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Lines of Force of a Point Charge near a Schwarzschild Black Hole*†

Richard Squier Hanni and Remo Ruffini

Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540

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The electric field generated by a charged particle at rest near a Schwarzschild black hole is analyzed using Maxwell's equations for curved space. After generalizing the definition of the lines of force to our curved background, we compute them numerically and graph them with the charge at $r = 4M$, $3M$, and $2.2M$. Particular attention is paid to the behavior of the lines of force near the event horizon and the smooth transition of the electric field to that of a Reissner-Nordström black hole.

I. INTRODUCTION

The formalism for gravitational perturbations away from a Schwarzschild background has been developed by Regge and Wheeler.[1] It was extended by Zerilli,[2] who has shown that perturbations corresponding to a change in the mass, the angular momentum, and the charge of a Schwarzschild black hole are well behaved. The decay of the non-well-behaved perturbations has been investigated by Price.[3] He has shown that any multipole $l \geq s$, where s is the spin of the field being examined, gets radiated away in the late stage of gravitational collapse and will die as $t^{-(2l+2)}$ for large t.

Instead of analyzing how higher-order multipoles are radiated away, we focus on how the allowed transition from a Schwarzschild to a Reissner-Nordström hole takes place through the capture of a charged particle in a given Schwarschild background. In this paper we neglect the electromagnetic radiation emitted during the fall of the particle and consider a succession of configurations in which the particle is momentarily at rest at decreasing distances from the Schwarzschild horizon ($r = 2M$ in geometrical units $G = c = 1$). The problem of examining the radiation emitted is, indeed, of great interest and has been presented elsewhere.[4]

The electric field of a charge at rest with respect to the Schwarzschild background can be developed in a multipole expansion centered about the black hole. For any finite separation of the charge from the black hole, the far-away observer will detect only the monopole term, the field corresponding to a Reissner-Nordström solution. In the region near the charge, however, the contribution of higher multipoles is important and the lines of force are no longer radial.

As the charge approaches the horizon, the strength of all multipoles, except the monopole term, tends to zero. The lines of force assume more and more their Reissner-Nordström pattern, only a very small region around the particle being significantly affected by the higher multipoles. We express the strengths of the multipole coefficients as functions of the distance of the charge from the horizon.

We generalize the concept of lines of force to curved space, and show that they are equivalent to the lines of constant flux. The concept of "induced charge" is introduced, and the *smooth* transition from the Schwarzschild to a Reissner-Nordström field is analyzed. The lines of force are evaluated numerically and presented graphically to give a complete picture of the electric field. Finally, we consider the behavior of the lines of force near the event horizon from a local and a global perspective.

II. ELECTROSTATIC FIELD IN A SCHWARZSCHILD BACKGROUND

The Schwarzschild metric can be expressed as:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + \left(1-\frac{2M}{r}\right)^{-1}dr^2 . \quad (1)$$

The only nonvanishing components of the electromagnetic field

$$F = F_{\mu\nu}dx^\mu \Lambda dx^\nu , \quad (2.1)$$

with

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} , \quad (2.2)$$

of a particle at rest with respect to the fixed background are:

$$F_{rt} = A_{t,r} = -F_{tr} \quad (3.1)$$

and

$$F_{\theta t} = A_{t,\theta} = -F_{t\theta} . \quad (3.2)$$

Using the relation,

$$\star F_{\mu\nu} = \tfrac{1}{2}\sqrt{-g}\,\epsilon_{\mu\nu\delta\gamma}F^{\delta\gamma} , \quad (4)$$

whereby $\epsilon_{\mu\nu\delta\gamma}$ we indicate the Levi-Civita symbol and $g = \det|g_{\mu\nu}|$, we find that the only nonvanishing components of the dual electromagnetic tensor are:

$$\star F_{\theta\phi} = r^2 \sin\theta A_{t,r} \quad (5.1)$$

and

$$\star F_{\phi r} = \sin\theta\left(1-\frac{2M}{r}\right)^{-1} A_{t,\theta} . \quad (5.2)$$

The only nonvanishing component of the four-current is:

$$j^t = \frac{q}{2\pi r^2}\delta(r-2Ma)\delta(\cos\theta - 1) , \quad (6)$$

where a is the value of the radial coordinate where the charge is located.

