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PHYSICAL REVIEW D VOLUME 8, NUMBER 1 1 JULY 1973

Unitary Multiperipheral Model with Diffractive Production

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(Received 13 March 1973)

A unitary model of multiparticle amplitudes with multiperipheral and diffractive production mechanisms is presented. The model has a bootstrap solution for the elastic amplitude of the form $\{(J-1)+[(J-1)^2-R_0t]^{1/2}\}^{-1}$, leading to constant total cross sections at high energies. Inelastic cross sections and multiplicity distributions may be predicted in qualitative agreement with experiment.

I. INTRODUCTION

It is generally accepted today that a realistic model of particle production in hadron collisions at high energies should contain both multiperipheral dynamics and diffractive fragmentation,[1] while at the same time satisfying the constraints imposed by the unitarity condition.[2] Unitary models of multiparticle amplitudes where particle production comes exclusively from multiperipheral mechanisms have been developed by several authors.[3] We present here a model in which the S matrix is unitary at high energies, but which includes both multiperipheral and diffractive production. Besides, a bootstrap solution exists which leads to a constant total cross section, in contrast with most previous models in which self-consistent solutions lead to $(\log s)^2$ behavior of the total cross section, thus saturating the Froissart bound.

The model is based on the eikonal approximation with the inclusion of the possibility of production of excited states of the external particles. The masses of these excited states are allowed to vary continuously within the limits imposed by the validity of the approximation. The excited states are allowed to decay into a nucleon and pions.

We thus have an infinite-channel model in the external "nucleons." In Sec. II we show how the inclusion of the possibility for excitation of the incoming particles leads to results similar to those obtained by Aviv, Sugar, and Blankenbecler (ASB),[3] but with the inclusion of an energy-dependent function $I(s)$, which is essentially an integral over the squares of the coupling functions for the excited "nucleons." The basic amplitude is specified by the diagram of Fig. 1.[4] All S-matrix elements are then shown to satisfy unitarity exactly at high energies. In Sec. III we show that there exists a bootstrap solution for the elastic amplitude which when combined with a restriction on the asymptotic behavior of the function $I(s)$ gives scattering amplitudes leading to constant total cross sections. We then compute inelastic cross sections and particle distributions, showing how a particular choice of coupling functions for the excited "nucleons" (which also determines their decay amplitudes) may lead to results in qualitative agreement with experiments.

Section IV contains a brief summary of the results and some concluding remarks.

II. THE MODEL

To define the kinematics, let us follow ASB and write

$$P_1 = m_1(\cosh y_1; 0, 0, \sinh y_1)\ ,$$

$$P_2 = m_2(\cosh y_2;\, 0,\, 0,\, \sinh y_2)\ ,$$

where P_1 and P_2 are the momenta of the colliding nucleons, characterized only by their rapidities y_1 and y_2 (we ignore spin and internal quantum numbers, and for the incident particles $m_1 = m_2 = m_0$), defined in the usual way:

$$y = \tfrac{1}{2}\ln\left(\frac{P_0 + P_z}{P_0 - P_z}\right). \tag{1}$$

At high energies,

$$s = (P_1 + P_2)^2 \cong m_0^{\,2}\, e^{|y_1 - y_2|} \equiv m_0^{\,2} e^{Y}. \tag{2}$$

Since multichain exchanges must be included to satisfy the unitarity condition, the S-matrix elements will have contributions from diagrams of the type shown on Fig. 2.

In the center-of-mass system the amplitudes are functions of Y, the rapidity difference between the incoming nucleons; $\vec{B}$, the impact parameter between the same nucleons; m_1, m_2, m_1', and m_2', the initial and final masses of the "nucleons"; and $\vec{q}_i$ and y_i, the transverse momentum and rapidity of the ith produced pion from the chains.

Let us for the moment consider the "nucleons" as stable against decay, and find the contribution to the scattering amplitude from quasielastic scattering (direct exchanges with no pion production) only. Each exchange contributes a factor $\delta(Y, \vec{B};\, m_1', m_2', m_1, m_2)$, where δ is the two-dimensional Fourier transform of the amplitude for a single exchange, i.e.,

$$\begin{aligned}\delta(Y, \vec{B};\, m_1', m_2', m_1, m_2) &= \int \frac{d^2\Delta}{(2\pi)^2}\, e^{-i\vec{\Delta}\cdot\vec{B}}\, \tilde{\delta}(Y, \vec{\Delta}) \\ &\quad \times G(m_1'^2, m_1^{\,2}, t)\, G(m_2'^2, m_2^{\,2}, t),\end{aligned} \tag{3}$$

where $t = -\vec{\Delta}^2$. In order to have a solvable model we now make the basic assumption that the coupling function $G(m_1'^2, m^2, t)$ is factorizable in each of the arguments and then neglect its t dependence:

$$G(m'^2, m^2, t) = g(m'^2)\, g(m^2)\ . \tag{4}$$

It follows that we can write[5]

$$\delta(Y, \vec{B};\, m_1', m_2', m_1, m_2) = \bar{\delta}(Y, \vec{B})\, g(m_1'^2)\, g(m_2'^2)\, g(m_1^{\,2})\, g(m_2^{\,2})\ .$$

