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Canonical Quantization of Relativistic Balls of Dust*

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The Hamiltonian form for the equations of a relativistic perfect fluid is considered and later specialized to the case of spherical symmetry and vanishing pressure. When comoving coordinates are used in the canonical formalism, one gets a reduced Hamiltonian which is independent of time. The continuous number of degrees of freedom are decoupled and the Schrödinger equation separates from a functional differential equation to a set of identical ordinary differential equations. Boundary conditions for these equations are naturally obtained by requiring that the minisuperspace be geodesically complete. The formalism remains the same whether one treats a closed nonhomogeneous universe or a collapsing star. The problem of singularities is discussed, and it is concluded that in this minisuperspace quantum formalism there is no inevitable singularity.

I. INTRODUCTION

The Hamiltonian methods developed by Arnowitt, Deser, and Misner[1] have been generalized by Schutz[2] to include perfect fluids. We specialize this formalism to spherically symmetric, pressureless dust. Spherically symmetric systems have the advantage of being sufficiently simple to admit explicit solutions, and of being sufficiently rich in structure not to require for their description a specific one-parameter family of hypersurfaces as is the case for homogeneous geometries. This allows the field aspect of general relativity to become explicit. Vacuum spherically symmetric systems have been shown to have no true dynamical degrees of freedom.[3] In this sense, the collapse of the Schwarzchild throat is a completely classical phenomenon, unchanged by quantum mechanics. The cloud of dust considered in this paper turns out be much more interesting, as it has a continuous number of true dynamical degrees of freedom, each one corresponding essentially to the radius of each spherical dust shell.

In Sec. II, the classical problem is reviewed from from the ADM point of view. The choice of comoving observers (vanishing shift, and lapse equal to unity) suggests a set of coordinate conditions (with matter acting as a clock) which make the super-Hamiltonian linear in the momentum canonically conjugate to time and also decouple the ∞^1 dynamical degrees of freedom. The reduced Hamiltonian is time-independent. The Schrödinger equation for the system is written down in Sec. III. Since all degrees of freedom are decoupled, each shell of dust moves independently of the matter that surrounds it and it is therefore enough to consider an ordinary differential equation in order to have a complete description of the problem.

The behavior of the quantized system near the classical singularity is discussed in Sec. III C. Our model remains pressureless at all times, even at advanced stages of collapse. This is surely unrealistic, but is has the advantage of being amenable to analytic treatment. In any case, it is of interest to see how in this simplified model one can answer a number of important questions. Quantized gravity has been conjectured to offer an escape from the singularities inherent to the classical theory.[4] The model studied here supports this conjecture, as the formalism is perfectly well-behaved at the classical singularity, and quantities that classically vanished at the singularity have nonvanishing expectation values in the quantum formalism.

II. THE CLASSICAL PROBLEM

Hamilton's equations for a perfect fluid in general relativity may be obtained by extremizing the action functional

$$S=\int d^3x dt\,(\pi^{ik}\dot g_{ik}+p^\phi\dot\phi+p^m q_m-\mathcal{H}), \tag{1}$$

with the Hamiltonian density

$$\mathcal{H}=N(\mathcal{H}^0+\mathcal{E})+N_i(\mathcal{H}^i+\mathcal{P}^i)+\lambda_m\phi_m, \tag{2}$$

upon variation of the variables π^{ik}, g_{ik}, p^ϕ, ϕ, q_m, λ_m, N, and N_i independently.[2] Let us specalize this to the case of the spherical symmetry. The line element for the spatial geometry may be written in the form

$$dl^2=e^{2\mu}dr^2+e^{2\lambda}(d\theta+\sin^2\theta\, d\phi^2), \tag{3}$$

with μ and λ being arbitrary functions of the radial coordinate r. If we define

$$\pi^{ik}=:\mathrm{diag}\left(\tfrac{1}{2}\pi_\mu e^{-2\mu}, \tfrac{1}{4}\pi_\lambda e^{-2\lambda}, \tfrac{1}{4}\pi_\lambda e^{-2\lambda}\sin^{-2}\theta\right)$$

and drop the variables λ_m and q_m since the radial flow is irrotational,[5] we obtain, after integration over the angular variables,

$$S = 4\pi \int dr dt [p^\phi \dot{\phi} + \pi_\mu \dot{\mu} + \pi_\lambda \dot{\lambda} - N(\mathcal{H}^0 + \mathcal{E}) - N_1(\mathcal{H}^1 + \mathcal{P}^1)] . \quad (4)$$