In covariant form Maxwell's equations are:

$$F^{\alpha\beta}{}_{;\beta} = -4\pi j^\alpha \quad (7.1)$$

and

$$\star F^{\alpha\beta}{}_{;\beta} = 0 . \quad (7.2)$$

(Greek indices here and in the following go from 1 to 4.) With (5) and (6) they yield a second-order differential equation:

$$\frac{\partial(r^2 A_{t,r})}{\partial r}\Big/ r^2 + \frac{\partial(\sin\theta A_{t,\theta})}{\partial\theta}\Big/ r^2\sin\theta\left(1-\frac{2M}{r}\right) = -4\pi j^t . \quad (8)$$

Using the axial symmetry of the problem and its regularity on the axis of symmetry, we can expand the solution in terms of Legendre polynomials

$$A_t = \sum_{l=0}^{\infty} f_l(r)P_l(\cos\theta) . \quad (9)$$

The functions $f_l(r)$ then satisfy the second-order differential equation

$$\sum_l \left\{\frac{d}{dr}\left[r^2\frac{d}{dr}f_l(r)\right] - l(l+1)f_l(r)\Big/\left(1-\frac{2M}{r}\right)\right\}P_l(\cos\theta) = q\delta(r-a)\delta(\cos\theta-1)/2\pi . \quad (10)$$

This equation was first solved for general electric fields in a Schwarzschild background by Israel.[5] He found the quadratic transformation under which Eq. (10) takes the standard form for the associated Legendre functions. We solved Eq. (10) using the method of Frobenius.[6,7] The resulting expansions are convenient for the physical interpretation of the conditions imposed by matching the boundary conditions.[8]

Substituting $g_l(r) = rf_l(r)$ and $z = r/2M$, Eq. (10) takes the hypergeometric form:

$$\frac{d^2g_l}{dz^2} - \frac{l(l+1)}{z(z-1)}g_l = 0 . \quad (11)$$

One of the two independent solutions of this equation is a polynomial of degree $l+1$ with coefficients given by the recursion relation

$$n(n+1)\alpha_{n+1}^l = [n(n-1) - l(l+1)]\alpha_n^l . \tag{12}$$

The first few are:

$$u_0 = z , \tag{13.1}$$

$$u_1 = z(1-z) , \tag{13.2}$$

$$u_2 = z(z-1)(2z-1) . \tag{13.3}$$

To obtain the other independent solution v_l we use the following relations, which follows from (11):

$$v_l u_l'' - u_l v_l'' = (v_l u_l' - u_l v_l')' = 0 \tag{14}$$

or

$$v_l u_l' - u_l v_l' = c . \tag{15}$$

Integrating,

$$v_l = c u_l \int \frac{dz}{u_l^2} . \tag{16}$$

From the explicit expressions for u_l we obtain for v_l:

$$v_0 = 1 , \tag{17.1}$$

$$v_1 = z(1-z)\{2\ln[z/(z-1)] - z^{-1} - (z-1)^{-1}\} , \tag{17.2}$$

$$\begin{aligned} v_2 = {} & z(z-1)(2z-1) \\ & \times\{-8(2z-1)^{-1} - (z-1)^{-1} \\ & + 6\ln[z/(z-1)] - z^{-1}\} , \end{aligned} \tag{17.3}$$

and so on for larger l. The most general solution for the potential can be cast in the form

$$A_t = \sum_l \frac{\alpha^l u_l(r) + \beta^l v_l(r)}{r} P_l(\cos\theta) . \tag{18}$$

All of the u_l's except the one corresponding to $l=0$ vanish at the horizon. In the region where $2M < r < a$ all the β^l's must vanish for the potential to be regular at the event horizon and the flux through a surface enclosing the hole but not the point charge, to be zero. Moreover, since $P_0(\cos\theta) = 1$, the horizon is an equipotential surface (A_t is constant).

For the potential to vanish as $r \to \infty$ all the α^l's must vanish in the region where $r > a$. As $r \to \infty$, the term v_0 is constant while all the other terms v_l decrease as r^{-l}. The monopole term dominates at infinity. Gauss's law gives us the magnitude of the spherically symmetric electric field and thus the weighting coefficient for the monopole term, $\beta^0 = q$. The field far away approaches that of a point charge located at the center of the black hole.