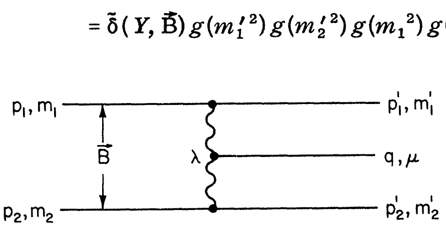

FIG. 1. The basic amplitude $W(Y, \vec{B};\, y, \vec{q};\, m_1', m_2', m_1, m_2)$.

With the usual treatment of the relativistic eikonal model[6] we readily find the matrix element for (quasi) elastic scattering:

$$\begin{aligned}M_{\rm qel}(Y, \vec{B};\, m_1', m_2', m_1, m_2) &= \frac{2is}{I(s)}\left(1 - e^{i\bar{\delta}(Y,\vec{B})I(s)/2s}\right) \\ &\quad \times g(m_1'^2)\, g(m_2'^2)\, g(m_1^{\,2})\, g(m_2^{\,2})\ ,\end{aligned} \tag{5}$$

where

$$I(s) = \left\{\int_{m_0^2}^{\mu\sqrt{s}} dm^2 [g(m^2)]^2\right\}^2. \tag{6}$$

The upper limit to the integration over m^2 is determined by the validity of the eikonal approximation. μ is the mass of the exchanged object.

Equation (5) can be written

$$M_{\rm qel}(Y, \vec{B};\, m_1', m_2', m_1, m_2) = 2is\left(1 - e^{i\delta(Y,\vec{B};\, m_1', m_2', m_1, m_2)/2s}\right) \tag{7}$$

if we regard this as a matrix equation in the (continuous) indices m_1', m_2', m_1, and m_2. Thus we can identify the two-body S matrix (between states of specified initial and final masses) as:

$$S_{\rm qel}(Y, \vec{B};\, m_1', m_2', m_1, m_2) = e^{i\delta(Y,\vec{B};\, m_1', m_2', m_1, m_2)/2s}\ . \tag{8}$$

Now let us turn to particle production. Using Eq. (4) for the coupling functions $G(m^2, m'^2, t)$, we can write the basic production amplitude of Fig. 1 as

$$\begin{aligned}W(Y, \vec{B};\, y, \vec{b};\, m_1', m_2', m_1, m_2) &= \tilde{W}(Y, \vec{B};\, y, \vec{b})\, g(m_1'^2)\, g(m_2'^2)\, g(m_1^{\,2})\, g(m_2^{\,2}),\end{aligned} \tag{9}$$

where $\vec{b}$ is the coordinate conjugate to the transverse momentum of the produced pion. We may write $\tilde{W}$ as a product of two-body amplitudes as follows:

$$\begin{aligned}\tilde{W}(Y, \vec{B};\, y, \vec{b}) &= \lambda \bar{M}(y_1 - y;\, \tfrac{1}{2}\vec{B} - \vec{b}) \\ &\quad \times \bar{M}(y - y_2;\, \tfrac{1}{2}\vec{B} + \vec{b})\ .\end{aligned} \tag{10}$$

Notice that $\bar{M}$ is a two-body amplitude without the external coupling functions and represents a process in which only one of the external particles

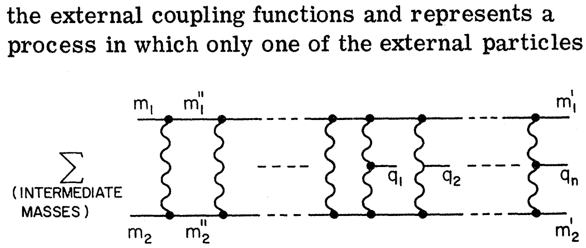

FIG. 2. A contribution to the amplitude $M_{0n}(Y, \vec{B};\, y_1, \vec{q}_1, \ldots y_n, \vec{q}_n;\, m_1', m_2', m_1, m_2)$.

can be excited.