Here, $\mathcal{H}^0$ and $\mathcal{H}^1$ are the super-Hamiltonian and supermomentum for the free gravitational field. Their explicit form is given by[6]:

$$\mathcal{H}^0 = e^{-\mu-2\lambda}\{\tfrac{1}{8}\pi_\mu{}^2 - \tfrac{1}{4}\pi_\mu\pi_\lambda + 2e^{4\lambda}[2\lambda'' - 2\lambda'\mu' + 3(\lambda')^2 - e^{2(\mu-\lambda)}]\} , \quad (5)$$

$$\mathcal{H}^1 = -e^{-2\mu}(\pi_\mu{}' - \mu'\pi_\mu - \lambda'\pi_\lambda) . \quad (6)$$

The coordinate densities of energy and of radial momentum $\mathcal{E}$ and $\mathcal{P}^1$, as measured by an observer moving in the direction perpendicular to the hypersurfaces of constant coordinate time, are

$$\mathcal{E} = g^{1/2}N^2[(\rho + p)(U^0)^2 + pg^{00}] , \quad (7)$$

$$\mathcal{P}^1 = g^{1/2}p^\phi g^{11}\phi_{,1} , \quad (8)$$

where p is the pressure and ρ the density of total mass energy. Here, ϕ is a scalar field related to the four-velocity U_ν through

$$U_\nu = h^{-1}\phi_{,\nu} , \quad (9)$$

where h is the specific enthalpy

$$h = \rho_0{}^{-1}(\rho + p) , \quad (10)$$

with ρ_0 being the rest-mass density. The momentum p^ϕ is given by

$$p^\phi = -16\pi N g^{1/2}\rho_0 U^0 . \quad (11)$$

N and N_1 are the lapse and radial-shift functions, respectively. The quantities $\mathcal{H}^2$, $\mathcal{H}^3$, $\mathcal{P}^2$, $\mathcal{P}^3$, U_2, and U_3 vanish identically on account of the spherical symmetry. Substituting (11) into (7) one gets

$$\mathcal{E} = (16\pi)^{-1}g^{-1/2}\rho_0{}^{-2}(\rho + p)(p^\phi)^2 - 16\pi p g^{1/2} . \quad (12)$$

Consider now the situation in which the fluid consists of pressureless dust. In this case, $p = 0$, $\rho = \rho_0$, $h = 1$. From the normalization of the four-velocity $U^\nu U_\nu = -1$ one gets

$$\rho_0 = (16\pi)^{-1}g^{-1/2}(1 + g^{11}\phi_{,1})^{-1/2}p^\phi , \quad (13)$$

with the consequence that

$$\mathcal{E} = p^\phi[1 + e^{-2\mu}(\phi')^2]^{1/2} \quad (14)$$

and

$$\mathcal{P}^1 = e^{-2\mu}p^\phi\phi' . \quad (15)$$

The prime denotes differentiation with respect to r.