III. MATCHING CONDITIONS AND STRENGTH OF THE MULTIPOLES

In order to calculate the electric field we must evaluate the weighting coefficients: α^1, β^1, α^2, $\beta^2, \ldots$. Substituting the expansion (18) of A_t into Poisson's equation (10), we relate the discontinuity in the slope of each radial function to the amount of charge concentrated at $r=a$, $\theta=0$, we have

$$\sum_l \left[\frac{d^2 g_l}{dz^2} - \frac{l(l+1)g_l}{z(z-1)}\right] P_l(\cos\theta) = \frac{-2q}{z}\delta(z-a)\delta(\cos\theta - 1) . \tag{19}$$

Integrating over a thin shell containing the point $z=a$ and using the orthogonality of the Legendre polynomials, we obtain

$$\left[\frac{dg_l}{dz}\right]_{z=a^-}^{z=a^+} = -(2l+1)q/a . \tag{20}$$

The continuity of the radial functions and the boundary conditions at the horizon and at infinity determine the weighting coefficients α_l and β_l uniquely.

Since the potential is continuous at the charge, we have

$$\begin{aligned} \alpha^l u_l(a) &= \beta^l v_l(a) \\ &= \beta^l u_l(a) \int_0^a \frac{dz}{u_l^2(z)} . \end{aligned} \tag{21}$$

The discontinuity in the field at the charge gives

$$\alpha^l u_l'(a) - \beta^l v_l'(a) = \frac{2l+1}{a} q ,$$

or, for $l=0$, $\beta^0 = q$ independent from the position of the charge, and $\alpha^0 = q/a$,

$$\begin{aligned} &\beta^1 = 3q(a-1) , \\ &\alpha^1 = 3q(1-a)\{2\ln[a/(a-1)] - a^{-1} - (a-1)^{-1}\} , \end{aligned} \tag{22}$$

$$\begin{aligned} &\beta^2 = 5q(2a-1)(a-1) , \\ &\alpha^2 = 5q(a-1)(2a-1) \\ &\quad \times\{6\ln[a/(a-1)] - 8(2a-1)^{-1} - (a-1)^{-1} - a^{-1}\} . \end{aligned} \tag{23}$$

Because of the complexity of higher-order terms we resort to numerical integrations to evaluate α^l, β^l, u_l, and v_l.

Figure 1 shows the behavior of the radial functions for selected values of l and selected values of the distance of the particle $a=3M$, $a=4M$, and $a=6M$. While the monopole coefficient remains constant, all the other multipoles vanish as the particle approaches the horizon. The decay of the higher multipoles ($l>0$) and the constancy of the

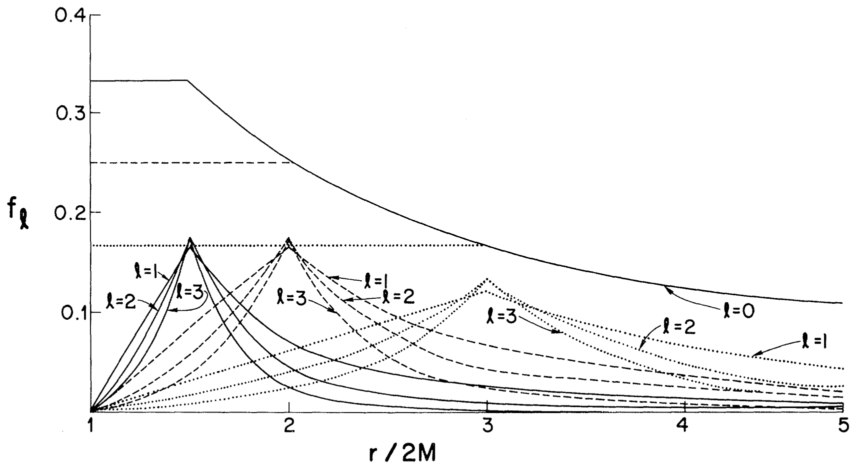

FIG. 1. Radial functions $f_l = [\alpha^l u_l(r) + \beta^l v_l(r)]/r$ plotted as a function of the radial Schwarzschild coordinate r for the selected values of l and of the distance of the particle from the Schwarzschild surface $a = 2, 1.5, 1.1$. The strength of the multipoles $\beta_l = f_l r^{l+1}$ is constant for $l = 0$ and increases with the distance of the particle from the black hole for $l > 0$.

monopole term determine the smooth transition from a Schwarzschild to a Reissner-Nordström geometry.