We can now write down the amplitude for production of n pions from the chains (pions produced in what we choose to denote as the "pionization region")[7]:

$$M_{no}(Y,\vec{B};y_1,\vec{b}_1,\ldots y_n,\vec{b}_n;m_1',m_2',m_1,m_2)$$
$$=e^{i\tilde{\delta}(Y,\vec{B})I(s)/2s}\left(\frac{iI(s)}{2s}\right)^{n-1}g(m_1'^2)g(m_2'^2)g(m_1^2)g(m_2^2)\prod_{j=1}^{n}\tilde{W}(Y,\vec{B};y_j,\vec{b}_j)\,, \quad (11)$$

where the exponential factor in front comes from the quasielastic two-body scattering. Similarly the connected part of M_{mn} is

$$M_{mn}^{(\mathrm{conn})}(Y,\vec{B};y_1',\vec{b}_1',\ldots y_m',\vec{b}_m';y_1,\vec{b}_1,\ldots y_n,\vec{b}_n;m_1',m_2',m_1,m_2)$$
$$=e^{i\tilde{\delta}(Y,\vec{B})I(s)/2s}\left(\frac{iI(s)}{2s}\right)^{n+m-1}\prod_{i=1}^{m}\tilde{W}(i)\prod_{j=1}^{n}\tilde{W}(j)g(m_1'^2)g(m_2'^2)g(m_1^2)g(m_2^2)\,, \quad (12)$$

where $\tilde{W}(i)$ stands for $\tilde{W}(Y,\vec{B};y_i,\vec{b}_i)$. The amplitude where some of the pions are disconnected is simply the connected amplitude multiplied by momentum conserving δ functions for the disconnected pions.

We now show which constraints unitarity imposes on the model. For diffraction scattering we expect the elastic amplitude to be purely imaginary at high energies, therefore defining

$$\tilde{A}(Y,\vec{B})\equiv-\frac{i}{2s}\tilde{\delta}(Y,\vec{B})\,, \quad (13)$$

and writing down the unitarity equation for the elastic two-body amplitude M_{00}, we find that unitarity is satisfied if

$$\tilde{A}(Y,\vec{B})=\frac{I(s)}{32\pi s^2}\int d^2b\,dy\,|\tilde{W}(Y,\vec{B};y,\vec{b})|^2\,. \quad (14)$$

Following ASB we write the S matrix in an explicitly unitary operator form:

$$S=e^{i(\chi+\chi^\dagger)}\,, \quad (15)$$

where

$$\chi=\frac{1}{8\pi s}\int d^2b\,dy\,\mathfrak{D}(Y,\vec{B};y,\vec{b};m_1',m_2',m_1,m_2)\,a(y,\vec{b})\,. \quad (16)$$

$a(y,\vec{b})$ is the pion destruction operator, in the normalization:

$$[a(y,\vec{b}),\,a^\dagger(y',\vec{b}')]=4\pi\delta(y-y')\,\delta^2(\vec{b}-\vec{b}')\,, \quad (17)$$

and

$$\mathfrak{D}(Y,\vec{B};y,\vec{b};m_1',m_2',m_1,m_2)=\int dm_1'^2\,dm_2'^2\,dm_1^2\,dm_2^2\,W(Y,\vec{B};y,\vec{b};m_1',m_2',m_1,m_2)$$
$$\times d_1^\dagger(m_1')\,|0_1\rangle\langle 0_1|\,d_1(m_1)\,d_2^\dagger(m_2')\,|0_2\rangle\langle 0_2|\,d_2(m_2)\,. \quad (18)$$

$|0_i\rangle$ is the vacuum state with respect to the ith "nucleon."

The "excited nucleon" creation and annihilation operators $d_i^\dagger$ and d_i satisfy the commutation relation

$$[d_i(m_i),\,d_j^\dagger(m_j')]=\delta_{ij}\,\delta(m_i^2-m_j'^2) \quad (19)$$

and matrix elements are to be taken between states labeled by the masses of the two "nucleons" and the pion coordinates, e.g.,

$$|m_1,m_2;y_1,\vec{b}_1,\ldots y_n,\vec{b}_n\rangle=d_1^\dagger(m_1)\,d_2^\dagger(m_2)\,a^\dagger(y_1,\vec{b}_1)\cdots a^\dagger(y_n,\vec{b}_n)\,|0\rangle\,. \quad (20)$$