We have thus arrived at a problem described by the field variables ϕ, μ, λ, their canonically conjugate momenta p^ϕ, π_μ, π_λ, and the Hamiltonian

$$H = \int dr \{N(\mathcal{H}^0 + \mathcal{E}) + N_1(\mathcal{H}^1 + \mathcal{P}^1)\}$$

given in terms of them by (5), (6), (14), and (15). The dynamical equations are (the dot denotes differentiation with respect to time):

$$\dot{\lambda} = \frac{\delta H}{\delta \pi_\lambda} = -\tfrac{1}{4}\pi_\mu N e^{-\mu-2\lambda} + N_1 e^{-2\mu}\lambda' , \quad (16)$$

$$\dot{\mu} = \frac{\delta H}{\delta \pi_\mu} = \tfrac{1}{4}Ne^{-\mu-2\lambda}(\pi_\mu - \pi_\lambda) + (N_1 e^{-2\mu})' + N_1 e^{-2\mu}\mu' , \quad (17)$$

$$\dot{\pi}_\mu = -\frac{\delta H}{\delta \mu} = N\mathcal{H}^0 - 4(Ne^{-\mu}\lambda' e^{2\lambda})' + 4Ne^\mu + Np^\phi e^{-2\mu}(\phi')^2[1 + e^{-2\mu}(\phi')^2]^{-1/2} + (N_1 e^{-2\mu}\pi_\mu)' + 2N_1 e^{-2\mu}p^\phi\phi' - 2N_1\mathcal{H}^1 , \quad (18)$$

$$\dot{\pi}_\lambda = -\frac{\delta H}{\delta \lambda} = 2N\mathcal{H}^0 - 8Ne^{-\mu+2\lambda}[2\lambda'' - 2\lambda'\mu' + 3(\lambda')^2 - e^{2\mu-2\lambda}] - 4(Ne^{-\mu+2\lambda})'' - 4(Ne^{-\mu+2\lambda}\mu')' + 12(Ne^{-\mu+2\lambda}\lambda')' - 4Ne^\mu - Np^\phi[1 + e^{-2\mu}(\phi')^2]^{1/2} + (N_1 e^{-2\mu}\pi_\lambda)' - N_1 e^{-2\mu}p^\phi\phi' , \quad (19)$$

$$\dot{\phi} = \frac{\delta H}{\delta p^\phi} = N[1 + e^{-2\mu}(\phi')^2]^{1/2} + N_1 e^{-2\mu}\phi' , \quad (20)$$

$$\dot{p}^\phi = -\frac{\delta H}{\delta \phi} = \{Np^\phi e^{-2\mu}\phi'[1 + e^{-2\mu}(\phi')^2]^{-1/2}\}' + (N_1 e^{-2\mu}p^\phi)' , \quad (21)$$

and the constraints are

$$\mathcal{H}^0 + \mathcal{E} = 0 , \quad (22)$$

$$\mathcal{H}^1 + \mathcal{P}^1 = 0 . \quad (23)$$

We are now in a position to apply the ADM reduction procedure. First of all, we need to specify a time variable. A time-coordinate built from the gravitational degrees of freedom alone would lead to a nonlocal reduced Hamiltonian, as it did for the Klein-Gordon geon.[7] This is because to obtain the reduced Hamiltonian, one must typically solve a differential equation. However, we observe that p^ϕ enters the constraint equations (22) and (23) linearly. This suggests to build a time variable from the matter variables.[9] Accordingly, we set

$$T(r, t) =: \phi(r, t) . \tag{24}$$

For this definition to make sense, it must be shown that this time runs monotonically. That is, the change δT must always have the same sign when passing from one slice to another, separated by a proper-time interval $\delta\tau = N\delta t$. It is enough to show this is so in a special coordinate system for each hypersurface: Set then $N_1 = 0$. From Eq. (20) we find

$$\delta T = [1 + e^{-2\mu}(\phi')^2]^{1/2} N\delta t , \tag{25}$$

or

$$\frac{\delta T}{\delta\tau} = [1 + e^{-2\mu}(\phi')^2]^{1/2} , \tag{26}$$

which is always positive, as desired.