IV. DEFINITION OF THE LINES OF FORCE

The generalization of Gauss's theorem to curved space is[9] given by the equation

$$\int_S {}^\star F = 4\pi q\,, \tag{24}$$

where q is the total charge within the surface S and

$${}^\star F = {}^\star F_{\mu\nu} dx^\mu \wedge dx^\nu\,. \tag{25}$$

Only the components (5.1) and (5.2) are nonzero, so

$${}^\star F = 2\left[r^2 \sin\theta \frac{\partial A_t}{\partial r} d\theta - \left(1 - \frac{2M}{r}\right)^{-1} \sin\theta \frac{\partial A_t}{\partial \theta} dr\right] \wedge d\phi\,. \tag{26}$$

If we consider an infinitesimal displacement in θ of the ring bounding a surface of azimuthal symmetry, we find that the change in flux is equal to the flux through the annular cap generated by the displacement

$$\begin{aligned}\delta\Phi &= \int {}^\star F_{\theta\phi} d\theta d\phi \\ &= 2\pi \int_\theta^{\theta+\delta\theta} r^2 \sin\theta A_{t,r} d\theta\,.\end{aligned} \tag{27.1}$$

So the flux depends on the angular coordinate of the ring as

$$\frac{\partial\Phi}{\partial\theta} = 2\pi r^2 \sin\theta A_{t,r}\,. \tag{27.2}$$

Similarly, the radial dependence is:

$$\frac{\partial\Phi}{\partial r} = -2\pi {}^\star F_{\phi r} = -2\pi \sin\theta \left(1 - \frac{2M}{r}\right)^{-1} A_{t,\theta}\,. \tag{27.3}$$

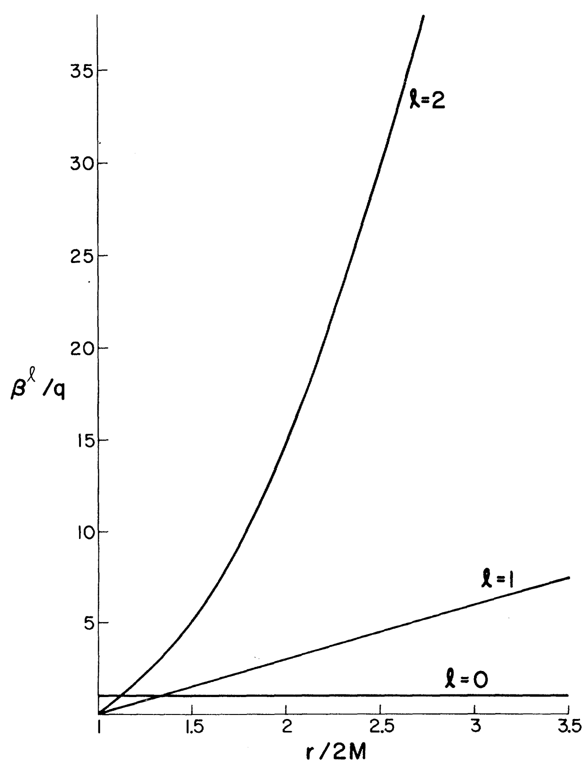

FIG. 2. Strength of the multipoles in units of the charge of the test particle q, as a function of the distance of the charge from the event horizon ($r = 2M$). As $a \to 1$, the monopole is constant, while the higher multipoles vanish.