To take matrix elements it is convenient to write S in normal ordered form:

$$S=e^{i\chi^\dagger}e^{-\alpha(Y,\vec{B};m_1',m_2',m_1,m_2)}e^{i\chi}\,, \quad (21)$$

where

$$\mathcal{A}(Y,\vec{B};m_1',m_2',m_1,m_2)=\tfrac{1}{2}[\chi,\chi^\dagger]$$
$$=\tilde{A}(Y,\vec{B})\int dm_1'^2\,dm_2'^2\,dm_1^2\,dm_2^2\,g(m_1'^2)g(m_2'^2)g(m_1^2)g(m_2^2)$$
$$\times d_1^\dagger(m_1')\,|0_1\rangle\langle 0_1|d_1(m_1)\,d_2^\dagger(m_2')\,|0_2\rangle\langle 0_2|d_2(m_2)\ . \tag{22}$$

One can easily see that this S matrix leads to the amplitudes (11) and (12).

In particular we write down the amplitude for producing two excited "nucleons" in the final state from two initial nucleons of mass m_0. Normalizing the coupling functions so that $g(m_0^2)=1$, we have:

$$M_{00}(Y,\vec{B};m_1',m_2',m_0,m_0)=2is\frac{g(m_1'^2)g(m_2'^2)}{I(s)}\left(1-e^{-\tilde{A}(Y,\vec{B})I(s)}\right)\ . \tag{23}$$

The corresponding amplitude for the production of two "excited nucleons" and n pions coming from the exchanged chains ("pionization region") is:

$$M_{n0}(Y,\vec{B};y_1\vec{b}_1,\ldots y_n,\vec{b}_n;m_1',m_2',m_0,m_0)=\left(\frac{iI(s)}{2s}\right)^{n-1}\prod_{j=1}^{n}\tilde{W}(Y,\vec{B};y_j,\vec{b}_j)\,e^{-\tilde{A}(Y,\vec{B})\cdot I(s)}\,g(m_1'^2)g(m_2'^2)\ . \tag{24}$$

Notice that everything so far reduces to the model of ASB if we set all "nucleon" masses equal to m_0 [and thus $I(s)\equiv 1$].

Let us now turn our attention to the production and decay of the "excited nucleons." In order to allow the "excited nucleons" to decay into an ordinary nucleon and pions, while at the same time retaining unitarity, we must consider the "excited nucleon" itself as a "resonant state" consisting of pions and one nucleon. The amplitude D for producing this state, must be closely related to the "coupling function," $g(m^2)$. Assuming for simplicity that the pions are produced independently of each other, we may write the amplitude for producing n pions with momenta q_i and one nucleon in a "resonant state" with invariant mass m [Fig. 3(a)] as[8]:

$$D_n(m,q_1,\ldots q_n)=\alpha(m^2)\prod_{i=1}^{n}h(q_i)\ , \tag{25}$$

where $\alpha(m^2)$ is some arbitrary function of m^2, and $h(q)$ is a function giving the pion distribution. The nucleon momentum will thus be determined by momentum conservation only. From Fig. 3(b) we see that the connection between g and D is:

$$\int_{m_0^2}^{\mu\sqrt{s}}dm^2[g(m^2)]^2=\frac{1}{\pi}\int_{m_0^2}^{\mu\sqrt{s}}dm^2\sum_{n=0}^{\infty}|D_n(m,q_1\ldots q_n)|^2\prod(\text{propagators})\,d'\phi_n^c\ . \tag{26}$$

$d'\phi_n^c$ is the phase space for the "cluster" of pions plus one nucleon. Making the approximation of putting all the particles on their mass shells, we can write

$$[g(m^2)]^2=\frac{1}{\pi}\int\sum_{n=0}^{\infty}|D_n(m,q_1\ldots q_n)|^2\,d\phi_n^c\ , \tag{27}$$

where

$$d\phi_n^c=\tfrac{1}{2}(2\pi)^4\,\delta^4\!\left(p^c-\sum_{i=1}^{n}q_i-p'\right)\prod_{i=1}^{n}\frac{d^3q_i}{(2\pi)^3\,2q_i^0}\ \frac{d^3p'}{(2\pi)^3\,2p'^0}\ . \tag{28}$$

p^c is the total cluster momentum and p' the momentum of the nucleon in the cluster.