With the conditions $T = -t$, $N_1 = 0$ it is possible to solve Eqs. (16)–(23) and to recover the solution for a spherically symmetric cloud of dust first given by Tolman.[9] In particular, (20) says that $N = 1$, and (21) says that $p^\phi = 0$. Using (23) and (16), (17) says that $\lambda' e^{\lambda-\mu}$ is a function of r only. The last fact guides us in the choice of a radial variable R to use in the reduced Hamiltonian. Define

$$R(r, t) =: \lambda' e^{\lambda-\mu} . \tag{27}$$

The transformation so defined should be canonical. This leads to the following changes in the momenta:

$$\pi_\mu \to \pi_R = -\pi_\mu(\lambda')^{-1} e^{\mu-\lambda} , \tag{28}$$

$$\pi_\lambda \to \bar\pi_\lambda = \pi_\lambda - e^\lambda [\pi_\mu(\lambda')^{-1} e^{-\lambda}]' . \tag{29}$$

The supermomentum constraint (23) now reads

$$\pi_R R' + \bar\pi_\lambda \lambda' + p^\phi \phi' = 0 . \tag{30}$$

If we impose the coordinate conditions $T = -t$, $R = r$ and solve the constraints (22) and (23), the action (4) becomes

$$S = 4\pi \int dt dr (\bar\pi_\lambda \dot\lambda + \pi_T) , \tag{31}$$

with

$$\begin{aligned} \pi_T &= p^\phi \\ &= -\tfrac{1}{8} r^2 e^{-3\lambda} \bar\pi_\lambda^2 + 2(1 - r^{-2}) e^\lambda \\ &\quad + (\text{total derivatives with respect to } r). \end{aligned} \tag{32}$$

Finally, after another canonical transformation,

$$x = (\tfrac{32}{9})^{1/2} e^{3\lambda/2} , \tag{33}$$

$$\pi_x = 8^{-1/2} e^{-3\lambda/2} \bar\pi_\lambda , \tag{34}$$

we get

$$S = 4\pi \int dt dr (\pi_x \dot x - \mathcal{H}_{\rm ADM}) , \tag{35}$$

with

$$\mathcal{H}_{\rm ADM} = r^2 \pi_x^2 - (\tfrac{9}{32})^{1/3} (1 - r^{-2}) x^{2/3} . \tag{36}$$

It is easy to see that this Hamiltonian yields the correct equations of motion, showing that the reduction procedure has been consistently carried out. The remarkable feature of the Hamiltonian density (36) is that it contains no derivatives of the field variables $x(r)$. This means then that the ∞^1 gravitational degrees of freedom effectively decouple from one another. We can then reduce a problem with ∞^1 degrees of freedom to an infinite number of identical problems, each one for a single degree of freedom. Note also that the Hamiltonian (36) is independent of time.

To conclude this section, we comment on the radial label R. Notice that e^λ is the circumference radius r_c, and $\lambda' e^{-\mu}$ is the inverse of the extrinsic radius of curvature r_e of the two-spheres at constant r and that therefore R is their ratio: $R = r_c/r_e$. In flat space, $R = 1$ for all spheres. If space is flat only asymptotically, then R must go to unity at infinity. If a space is to be regular, then R must also go to unity with $r_c = 0$. In a time-symmetric section of a Kruskal metric, the horizon is a minimal area surface which therefore has zero extrinsic curvature so that R vanishes at the horizon.

III. THE QUANTUM PROBLEM

Let us remark, before going into the computations and drawing conclusions from them, that what we are considering here is only a model for quantum geometrodynamics will exhibit the same detailed behavior as the models attempting to represent it in stripped-down form. However, the insight gained from the study of simplified models is considerable, and the hope remains that at least the general features of the results will remain valid in the full theory. This is also the motivation for studying models of increasing generality.

A. Inhomogeneous Case

In Sec. II we derived the reduced Hamiltonian

$$H = \int r^2 dr [\pi_x^2 - (\tfrac{9}{32})^{1/3} r^{-2} (1 - r^{-2}) x^{2/3}] \tag{37}$$

which means that $x(r)$ are Cartesian coordinates in midisuperspace. The natural choice of representation is then

$$\pi_x(r) \to \frac{1}{i}\frac{\delta}{\delta x(r)}, \tag{38}$$

which leads to the Schrödinger equation

$$i\frac{\partial\Psi}{\partial t} = \int r^2 dr\left(-\frac{\delta^2\Psi}{\delta x^2} - (\tfrac{9}{32})^{1/3} r^{-2}(1-r^{-2})x^{2/3}\Psi\right), \tag{39}$$

where $\Psi = \Psi[x(r); t]$ is a functional of the field variable $x(r)$ and a function of time t. $x^{2/3}$ is the circumference radius of a 2-sphere of surface area $4\pi x^{4/3}$.

