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Comment on the Decay of Neutral Kaons into Muon Pairs

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The contributions of the $2\pi\gamma$ and 2π intermediate states to the absorptive part of the $K_1^0 \to \mu^+\mu^-$ decay amplitude are calculated. The result obtained for $\text{Abs}F^{(2\pi\gamma)}(K_1^0 \to \mu^+\mu^-)$ differs significantly from that of Martin, de Rafael, and Smith. The value of $\text{Abs}F(K_2^0 \to \mu^+\mu^-)$ is also discussed.

I. INTRODUCTION

In connection with the negative result of the search for the $K_L^0 \to \mu^+\mu^-$ decay[1] and the publication of different theoretical models explaining the observed suppression of this decay rate,[2] independent estimation of the $K_{1,2} \to \mu^+\mu^-$ decay rates is of definite interest.

The most general expression for the decay amplitude $K^0 \to \mu^+\mu^-$ is as follows:

$$A[K^0 \to \mu^+(p')\mu^-(p)] = \bar{u}(p)[F_1 + \gamma^5 F_2]v(p') .$$

Then the $K_1^0 \to \mu^+\mu^-$ decay amplitude conserving CP is equal to $\sqrt{2}\, F_1\, \bar{u}(p)v(p')$, and that of $K_2^0 \to \mu^+\mu^-$ is $\sqrt{2}\, F_2\, \bar{u}(p)\gamma^5 v(p')$.

In the present paper we use the notations adopted in the paper of Martin, de Rafael, and Smith[3] (hereafter referred to as MRS) if not stated to the contrary.

In the papers of Refs. 3–5 the values of $\text{Abs}F_1$ and $\text{Abs}F_2$ were calculated from the unitarity condition taking into account the main intermediate states. Unfortunately, in our opinion, the $\pi^+\pi^-\gamma$ intermediate-state contributions to $\text{Abs}F_1$ and to $\text{Abs}F_2$ found in MRS and Ref. 5, respectively, are erroneous.

In Sec. II of this paper we represent briefly our calculations of $\text{Abs}F_1^{(2\pi\gamma)}$ and point out the errors contained in MRS.

In Sec. III the $\pi^+\pi^-\gamma$ intermediate-state contribution to $\text{Abs}F_2$ is considered, and the result for $\text{Abs}F_2^{(2\pi\gamma)}$ obtained in Ref. 6 is presented. The 3π intermediate-state contribution to $\text{Abs}F_2$ is also discussed in the same section.

In the present paper we follow all the assumptions accepted in Refs. 3 and 5, particularly the conservation of CP.

II. PERTURBATION-THEORY LOWER BOUND TO THE DECAY RATE $K_1^0 \to \mu^+\mu^-$

The value of $\text{Abs}F_1$ is determined mainly by the contributions of the 2γ, 2π, and $2\pi\gamma$ states. Since our result for $\text{Abs}F_1^{(2\pi\gamma)}$ differs from that of MRS, we regard in this section the $2\pi\gamma$ intermediate-state contribution, which is equal to (according to the unitarity condition)

$$\text{Abs}\sqrt{2}\, F_1^{(2\pi\gamma)}\, \bar{u}(p)v(p') = \tfrac{1}{2}\sum_{\epsilon_\gamma} \int dV(2\pi\gamma) A^*(2\pi\gamma \to K_1^0) \times A(2\pi\gamma \to 2\mu) .$$

Here $dV(2\pi\gamma)$ is the phase space of the $2\pi\gamma$ system coming from the $K_1^0 \to 2\pi\gamma$ decay. Summing up over the γ quantum polarization, and integrating over all variables except ω (ω is the γ quantum energy in the K_1^0 rest frame), we obtain

$$\text{Abs}F_1^{(2\pi\gamma)} = C\int_\lambda^{\omega_{\max}} \frac{d\omega}{\omega} f(\lambda, \omega) . \tag{1}$$

Here $\omega_{\max} = (M_K^2 - 4m_\pi^2 + \lambda^2)/2M_K$, λ is the mass of the γ quantum, C is the constant

$$C = -\frac{A(+-)}{M_K}\,\frac{\alpha^2}{\pi}\,\frac{m_\mu}{M_K}\,\frac{1}{\beta_\mu^3}, \tag{2}$$

and $f(\lambda, \omega)$ is the function defined below in Eq. (6) provided that $\omega \geq \lambda$.