We can now write down the full scattering amplitude for the production of N pions, of which n_π come from the chains and n_1 and n_2 from the "clusters"; with cluster masses m_1 and m_2:

$$\mathfrak{M}_{N0}(Y,\vec{\Delta};q_1\ldots q_{n_\pi};q_1'\ldots q_{n_1}',q_1''\ldots q_{n_2}'')=M_{n_\pi 0}(Y,\vec{p};q_1\ldots q_{n_\pi};m_1,m_2)\,D_{n_1}(m_1,q_1'\ldots q_{n_1}')\,D_{n_2}(m_2,q_1'',\ldots q_{n_2}''), \tag{29}$$

where

$$\vec{p}=\tfrac{1}{2}(\vec{p}^{\,c_1}-\vec{p}^{\,c_2})$$

is the transverse momentum transfer conjugate to $\vec{B}$.

III. PROPERTIES OF THE MODEL

A. Bootstrap Solution

As we have shown, the model is completely specified once the function $W(Y,\vec{B};y,\vec{b};m_1',m_2',m_1,m_2)$ and the cluster decay amplitudes $D_n(m,q_1\ldots q_n)$ are given. According to Eq. (10) we can write $\tilde{W}$ as a product of two "basic" two-body amplitudes $\tilde{M}$, where each of these corresponds to a process in which only one side of the diagram has a nucleon-type particle that can be excited. Therefore, with our definition of the S operator, these amplitudes have the form [see Eq. (23)]:

$$\tilde{M}(y_1-y;\tfrac{1}{2}\vec{B}-\vec{b})=\frac{2is_1}{[I(s)]^{1/2}}\left(1-e^{-\tilde{A}(y_1-y;\,\frac{1}{2}\vec{B}-\vec{b})}\right), \tag{30}$$

where $s_1=m_1'\mu\, e^{|y_1-y|}$ is the subenergy between the pion and the "excited nucleon"; with a corresponding expression for $\tilde{M}(y-y_2;\frac{1}{2}\vec{B}+\vec{b})$. $\tilde{A}$ is given by Eq. (14).

We then choose an input function of the form

$$(1-e^{-\tilde{A}(Y,\vec{B})})^{\text{in}}=\theta(R_0Y-B) \tag{31}$$

which through Eqs. (30), (10), and (14) gives:

$$\tilde{A}^{\text{out}}(Y,\vec{B})=\frac{\lambda^2\mu^4}{24R_0I(s)}(R_0^2Y^2-B^2)^{3/2}\,\theta(R_0Y-B)\,. \tag{32}$$

For the input and output forms to be consistent, we must require that $e^{-\tilde{A}}$ goes to zero *at least like a negative power of s*. This implies that $I(s)$ cannot increase faster than a power of $\ln s$, i.e.,

$$I(s)\lesssim\kappa(\ln s)^{\eta}, \tag{33}$$

where

$$0\leqslant\eta\leqslant2\,. \tag{34}$$

For this asymptotic behavior of $I(s)$ the bootstrap is complete, except in a ring of radius $B=R_0Y$ and width proportional to $Y^{(2\eta/3-1)}$. The area of the "grey ring" in which the bootstrap fails, increases with energy (except for $\eta=0$, in which case it remains constant), but becomes a negligible fraction of the total interaction area $\pi R_0^2Y^2$ for large s.

B. Cross Sections

The elastic amplitude now becomes [Eq. (23)]:

$$M(Y,\vec{B})_{\text{elastic}}\equiv M_{00}(Y,\vec{B};m_0,m_0,m_0,m_0)=\frac{2is}{I(s)}\,\theta(R_0Y-B)\,. \tag{35}$$

The total scattering cross section is therefore:

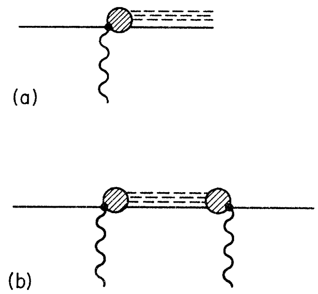

FIG. 3. (a) The amplitude $D_n(m,q_1,\ldots q_n)$ for producing a "resonant state" of invariant mass m, consisting of one nucleon and n pions. (b) The sum over all intermediate "resonant states" in this diagram gives the connection between $g(m^2)$ and $D_n(m,q_1\ldots q_n)$.

$$\sigma_{\text{tot}}(s)=\frac{1}{s}\int d^2B\,\text{Im}M(Y,\vec{B})_{\text{elastic}}=\frac{2\pi R_0^2Y^2}{I(s)}\,. \tag{36}$$

The elastic cross section is:

$$\sigma_{\text{elastic}}(s)=\frac{1}{4s^2}\int d^2B\,|M(Y,\vec{B})_{\text{elastic}}|^2=\frac{\pi R_0^2Y^2}{[I(s)]^2}\,. \tag{37}$$