Equation (39) implies a continuity equation

$$\frac{\partial|\Psi|^2}{\partial t} + \int r^2 dr \frac{\partial}{\partial x}\left[\frac{1}{i}\left(\Psi^*\frac{\delta\Psi}{\delta x} - \Psi\frac{\delta\Psi^*}{\delta x}\right)\right] = 0, \tag{40}$$

where $\int r^2 dr\, \delta/\delta x(r)$ is a divergence in x space (midisuperspace).

Let us restrict now our attention to the interval $0 \le r \le r_0 \le 1$, with r_0 fixed. Divide this interval into M equal intervals of width ϵ, thus reducing the continuous variables $x(r)$ to a set of M variables x_j, $j = 1, \ldots, M$. Equation (39) may then be written

$$i\frac{\partial\Psi(x_1,\ldots,x_M,t)}{\partial t} = \sum_{j=1}^{M}\left(-\frac{\partial^2\Psi(x_1,\ldots,x_M,t)}{\partial x_j^2} + a_j x_j^{2/3}\Psi(x_1,\ldots,x_M,t)\right), \tag{41}$$

with a_j a positive constant.

If we let

$$\Psi(x_1,\ldots,x_M,t) = \psi_1(x_1,t)\cdots\psi_M(x_M,t),$$

we see that each ψ_j obeys the equation

$$i\frac{\partial\psi_j(x_j)}{\partial t} = -\frac{\partial^2\psi_j(x_j)}{\partial x_j^2} + a_j x_j^{2/3}\psi_j(x_j). \tag{42}$$

The continuity equation (40) will now be

$$\frac{\partial|\Psi|^2}{\partial t} + \frac{1}{i}\vec{\nabla}\cdot(\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*) = 0, \tag{43}$$

where

$$\frac{d}{dt}\int dx_1\cdots dx_M|\Psi|^2 = 0, \tag{44}$$

and $|\Psi|^2$ is the conserved probability density.

As it will become apparent below, Eq. (42) is telling us that each shell of circumference radius $x^{2/3}$ changes in time quite independently of the other shells, and in the same fashion as if it were a part of a Friedmann universe. This has the consequence that the evolution of an inhomogeneous distribution of dust can be deduced from the equations valid for the homogeneous case. To this we turn now our attention.

B. Homogeneous Case

Although the quantized Friedman universe has been treated by a number of authors,[10,11] it is instructive to reexamine it from the point of view adopted in this work, namely, to apply the ADM method with incoherent dust acting as a clock. Going back to (3), we impose now

$$r = \chi, \tag{45}$$

$$e^{2\lambda} = e^{2\mu}\sin^2\chi, \tag{46}$$

with λ and μ functions of t only. The metric tensor in then

$$g_{ij} = \mathrm{diag}(e^{2\mu}, e^{2\mu}\sin^2\chi, e^{2\mu}\sin^2\chi\sin^2\theta),$$

and the momenta are

$$\pi^{ik} = \mathrm{diag}(\tfrac{1}{6}e^{-2\mu}\pi_\mu, \tfrac{1}{6}e^{-2\mu}\pi_\mu\sin^{-2}\chi, \tfrac{1}{6}e^{-2\mu}\pi_\mu\sin^{-2}\chi\sin^{-2}\theta).$$