Let the flux equal

$$\Phi = -2\pi \sum_{l \neq 0} \frac{r^2}{l(l+1)} \frac{df_l}{dr} \sin\theta \frac{dP_l}{d\theta}. \qquad (28.1)$$

Recalling the differential equation for the Legendre polynomials, we see that

$$\begin{aligned} \frac{\partial \Phi}{\partial \theta} &= +2\pi \sum_{l \neq 0} r^2 \frac{df_l}{dr} \sin\theta\, P_l \\ &= 2\pi r^2 \sin\theta A_{t,r}\,. \end{aligned} \qquad (28.2)$$

Recalling the differential equation for the radial functions (10), we note that

$$\begin{aligned} \frac{\partial \Phi}{\partial r} &= -2\pi \sum_{l \neq 0} \left(1 - \frac{2M}{r}\right)^{-1} f_l \sin\theta \frac{dP_l}{d\theta} \\ &= -\frac{2\pi \sin\theta}{1 - 2M/r} A_{t,\theta}\,. \end{aligned} \qquad (28.3)$$

Clearly this analytic expression for the flux satisfies our conditions, (27.2) and (27.3), and is determined by them up to an arbitrary constant. Since the slope of the lines of constant flux depends only on the radial and angular dependence of the flux, it is uniquely determined.

We define a line of constant flux as the locus of points with a given value of Φ. At any given point, the slope of the line of constant flux is:

$$\begin{aligned} \frac{dr}{d\theta} &= -\frac{\partial \Phi}{\partial \theta} \Big/ \frac{\partial \Phi}{\partial r} \\ &= \left(1 - \frac{2M}{r}\right) r^2 \frac{\partial A_t}{\partial r} \Big/ \frac{\partial A_t}{\partial \theta}. \end{aligned} \qquad (29)$$

This is equivalent to the operational definition of lines of force, introduced by Christodoulou and Ruffini.[10] They define a line of force as the line tangent to the direction of the electric force measured by a free-falling observer momentarily at rest; this definition is also valid for stationary metrics. We than have

$$\epsilon_{ijk} F^{j}{}_{\alpha} u^{\alpha} dx^{k} = 0 \qquad (30)$$

or for this problem

$$F^{r}{}_{t} u^{t} d\theta - F^{\theta}{}_{t} u^{t} dr = 0 \qquad (30.1)$$

which also gives expression (29) for the slope of the lines of force if we assume an inertial observer with four-velocity

$$u = (0, 0, 0, (1 - 2M/r)^{-1/2}).$$

Figures 3, 4, and 5 show the lines of force in Schwarzschild coordinates, as the charged particle approaches the event horizon. When the charge is at $r = 2.2M$ (Fig. 5), the field far away ($r \gtrsim 10M$) from the hole is nearly radial about the center of the hole. The monopole term clearly dominates. The contribution of the higher multipoles is dominant in the region near the charge.

The lines of force intersect the event horizon. We interpret this as a charge induced on the surface of the hole, which is proportional to the electric field normal to the surface. The total flux through the Schwarzschild surface, and thus the net induced charge, is zero.

Let us assume that the point charge is positive. At angles less than a critical angle $\theta_{\rm crit}$, the induced charge is negative and the lines of force go toward the event horizon. At the critical angle there is no induced charge and the line of force is

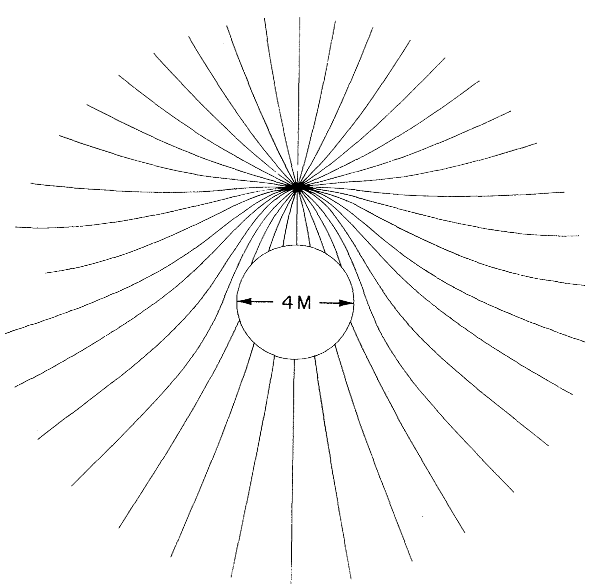

FIG. 3. Lines of force with the test charge at rest at $r = 4M$.