Asymptotically, $\sigma_{\text{tot}}(s)\sim(\ln s)^{2-\eta}$ and $\sigma_{\text{elastic}}(s)\sim(\ln s)^{2-2\eta}$. For $\eta=2$ the total cross section becomes constant, while σ_{elastic} falls off like $(\ln s)^{-2}$. For $\eta=0$, on the other hand, both the total and the elastic cross sections increase like $(\ln s)^2$, saturating the Froissart bound, while σ_{elastic} becomes a constant fraction of σ_{tot}. We shall show below that the production and decay mechanism implicit in the nova model[7] corresponds to coupling functions $g(m^2)$ that produce $\eta=2$. It is also interesting to notice that if there exists a mechanism which strongly suppresses production of clusters with mass larger than some M_0, a total cross section which is constant for $s<M_0^4/\mu^2$ would start to increase for larger s, asymptotically growing like $(\ln s)^2$.[9]

For the case $\eta=2$ we compute the J-plane structure of the amplitude, and find:

$$M(J,t)_{\text{elastic}}=\frac{4\pi iR_0^2}{\kappa\{(J-1)+[(J-1)^2-R_0^2t]^{1/2}\}}+C(J,t), \tag{38}$$

where $C(J,t)$ is a nonsingular function near $J=1$. We thus have a branch cut in the J plane between the two points $J=1\pm iR_0\sqrt{-t}$. The cut collapses to a simple pole at $J=1$ for $t=0$, giving rise to constant total cross sections.

If we treat the pions in one cluster as distinguishable from those in the other cluster and from those in the pionization region, we can write the phase space for n pions and two nucleons as follows[8]:

$$d\phi_n = \frac{1}{\pi^2}\, dm_1^2\, dm_2^2\, d\phi^{c_1}_{n_1}\, d\phi^{c_2}_{n_2}\, d\phi^{\pi}_{n_\pi}.$$

$d\phi^{c_1}_{n_1}$ and $d\phi^{c_2}_{n_2}$ are given by Eq. (28), and represent the phase spaces for two clusters containing n_1 and n_2 pions, and with invariant masses m_1 and m_2, respectively. $d\phi^{\pi}_{n_\pi}$ is the phase space for two clusters of momenta p^{c_1} and p^{c_2} and n_π pions in the pionization region (see ASB):

$$d\phi^{\pi}_{n_\pi} = \frac{1}{n_\pi!}\,\tfrac{1}{2}(2\pi)^4\delta^4\Big(P_1+P_2-p^{c_1}-p^{c_2}-\sum_{i=1}^{n_\pi} q_i\Big)\prod_{i=1}^{n_\pi}\frac{d^3q_i}{(2\pi)^3 2q_i^0}\,\frac{d^3p^{c_1}}{(2\pi)^3 2p_0^{c_1}}\,\frac{d^3p^{c_2}}{(2\pi)^3 2p_0^{c_2}}$$

$$\simeq \frac{1}{n_\pi!}\,\frac{1}{4s}\prod_{i=1}^{n_\pi}\frac{d^2q_i}{(2\pi)^2}\,\frac{dy_i}{4\pi}\,\frac{d^2p}{(2\pi)^2}\ . \tag{39}$$

The cross section for production of n pions is therefore given by:

$$\sigma_n(s) = \frac{1}{\pi^2 s}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_\pi=0}^{\infty}\int_{m_0^2}^{\mu\sqrt{s}} dm_1^2\int_{m_0^2}^{\mu\sqrt{s}} dm_2^2\, |\tilde{M}_{n_\pi,0}(Y,\vec{p};\vec{q}_1,y_1,\ldots\vec{q}_{n_\pi},y_{n_\pi})|^2\, d\phi^{\pi}_{n_\pi}\, |D_{n_1}(m_1;q_1',\ldots q_{n_1}')|^2$$
$$\times d\phi^{c_1}_{n_1}\, |D_{n_2}(m_2;q_1'',\ldots q_{n_2}'')|^2\, d\phi^{c_2}_{n_2}\, \delta_{n,n_1+n_2+n_\pi}\ . \tag{40}$$

This can be written as:

$$\sigma_n(s) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_\pi=0}^{\infty}\sigma_c(n_1)\,\sigma_c(n_2)\,\sigma_\pi(n_\pi)\,\delta_{n,n_1+n_2+n_\pi}\ , \tag{41}$$

where

$$\sigma_c(n) \equiv \frac{1}{\pi}\int_{m_0^2}^{\mu\sqrt{s}} dm^2\int |D_n(m,q_1\ldots q_n)|^2\, d\phi^c_n \tag{42}$$

and

$$\sigma_\pi(n) \equiv \frac{1}{s}\int |\tilde{M}_{n,0}(Y,\vec{p};\vec{q}_1,y_1;\ldots\vec{q}_n,y_n)|^2\, d\phi^{\pi}_n\ . \tag{43}$$