At $\lambda = 0$ integral (1) diverges logarithmically, and we are faced with the problem of the infrared-divergence separation. The correct way of solving the problem is, for example, the following.

Rewriting expression (1) in the identical form, we get

$$\text{Abs}F_1^{(\pi^+\pi^-\gamma)} = C\int_\lambda^\delta \frac{d\omega}{\omega}[f(\lambda, \omega) - f(0, 0)] + C\int_\delta^{\omega_{\max}} \frac{d\omega}{\omega}[f(\lambda, \omega) - f(0, 0)] + C\int_\lambda^{\omega_{\max}} \frac{d\omega}{\omega} f(0, 0) , \tag{3}$$

where $f(0, \omega) = \lim_{\lambda \to 0} f(\lambda, \omega)$ at fixed $\omega > 0$, $f(0, 0) = \lim_{\omega \to 0} f(0, \omega)$, and δ is an arbitrary small fixed mass: $\delta \ll \omega_{\max}$.

If we let $\lambda \to 0$ at fixed δ and after that $\delta \to 0$, then the second and the third terms in (3) tend to

$$C\int_0^{M_K\beta_\pi^2/2} \frac{d\omega}{\omega}[f(0, \omega) - f(0, 0)] + Cf(0, 0)\ln\left(\frac{M_K}{\lambda}\,\frac{\beta_\pi^2}{2}\right) . \tag{4}$$

The terms analogous to the first term in (3) are omitted in MRS [see Eqs. (4.79)–(4.82) of MRS]. However, they should be taken into account because the integral

$$\mathcal{g}=\int_\lambda^\delta \frac{d\omega}{\omega}[f(\lambda,\omega)-f(0,0)] \quad (5)$$

does not vanish, since the integrand converges to $[f(0,\omega)-f(0,0)]/\omega$ nonuniformly in ω as $\lambda\to 0$ ($\omega\sim\lambda$ is of importance at small λ), and hence the order of integration over ω and taking the limit $\lambda\to 0$ cannot be interchanged.

We obtain[7]

$$f(\lambda,\omega)=\frac{1}{\gamma'^3\gamma}\left\{\left[\frac{3-2\gamma\gamma'}{\gamma}\left(\beta_\pi'\gamma'+\tfrac12\ln\frac{1-\beta_\pi'\gamma'}{1+\beta_\pi'\gamma'}\right)-\tfrac12\beta_\pi'^2\gamma'\ln\frac{1-\beta_\pi'\gamma'}{1+\beta_\pi'\gamma'}\right]\left(\beta_\mu\gamma+\tfrac12\ln\frac{1-\beta_\mu\gamma}{1+\beta_\mu\gamma}\right)\right.$$
$$\left.-\left[\beta_\pi'\gamma'+\tfrac12(1-\beta_\pi'^2\gamma'^2)\ln\frac{1-\beta_\pi'\gamma'}{1+\beta_\pi'\gamma'}\right]\left[\tfrac12\beta_\mu{}^2\gamma\ln\frac{1-\beta_\mu\gamma}{1+\beta_\mu\gamma}+\frac{2\gamma'\omega}{M_K}\left(\beta_\mu\gamma+\tfrac12\ln\frac{1-\beta_\mu\gamma}{1+\beta_\mu\gamma}\right)\right]\right\}, \quad (6)$$

and $f(0,\omega)$ can be derived from (6) if one sets γ and γ' equal to 1 in the right-hand side of Eq. (6).