The constraint $\mathcal{H}^1 + \mathcal{P}^1 = 0$ is identically satisfied due to the homogeneity requirement. The action becomes

$$S = 2\pi^2\int dt[\pi_\mu\dot\mu + p^\phi\dot\phi - N(\mathcal{H}^0 + \mathcal{E})], \tag{47}$$

with

$$\mathcal{H}^0 = -\tfrac{1}{24}e^{-3\mu}\pi_\mu^2 - 6e^\mu, \tag{48}$$

and

$$\mathcal{E} = p^\phi. \tag{49}$$

The coordinate condition $\phi = -t$ yields the Hamiltonian

$$H = p^\phi = \tfrac{1}{24}e^{-3\mu}\pi_\mu^2 + 6e^\mu, \tag{50}$$

which, after setting $x = \frac{4}{3}6^{1/2}e^{3\mu/2}$ goes into

$$H = \pi_x^2 + (\tfrac{9}{96})^{1/3}x^{2/3}, \tag{51}$$

and the Schrödinger equation becomes

$$i\frac{\partial\Psi}{\partial t} = -\frac{\partial^2\Psi}{\partial x^2} + (\tfrac{9}{96})^{1/3}x^{2/3}\Psi, \tag{52}$$

which has exactly the same structure as (42). Let us remark here that had we not integrated χ from 0 to π but to some $\chi_0 < \pi$, the formalism would stand unchanged, but we could interpret the physical system not as a closed universe but as a collapsing finite ball of dust of uniform density. Thus any conclusion obtained from the equations holds just as well for the Friedmann universe as for the Oppenheimer-Snyder solution.[12] In fact, it is well known that in the classical case the geometry of a finite ball of dust is independent of what happens outside it.[13]

We remark now on the factor ordering problem. Just as $x(r)$ were Cartesian coordinates in midi-superspace, x is now a Cartesian coordinate in one-dimensional minisuperspace, so that its conjugate momentum is to be represented by $\pi_x \to (1/i)\partial/\partial x$. This, however, depends heavily on the choice of time coordinate. In fact, it is our choice $\phi = -t$ that brought the Hamiltonian to the form (51). Other choices of time may make a scaling of the super-Hamiltonian desirable. This is exactly what happens with the Friedmann universe as treated by Misner.[9] He finds it convenient to write the super-Hamiltonian constraint in the form $g^{1/2}(\mathcal{H}^0 + \mathcal{E}) = 0$ rather than in the form $\mathcal{H}^0 + \mathcal{E} = 0$, due to his choice of time variables: $\mu = t$. In this way he arrives at the reduced Hamiltonian

$$H^2 = -144e^{4\mu} + 24p^{\phi}e^{3\mu} \,. \tag{53}$$

In Eq. (53), it is μ which plays the role of Cartesian coordinate, making the representation $\pi_\mu \to (1/i)\partial/\partial\mu$ the natural one. This ambiguity will stay until a canonical formalism invariant under rescaling of the constraints is developed. Since the representation $\phi = -t$ makes the Hamiltonian separable and time independent in the infinite-dimensional case, we shall treat it as fundamental in this paper.

Going back to (42), we see that due to Eq. (33) we have $x > 0$. However, if x is to be a Cartesian coordinate in minisuperspace, this restriction makes minisuperspace geodesically incomplete. Already in flat space, when quantizing a scalar field in a uniformly accelerated frame of reference, geodesic incompleteness may cause inconsistencies.[14] Kuchař has suggested[15] that in such cases one should always quantize on a geodesically complete manifold and therefore, whether one treats a field in a prescribed spacetime or a geometry in minisuperspace, one should always first care about an analytic extension of the metric. This is easily done in our case, by allowing x to run from $-\infty$ to $+\infty$, with $x = e^{3\lambda/2}$ for $x > 0$ and $x = -e^{3\lambda/2}$ for $x < 0$. The natural boundary conditions for (52) are then $\psi \to 0$ as $x \to \pm\infty$, since these are classically forbidden regions. This is to be contrasted with DeWitt's procedure[10] which imposes $\psi(x=0) = 0$ to preserve $x > 0$.