To compute the inelastic partial cross sections σ_n, we introduce the generating function:

$$\sigma(x) \equiv \sum_{n=0}^{\infty} x^n\sigma_n = \Big[\sum_{n_1=0}^{\infty} x^{n_1}\sigma_c(n_1)\Big]\Big[\sum_{n_2=0}^{\infty} x^{n_2}\sigma_c(n_2)\Big]\Big[\sum_{n_\pi=0}^{\infty} x^{n_\pi}\sigma_\pi(n_\pi)\Big]\ . \tag{44}$$

$\sigma_n(s)$ is then given by:

$$\sigma_n(s) = \frac{1}{n!}\frac{d^n}{dx^n}\sigma(x)\Big|_{x=0} = \sum_{p=0}^{n}\sum_{k=0}^{n-p}\sigma_c(p)\,\sigma_c(n-p-k)\,\sigma_\pi(k)\ . \tag{45}$$

To compute σ_c and σ_π explicitly we need to make a specific assumption about the form of the decay amplitude D. Note that σ_c is an invariant, so we can consider the decay in the cluster's rest frame. Following Hwa[8] we assume a Gaussian pion distribution in the cluster rest frame and write Eq. (25) as

$$\bar{D}_n(m;k_1,\ldots k_n) = \alpha(m^2)\prod_{i=1}^{n} f(k_i)\ , \tag{46}$$

where

$$|f(k)|^2 = \left(\frac{6\pi}{E^2}\right)^{3/2} 2k_0\, e^{-3\vec{k}^2/2E^2}\ . \tag{47}$$

$\bar{D}$ is the decay matrix element in the cluster rest frame, k_i is the momentum of the ith pion in the same frame, and $E = \langle\vec{k}^2\rangle^{1/2}$ can be taken to be 350 MeV.

Now we can compute $[g(m^2)]^2$ from Eq. (27), obtaining:

$$[g(m^2)]^2 = \sum_{n=0}^{\infty} |\alpha(m^2)|^2\,\frac{m}{p_0(n)}\,\delta(m^2-m_n^2)\ , \tag{48}$$

where $p_0(n)$ is the average value of the zeroth component of the four-momentum of the final nucleon in the cluster;

$$p_0(n)=(nE^2+m_0^2)^{1/2}\ , \tag{49}$$

and

$$m_n=n\,\omega+p_0(n)\ , \tag{50}$$

$\omega\equiv(E^2+\mu_\pi^2)^{1/2}$, and the δ functions arise from making the approximation of using average values for the zeroth components of momenta in the momentum conservation δ function.

To obtain a constant total cross section we must demand that $[g(m^2)]^2\sim m^{-2}$ for large m. Otherwise the coupling functions are nearly arbitrary. We now show that a function similar to the one used in the nova model[8] shows that behavior for $[g(m^2)|^2$, namely

$$[g(m^2)]^2=\sum_{n=0}^{\infty}\left(\rho\delta_{n,0}+\gamma\frac{\exp[-\beta/(m^2-m_0^2)]}{m^2-m_0^2}\right)\delta_{n,n(m)}\ , \tag{51}$$

where ρ, γ, and β are parameters, and the $\delta_{n,0}$ takes care of the case of no excitation. Our normalization

$$[g(m_0^2)]^2\equiv\int_{m_0^2-\Delta m^2}^{m_0^2+\Delta m^2}dm^2[g(m^2)]^2=1$$

demands $\rho=(2m_0\omega+E^2)^{-1}$, and we can write (51) as:

$$[g(m^2)]^2=\sum_{n=0}^{\infty}\left(\delta_{n,0}+\gamma\frac{m_n[2p_0(n)\omega+E^2]}{p_0(n)(m_n^2-m_0^2)}\exp[-\beta/(m_n^2-m_0^2)]\right)\delta(m^2-m_n^2)\ . \tag{52}$$

Comparing with Eq. (48) we see that $\alpha(m^2)$ is determined by

$$|\alpha(m_n^2)|^2=\delta_{n,0}+\gamma\frac{2p_0(n)\omega+E^2}{m_n^2-m_0^2}\exp[-\beta/(m_n^2-m_0^2)]\ . \tag{53}$$