The substitution $x=(\omega^2-\lambda^2)^{1/2}/\omega$ results in the following form of the integral (5):

$$\lim_{\lambda\to 0}\mathcal{g}=\int_0^1\frac{x\,dx}{1-x^2}[F(x)-F(1)], \quad (7)$$

where

$$F(x)=\frac{1}{x^4}\left\{\left[\frac{3-2x^2}{x}\left(\beta_\pi x+\tfrac12\ln\frac{1-\beta_\pi x}{1+\beta_\pi x}\right)-\tfrac12\beta_\pi{}^2x\ln\frac{1-\beta_\pi x}{1+\beta_\pi x}\right]\left(\beta_\mu x+\tfrac12\ln\frac{1-\beta_\mu x}{1+\beta_\mu x}\right)\right.$$
$$\left.-\left[\beta_\pi x+\tfrac12(1-\beta_\pi{}^2x^2)\ln\frac{1-\beta_\pi x}{1+\beta_\pi x}\right]\tfrac12\beta_\mu{}^2x\ln\frac{1-\beta_\mu x}{1+\beta_\mu x}\right\}. \quad (8)$$

The numerical integration in Eq. (7) yields

$$\int_0^1\frac{x\,dx}{1-x^2}[F(x)-F(1)]=-0.067\,.$$

We now present the final result for the value of $\mathrm{Abs}F_1^{(2\pi\gamma)}$. Using Eqs. (1)–(4), (9), and taking into account that

$$\int_0^{M_K\beta_\pi{}^2/2}\frac{d\omega}{\omega}[f(0,\omega)-f(0,0)]=-0.115\,,$$

$$f(0,0)\ln\left(\frac{M_K}{\lambda}\frac{\beta_\pi{}^2}{2}\right)=-Q(t)\ln\frac{M_K}{\lambda}-0.310\,,$$

we obtain

$$\mathrm{Abs}F_1^{(2\pi\gamma)}=C\left[-0.492-Q(t)\ln\frac{M_K}{\lambda}\right]$$
$$=1.35\times10^{-12}+\frac{A(+-)}{M_K}\frac{\alpha^2}{\pi}\frac{m_\mu}{M_K}\times\frac{1}{\beta_\mu{}^3}Q(t)\ln\frac{M_K}{\lambda}\,. \quad (10)$$

Notice that the term in $\mathrm{Abs}F_1^{(2\pi\gamma)}$ containing the infrared divergence is canceled out by the similar term coming from the 2π intermediate state (see Table I.)

Besides the $2\pi\gamma$ intermediate-state contribution, we have calculated the 2γ and 2π intermediate-state contributions. Our results for $\mathrm{Abs}F_1^{(2\gamma)}$ and

TABLE I. The summary table.

n	2γ	$2\pi\gamma$	2π	3π
$\mathrm{Abs}F_1^{(n)}$	-1.12×10^{-12} [a]	$1.35\times10^{-12}+C'\ln(M_K/\lambda)$	$0.16\times10^{-12}-C'\ln(M_K/\lambda)$ [a]	...
$\mathrm{Abs}F_2^{(n)}$	1.4×10^{-12} [b]	$\le 0.9\times10^{-2}\,\mathrm{Abs}F_2^{(2\gamma)}$ [c]	...	$\le 6\times10^{-5}\,\mathrm{Abs}F_2^{(2\gamma)}$ [d]

[a] The result is obtained in MRS (see also Ref. 9).

[b] The result is obtained in Ref. 4 with the use of the experimental value of the $K_L\to2\gamma$ decay rate.

[c] The result is obtained in Ref. 6 taking account of the form factors and the experimental upper bound for the $K_L\to\pi^+\pi^-\gamma$ decay rate: $\Gamma(K_L\to\pi^+\pi^-\gamma)\le4\times10^{-4}\Gamma(K_L\to\mathrm{all})$. Neglecting the form factors gives $|\mathrm{Abs}F_2^{(2\pi\gamma)}/\mathrm{Abs}F_2^{(2\gamma)}|\simeq0.5\times10^{-2}$.

[d] The result is obtained in Ref. 11 (see also Ref. 12).

$\mathrm{Abs}F_1^{(2\pi)}$ coincide[8] with those of MRS. According to Eq. (10) and the appropriate equations given in MRS, we have[9]

$$\mathrm{Abs}F_1 = \mathrm{Abs}F_1^{(2\gamma)} + \mathrm{Abs}F_1^{(2\pi)} + \mathrm{Abs}F_1^{(2\pi\gamma)} \approx 0.4\times 10^{-12}, \quad (11)$$
$$\Gamma(K_1^0 \to \mu^+\mu^-) \geq 0.8\times 10^{-12}\,\Gamma(K_1^0 \to \mathrm{all}).$$

This result for $\mathrm{Abs}F_1$ is about 1.5 times smaller than that of MRS, and is of the opposite sign.