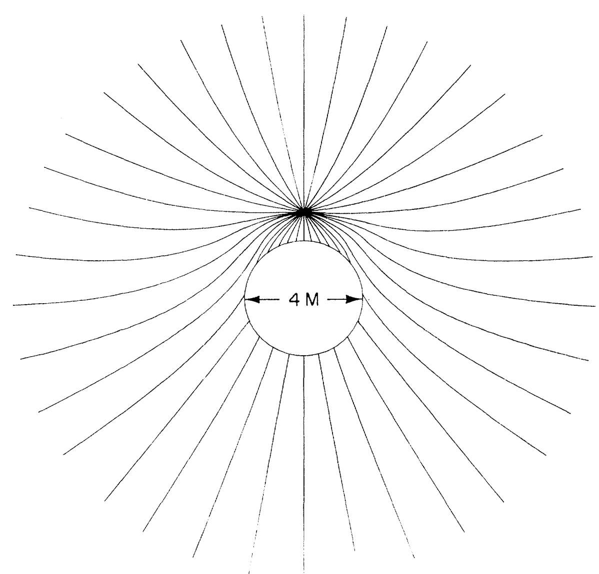

FIG. 4. Lines of force with the test charge at rest at $r = 3M$.

tangent to the Schwarzschild surface. At angles greater than the critical angle, the induced charge is positive and the lines of force are directed away from the horizon. Figure 6 shows the ratio of the positive charge induced to that of the point charge and the critical angle, as a function of the radial coordinate of the charge.

The behavior of this induced charge shows clearly the smooth transition from a Schwarzschild to a Reissner-Nordström hole. As the charge approaches the horizon, the magnitude of the induced charge approaches that of the point charge and the negative induced charge is crowded into a decreasingly small area around the pole. In contrast, the positive charge disperses itself more and more evenly over the rest of the surface of the sphere. The asymptotic limit of this evolution is a dipole with no strength at the pole, and a positive charge, equal in magnitude to that of the point charge, distributed evenly over the Schwarzschild surface. This charge distribution generates the monopole field of the Reissner-Nordström solution.

It is essential to the interpretation of Fig. 6, that as seen by a far-away observer, the charge approaches, but never reaches, the horizon ($r=2M$). The time as a function of the radius for a freely falling nonradiating particle with no angular momentum about the center of the hole is

$$T(r) = -\tfrac{4}{3}\left(\frac{r}{2M}\right)^{3/2} - 4\left(\frac{r}{2M}\right)^{1/2} + 2\ln\left\{\left[\left(\frac{r}{2M}\right)^{1/2}+1\right]\left[\left(\frac{r}{2M}\right)^{1/2}-1\right]^{-1}\right\}.$$

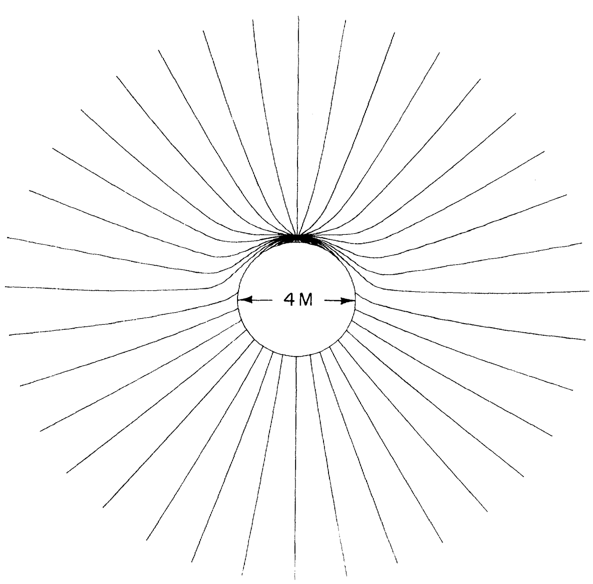

FIG. 5. Lines of force with the test charge at rest at $r=2.2M$.

The ratio of the induced charge to the point charge approaches, but never reaches, unity. The critical angle approaches, but never reaches, zero. The coordinate velocity,

$$\frac{dr}{dt} = (1-2M/r)(2M/r)^{1/2},$$

near the hole decreases with distance and decreases exponentially with time. Thus the system we have investigated is a good approximation of a charged particle in radial free fall, when the charge is sufficiently close to the hole.