C. Singularities[16]

We have passed from the classical description of our system by a pair of canonically conjugate variables $x(r)$, $\pi_x(r)$ satisfying Hamilton's equations (16)–(23) to a quantum description by a state functional $\Psi[x(r)]$ satisfying Schrödinger's Eq. (39). The classical description predicts that after a finite amount of proper time t, $x(r)$ will vanish, the volume of each shell shrinking to zero and the density of dust contained in it growing without limit. How does the quantum picture affect this prediction? Note once more that the equation obeyed by a shell of dust is the same whether it belongs to a Friedmann universe, to a finite ball of homogeneous dust, to a nonhomogeneous, closed universe, or to a finite, nonhomogeneous ball of dust. Accordingly, we base our analysis on Eq. (52) with the boundary condition that ψ vanishes when $x \to \pm\infty$. The formalism is then the same as that of a particle moving in one dimension under the potential $V(x) = x^{2/3}$.

First of all: Does the predictive power of Schrödinger's equation break down at $x = 0$? Clearly it does not; all terms appearing in it are perfectly regular, and the origin of coordinates in our problem is no more privileged than the origin of coordinates for a harmonic oscillator.

Consider now some initial state $\psi(x, 0)$. This will evolve in time to some $\psi(x, t)$, and [even if $\psi(0,0) = 0$ originally] we shall have in general $\psi(0, t) \neq 0$. Does this mean that it is inevitable to find the particle at $x = 0$? In other words: Will it happen that $\psi(x, t) = \delta(x)$ for some $t > 0$? The answer is no. More precisely, of all the possible (square integrable) functions $\psi(x, 0)$ which may be chosen as the initial state, one and only one will evolve into $\psi(x, t) = \delta(x)$, a state perfectly localized at the origin. This particular initial state is just the complex conjugate at time $(-t)$ of the function one would find at time t had $\delta(x)$ been the initial state. That is, only one out of the ∞^1 possibilities for the initial state will necessarily run into the singularity. It is a set of measure zero. In other words, it is practically impossible to construct such a state.

Ask now the question: Does quantum mechanics prevent the occurence of a singularity? To answer, we rephrase the question in a more precise way: To find the system at $x = 0$ means to find an infinite density shell of dust of vanishing radius. Given a state $\psi(x)$, what is the probability to find the system at $x = 0$? Now, the probability to find the system between x and $x + dx$ is given by $|\psi(x)|^2 dx$. If $\psi(0) = 0$, it is impossible to find the shell in the singular state. If $\psi(0) \neq 0$, the probability to find the system at $x = 0$ (or rather, at an interval dx about $x = 0$) is $|\psi(0)|^2 dx$. This, however, is negligible compared with the probability of *not* finding it at $x = 0$, since $\psi(0)$ is finite. In other words, the probability that a singular state occurs is negligible compared with the probability that a nonsingular state occurs.

Take now the radius of a dust shell, $x^{2/3}$. Classically, this vanishes at $x=0$. Does its mean value vanish when it is turned into an operator in the quantum formalism? We have

$$\langle x^{2/3}\rangle = \int dx\, x^{2/3}|\psi(x)|^2, \qquad (54)$$

and the answer is no, unless $\psi(x)=\delta(x)$ which we have seen is practically impossible. Let us remark here that to get meaningful answers to this type of question it is necessary to look at the mean values of operators that would classically vanish, not grow without limit, at the singularity. In fact, although $\langle x^{2/3}\rangle$ is nonzero, the mean value of the curvature, $6\langle x^{-2/3}\rangle$, is divergent. This is due to the fact that when averaging, a vanishing quantity will not contribute to the sum while a divergent one will dominate it. What we want to say is that states such as the ground state, which will have a wave function similar to $\psi(x)=e^{-x^2}$, not only give a nonvanishing probability for the shell not to be found at the singularity, but that they also give "nonsingular" expectation values to operators that classically vanish at the singularity.

In one sentence, we may say that quantum fluctuations prevent the occurence of a singularity. They do not allow the system to be sufficiently localized.

The considerations of this section are summarized in Table I.

TABLE I. Quantum dynamics of relativistic dust and singularities.