Notice that $\mathrm{Abs}F_1^{(2\pi)}$ was calculated in MRS by means of the double dispersion relations, but the question about necessary subtractions was not discussed. Our calculations show that the contributions of both diagrams 8(a) and 8(b) of MRS to the amplitude $A(\pi^+\pi^- \to \mu^+\mu^-)$ need subtractions, but after summing up, the subtraction terms cancel out (when the muons are on their mass shells), and thus the result for $\mathrm{Abs}F_1^{(2\pi)}$ obtained by Martin, de Rafael, and Smith appears to be correct.

III. THE ABSORPTIVE PART OF THE $K_2^0 \to \mu^+\mu^-$ DECAY AMPLITUDE

The two-photon intermediate state is known to give the main contribution to $\mathrm{Abs}F_2$:

$$\mathrm{Abs}F_2^{(2\gamma)} \simeq 1.4\times 10^{-12}$$

(see Ref. 10). The corrections to $\mathrm{Abs}F_2$ due to the other intermediate-state contributions were calculated earlier in MRS and Ref. 5. Martin, de Rafael, and Smith (MRS) estimated the $2\pi\gamma$ and 3π intermediate-state contributions to $\mathrm{Abs}F_2$ from dimensional considerations and obtained

$$|\mathrm{Abs}F_2^{(2\pi\gamma)}/\mathrm{Abs}F_2^{(2\gamma)}| \lesssim 6\times 10^{-2}; \quad (12)$$
$$|\mathrm{Abs}F_2^{(3\pi)}/\mathrm{Abs}F_2^{(2\gamma)}| \lesssim 3\times 10^{-2}.$$

The contribution of the $2\pi\gamma$ intermediate state to $\mathrm{Abs}F_2$ was calculated by Gaillard,[5] the form factors in the amplitudes $A(K_2^0 \to \pi^+\pi^-\gamma)$ and $A(\pi^+\pi^-\gamma \to \mu^+\mu^-)$ being taken into consideration, namely, $1+\lambda Q^2/m_\pi^2$ in the former amplitude, and $1+\sigma Q^2/m_\pi^2$ in the latter one. Here $\sqrt{Q^2}$ is the invariant mass of the $\pi^+\pi^-$ system, and λ and σ are parameters varying in the following limits (see Ref. 5):

$$-2\frac{m_\pi^2}{m_\rho^2} \leq \lambda \leq 3\frac{m_\pi^2}{m_\rho^2}; \quad 0 \leq \sigma \leq 3\frac{m_\pi^2}{m_\rho^2}.$$

Gaillard, using the experimental upper bound for the $K_L \to 2\pi\gamma$ decay rate and neglecting the bremsstrahlung contribution to the amplitude $A(K_L \to \pi^+\pi^-\gamma)$, obtained the following result:

$$|\mathrm{Abs}F_2^{(2\pi\gamma)}/\mathrm{Abs}F_2^{(2\gamma)}| \leq 5.5\times 10^{-2}. \quad (13)$$

As was shown in Refs. 6 and 11 (published later), the results (12)–(13) are essentially overestimated as a matter of fact. The result of Ref. 11, obtained within the framework of current algebra and with the help of the dispersion relation over the 3π-system mass, is the following[12]:

$$|\mathrm{Abs}F_2^{(3\pi)}/\mathrm{Abs}F_2^{(2\gamma)}| \leq 6\times 10^{-5}.$$

The contribution of the $2\pi\gamma$ intermediate state to $\mathrm{Abs}F_2$ was calculated in Ref. 6, all assumptions made in Ref. 5 being accepted. It was found there that

$$|\mathrm{Abs}F_2^{(2\pi\gamma)}/\mathrm{Abs}F_2^{(2\gamma)}| \leq 0.9\times 10^{-2}.$$

IV. CONCLUSION

In conclusion we present in Table I a summary of the different intermediate-state contributions to $\mathrm{Abs}F_{1,2}$ calculated under conventional assumptions.

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[1] A. R. Clark *et al.*, Phys. Rev. Lett, 26, 1967 (1971).

