summation is taken over the field states. Having described the commutator in the second term, and using the trace invariance with reference to cyclic commutations, we obtain

$$\mathrm{Tr}_{\gamma x}\rho R\int_0^t C^{-1}(\tau)[H_{0R}(\tau), \vec{F}_{\mathrm{op}}(t)]\,C(\tau)\,d\tau = \mathrm{Tr}_{\gamma x}\int_0^t [H_{0R}(\tau), C(\tau)\rho R C^{-1}(\tau)]\,d\tau\,\vec{F}_{\mathrm{op}}(t),$$

from which it follows that the density matrix $\rho(t)$ is defined by the following relation:

$$\rho(t) = \rho - \frac{i}{\hbar}\mathrm{Tr}_x\int_0^t [H_{0R}(\tau), C(\tau)\rho R C^{-1}(\tau)]\,d\tau\,. \quad (44)$$

The C operator satisfies the Schrödinger Eq. (25), where the field in the interaction Hamiltonian is the solution of the Maxwell equations. If one knows the solution of the Maxwell equations (26) one may find the C operator, by means of which the density matrix in the semiclassical case is found from the relation (44).

The analyzed examples show that the eigenfunctions of the vector-potential operator ψ_a form a "natural" basis for the description of the interaction of a strong electromagnetic field with quantum systems.

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PHYSICAL REVIEW D VOLUME 8, NUMBER 10 15 NOVEMBER 1973

New Two-Metric Theory of Gravity with Prior Geometry*

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(Received 14 May 1973; revised manuscript received 23 July 1973)

We present a Lagrangian-based metric theory of gravity with three adjustable constants and two tensor fields, one of which is a nondynamical "flat-space metric" η. With a suitable cosmological model and a particular choice of the constants, the "post-Newtonian limit" of the theory agrees, in the current epoch, with that of general relativity theory (GRT); consequently our theory is consistent with current gravitation experiments. Because of the role of η, the gravitational "constant" G is time-dependent and gravitational waves travel null geodesics of η rather than the physical metric g. Gravitational waves possess six degrees of freedom. The general exact static spherically-symmetric solution is a four-parameter family. Future experimental tests of the theory are discussed.

I. INTRODUCTION AND SUMMARY

Within the past few years an elegant theoretical formalism, the "parametrized post-Newtonian" (PPN) framework, has been developed[1] to analyze metric[2] theories of gravity. The PPN framework is structured around the "weak gravitational fields" and low velocities of the gravitational matter which characterize typical solar-system tests of gravity. It classifies each gravitation theory as to its form "in the post-Newtonian (PN) limit." At first it was hoped, and indeed seemed to be true, that the PN limit of each theory of gravity is unique—thus by solar-system experiments alone, one could, in principle, determine the "correct PN limit," which would then correspond to one and only one "correct theory of gravity." In addition, it was hoped and is hoped that the "correct PN limit" is that of general relativity theory (GRT) (although we try not to let this fact prejudice our investigations). To play devil's advocate, a program was initiated to attempt to formulate theories of gravity with the same PN limit (and hence PPN parameters[1]) as GRT. The aims of such a program are twofold, as one can ask the following questions: (i) If such theories exist, how complex and contrived are their formulations? (ii) Do such theories have anything in common and in what respect do they differ from GRT outside of the PN limit? The first question is primarily only of aesthetic interest. But the second has the possibility of identifying powerful new theoretical and experimental tools for testing relativistic gravity—indeed, that has been the case (see Sec. V and Refs. 3 and 4).

In this paper[5] we present and analyze a new theory of gravity—one which has the same PN limit (for the current epoch) as GRT, given a suitable cosmological model and a particular choice of the adjustable constants. Analysis of our new theory provides partial answers to questions (i) and (ii) above.

A further motivation for study of this particular theory is to analyze in detail the role of prior geometry,[2] and its influence through cosmological boundary values, in gravitation theories, a role which will be investigated in more general terms in another paper.[6]

To date the authors are aware of three other new metric theories which are candidates for sharing the property of having the same PN limit as GRT (candidates in the sense of contingency upon the existence of special but acceptable cos-