Question	Answer
1. Does the predictive power of the Schrödinger equation break down at the classical singularity, $x=0$?	1. No.
2. Is there a hard wall at $x=0$ that prevents collapse by forcing the wave function to vanish there?	2. No.
3. How does the probability to find a shell in the vicinity of a singular state compare with the probability to find it in the vicinity of a nonsingular state?	3. It is negligible.
4. How many states evolve to be "singular quantum states," that is, such that $\psi(x,t)=\delta(x)$ at some time t?	4. A set of measure zero.
5. Consider a particular shell of dust. Does its radius have a vanishing expectation value for some state, in particular for one centered at $x=0$?	5. No.

IV. CONCLUDING REMARKS

The evolution of a finite ball of dust, be it homogeneous or not, is the same whether it is lying in an otherwise empty space or in an otherwise dust-filled space. To see what happens at the late stages of collapse, it is of no use to be standing outside the collapsing matter. Observers have to be collapsing themselves. Thus the evolution of finite balls of dust can be viewed on the same footing as dust-filled universes, homogeneous or not.

Under suitable coordinate conditions, the geometry of the shells of dust decouple from one another and it is possible to analyze the motion of each one as the motion of a particle in one dimension under the influence of the potential $V(x)=x^{2/3}$. To have a geodesically complete minisuperspace we have let x take all real values, not just positive values. This results in a doubling of the number of eigenstates of the Hamiltonian in comparison with the situation in which there is a hard wall at $x=0$, and lifts the restriction $\psi(0)=0$. This extension of superspace is analogous to the one considered by DeWitt.[17] In our case it is possible to carry it out since the metric in minisuperspace implicit in the kinetic term of the Hamiltonian (51) is regular at $x=0$. In DeWitt's treatment of the Friedmann universe[10] this is not the case, as his supermetric is singular at the point where classical collapse occurs and no natural extension across such a singular frontier is possible. In such an extension, a particular three-geometry corresponds to several points in the extended superspace. Thus, in our case, x and $-x$ represent the same geometry for a given shell of dust. Classically, a particle moving from positive to negative x represents a collapsing shell reexploding back into the same universe from which it came, and not appearing suddenly into another universe otherwise disconnected from the first one.

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Lines of Force of a Point Charge near a Schwarzschild Black Hole*†

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The electric field generated by a charged particle at rest near a Schwarzschild black hole is analyzed using Maxwell's equations for curved space. After generalizing the definition of the lines of force to our curved background, we compute them numerically and graph them with the charge at $r = 4M$, $3M$, and $2.2M$. Particular attention is paid to the behavior of the lines of force near the event horizon and the smooth transition of the electric field to that of a Reissner-Nordström black hole.

I. INTRODUCTION

The formalism for gravitational perturbations away from a Schwarzschild background has been developed by Regge and Wheeler.[1] It was extended by Zerilli,[2] who has shown that perturbations corresponding to a change in the mass, the angular momentum, and the charge of a Schwarzschild black hole are well behaved. The decay of the non-well-behaved perturbations has been investigated by Price.[3] He has shown that any multipole $l \geq s$, where s is the spin of the field being examined, gets radiated away in the late stage of gravitational collapse and will die as $t^{-(2l+2)}$ for large t.

Instead of analyzing how higher-order multipoles are radiated away, we focus on how the allowed transition from a Schwarzschild to a Reissner-Nordström hole takes place through the capture of a charged particle in a given Schwarschild background. In this paper we neglect the electromagnetic radiation emitted during the fall of the particle and consider a succession of configurations in which the particle is momentarily at rest at decreasing distances from the Schwarzschild horizon ($r = 2M$ in geometrical units $G = c = 1$). The problem of examining the radiation emitted is, indeed, of great interest and has been presented elsewhere.[4]

The electric field of a charge at rest with respect to the Schwarzschild background can be developed in a multipole expansion centered about the black hole. For any finite separation of the charge from the black hole, the far-away observer will detect only the monopole term, the field corresponding to a Reissner-Nordström solution. In the region