[2] The detailed discussion of the $K_L \to 2\mu$ problem and the relevant references are given in the review of A. D. Dolgov, V. I. Zakharov, and L. B. Okun, Usp. Fiz. Nauk. 107, 537 (1972) [Sov. Phys.–Usp. 15, 404 (1973)].

[3] B. R. Martin, E. de Rafael, and J. Smith, Phys. Rev. D 2, 179 (1970) and 3, 272(E) (1971), hereafter referred to as MRS.

[4] L. M. Sehgal, Nuovo Cimento 45, 785 (1966); Phys. Rev. 183, 1511 (1969).

[5] M. K. Gaillard, Phys. Lett. 35B, 431 (1971).

[6] A. D. Dolgov, M. A. Shifman, and M. Zh. Shmatikov, Yad. Fiz. 16, 149 (1972) [Sov. J. Nucl. Phys. 16, 80 (1973)].

[7] Notice that although the function $f(\lambda,\omega)$ is not presented in MRS in the evident form, this function seems to be calculated erroneously, since Eq. (4.76) of this paper contains a mistake: The coefficient before $\cos^2\theta\cos^2\theta'$ must be equal to $[3ts - \gamma\gamma'(3ts - s^2 + s\lambda^2)]$.

[8] The finite result for $\mathrm{Abs}\,F_1^{(2\pi)}$ can be written in a more simple form than that in MRS if we take into considera-

tion the following:

$$\chi(x') - \chi(x'') = \mathrm{Li}_2\left(-\frac{1+\beta_\pi}{1-\beta_\pi}\right) - \mathrm{Li}_2\left(-\frac{1-\beta_\pi}{1+\beta_\pi}\right).$$

[9]The $K_S^0 \to 2\gamma$ decay has not been observed experimentally. The perturbation-theory estimate of the value of Abs $F_1^{(2\gamma)}$ is obtained in the model where the K_1^0 meson decays into two γ quanta through the pion loop.

[10]The result is obtained in Ref. 4 using the experimental decay rate for $K_L \to 2\gamma$: $\Gamma(K_L \to 2\gamma) \approx 5\times 10^{-4}\Gamma(K_L \to \text{all})$.

[11]S. L. Adler, G. R. Farrar, and S. B. Treiman, Phys. Rev. D 5, 770 (1972).

[12]The simple dimensional estimate for the amplitude $A(3\pi \to 2\mu)$,

$$A(3\pi \to 2\mu) = (\alpha/M_K^2)\bar{\mu}\gamma^5\mu\varphi_\pi^3,$$

gives a result which is about an order of magnitude.

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Erratum

Erratum: Hyperon-Nucleon Scattering. I. Invariant and Helicity Amplitudes [Phys. Rev. D 6, 2513 (1972)].

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1. The right-hand side of Eqs. (5) and (19) should be multiplied by $\chi(\lambda_1)$ and $\chi(-\mu_2')$, respectively, where $\chi(+) = \binom{1}{0}$ and $\chi(-) = \binom{0}{1}$.

2. In f_4 of Eq. (9), the factor $p_C p_{AB} E_{CD}$ multiplying F_6 should read $p_C \Delta_{AB} E_{CD}$.

3. Equation (12) should read $\cos\theta_s = [s(t-u) + \cdots]$, while Eq. (34) should read $\cos\theta_u = [u(t-s) - \cdots]$.

4. The factor s in $\bar{f}_1$ and $\bar{f}_2$ of Eq. (14) should be absent; similarly for the factor t in $\bar{g}_1$ and $\bar{g}_2$ of Eq. (24).

5. In Eq. (27), the right-hand side of b_3 should be multiplied by -1, while the right-hand side of b_7 and b_8 should both be multiplied by $\frac{1}{4}$.

6. The positions of Γ_b and Γ_c in c_{ij}' of Eq. (45) should be interchanged.

7. When performing an st or su crossing, the angles θ_t and θ_u defined in this article should be modified as $-\theta_t$ and $-\theta_u$, respectively, so that $\cos\theta_t \to \cos\theta_t$, $\sin\theta_t \to -\sin\theta_t$, etc. See, for example, Fig. 1 of Y. Hara, Prog. Theor. Phys. 45, 584 (1971).