V. BEHAVIOR OF THE LINES OF FORCE NEAR THE EVENT HORIZON

We have expanded the potential in the region within the charge in multipoles:

$$A_t = \alpha^0 + \alpha^1(z-1)\cos\theta + \tfrac{1}{2}\alpha^2(z-1)(2z-1)(3\cos^2\theta - 1) + \cdots . \quad (30.2)$$

Since the Schwarzschild metric is diagonal, the physical components of the Lorentz force take the form

$$L_{\hat{r}} = \sqrt{g^{rr}}\,F_{rt}U^t = A_{t,r}, \quad (30.3)$$

$$L_{\hat{\theta}} = \sqrt{g^{\theta\theta}}\,F_{\theta t}U^t = (r^2 - 2Mr)^{-1/2}A_{t,\theta}. \quad (30.4)$$

As we approach the event horizon, the θ physical component of the Lorentz-force vanishes as $(1-2M/r)$, while the radial physical component remains finite. Consequently, the lines of force must intersect the horizon, and they must do so

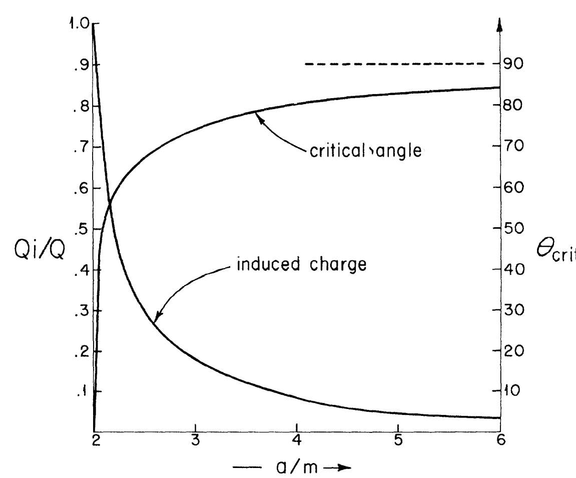

FIG. 6. The critical angle, θ_{crit}, where the line of force is tangent to the Schwarzschild surface and the induced surface charge vanishes, as a function of the radial coordinate of the test particle. Also given is the ratio of the charge induced on the section of the event horizon where $\theta > \theta_{crit}$ to the charge of the test particle.

orthogonally. This result seems inconsistent with the lines of force calculated using the flux method, shown in Figs. 4, 5, and 6.

To resolve this apparent contradiction, we must look at things in the curved background. Since such a four-dimensional manifold is difficult to visualize, we will exploit its static nature and axial symmetry, by suppressing the temporal and azimuthal dependence. Then the exterior Schwarzschild solution can be visualized as a two-dimensional hyperboloid embedded in the usual three-space. A simple application of the Pythagorean theorem to a radial trajectory suffices to determine the radial metric coefficient:

$$d\sigma^2 = dr^2 + dz^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 = g_{rr} dr^2 . \qquad (30.5)$$

Integration of this differential relation yields an analytic formula for the embedding in regular cylindrical coordinates:

$$dz = \left(\frac{r}{2M} - 1\right)^{-1/2} dr ,$$

or (30.6)

$$r = 2M + \frac{z^2}{8M} .$$

$z = 0$ and $r = 2M$ at the event horizon.

It is in the curved space that the lines of force intersect the event horizon orthogonally. As we have already shown, the Lorentz-force is tangent to the lines of force in the curved space. The θ physical component of the Lorentz force vanishes at the horizon, while the radial physical component remains finite. The azimuthal physical component is everywhere zero. Thus the field is strictly radial at the event horizon in curved space. When we look at the lines of force in Schwarzschild coordinates, however, we are projecting the embedding diagram onto the plane $z = 0$, which contains the event horizon. This is how they would be represented by an observer at infinity. The lines of force, as seen from infinity, or equivalently projected onto the plane $z = 0$, may even be tangent to the event horizon. This is because the slope of the hyperboloid,

$$\frac{dz}{dr} = \frac{4M}{z} \qquad (30.7)$$

or equivalently the radial component of the metric, which relates the coordinate distance to the proper distance measured by a local observer, diverges at the event horizon.

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