We can now write down $\sigma_c(n)$ directly from (42), (46), and (53) as:

$$\sigma_c(n)=\delta_{n,0}+\gamma\frac{m_n[2p_0(n)\omega+E^2]}{p_0(n)(m_n^2-m_0^2)}\exp[-\beta/(m_n^2-m_0^2)]\,\theta(n^c_{\max}-n)\ , \tag{54}$$

where

$$n^c_{\max}\lesssim\frac{\mu^{1/2}}{\omega}s^{1/4}\ . \tag{55}$$

Furthermore, for $n>0$:

$$\begin{aligned}\sigma_\pi(n)&=\frac{1}{s}\int|\tilde{M}_{n,0}(Y,\vec{p};y_1,\vec{q}_1,\ldots y_n,\vec{q}_n)|^2\,d\phi_n^\pi\\&=\frac{1}{n!}\frac{1}{[I(s)]^2}\int d^2B\,e^{-2\tilde{A}(Y,\vec{B})I(s)}[2\tilde{A}(Y,\vec{B})I(s)]^n\,\theta(n^\pi_{\max}-n)\\&\simeq\frac{1}{[I(s)]^2}\frac{\pi}{n!}\times\tfrac{2}{3}(2c)^{-2/3}\,\Gamma(n+\tfrac{2}{3})\,\theta(n^\pi_{\max}-n)\ ,\end{aligned} \tag{56}$$

where

$$c=\frac{\lambda^2\mu^4}{24R_0} \tag{57}$$

and $n^\pi_{\max}$ is a cutoff which we may choose as $n^\pi_{\max}\simeq c(R_0Y)^3$ to ensure that the pion production is peripheral. For $n=0$, $\sigma_\pi(0)=\sigma_{\text{elastic}}$.

Putting all this together we arrive at the following expression for the n-particle cross section:

$$\sigma_n(s)=\frac{\pi R_0^2 Y^2}{[I(s)]^2}\sum_{p=0}^{n}\left(\delta_{p,0}+\gamma\frac{2p_0(p)\omega+E^2}{m_p^2-m_0^2}\exp[-\beta/(m_p^2-m_0^2)]\right)$$
$$\times\left(\delta_{n-p,0}+\gamma\frac{2p_0(n-p)\omega+E^2}{m_{n-p}^2-m_0^2}\exp[-\beta/(m_{n-p}^2-m_0^2)]\right)\theta(n^c_{\max}-p)\,\theta(n^c_{\max}-n+p)$$
$$+\frac{\pi}{[I(s)]^2}\tfrac{2}{3}(2c)^{-2/3}\sum_{p=0}^{n-1}\sum_{k=1}^{n-p}\left(\delta_{p,0}+\gamma\frac{2p_0(p)\omega+E^2}{m_p^2-m_0^2}\exp[-\beta/(m_p^2-m_0^2)]\right)$$
$$\times\left(\delta_{n-p-k,0}+\gamma\frac{2p_0(n-p-k)\omega+E^2}{m_{n-p-k}^2-m_0^2}\exp[-\beta/(m_{n-p-k}^2-m_0^2)]\right)\frac{\Gamma(k+\frac{2}{3})}{\Gamma(k+1)}$$
$$\times\theta(n^c_{\max}-p)\,\theta(n^c_{\max}-n+p+k)\,\theta(n^\pi_{\max}-k)\,. \tag{58}$$

The first sum is the contribution from the amplitudes with all the pions produced in the clusters, while the second (double) sum corresponds to the case where at least one of the pions is produced from the chains.

IV. CONCLUSIONS

With the choice of coupling functions suggested by the nova model, our model has four free parameters. These correspond to the strength of the coupling functions (γ), the most probable mass of the excited states (β), the coupling constant for producing a pion from a chain (λ), and a parameter related to the radius of the absorbing disk(R_0). There are also two cutoffs forced upon us by the validity of the eikonal approximation, although the results are only dependent on one of them; the cutoff in the cluster mass.

In Figs. 4 and 5 we show a set of partial cross sections obtained from Eq. (58). While the parameter β was taken directly from Jacob and Slansky,[8] the remaining three parameters were determined by requiring a constant total cross section, both as obtained from the optical theorem [Eq. (36)] and from the sum of partial cross sections:

$$\sigma_{\text{tot}}(s)=\sum_{n=0}^{\infty}\sigma_n(s)\,.$$

It can be observed that at laboratory energies of the order of 2–300 GeV diffractive production dominates the low multiplicity region, while at higher energies diffraction becomes important also for larger values of n. The tail of the partial cross sections is always dominated by the production of pions from the chains, and therefore the behavior of the average multiplicity depends strongly on the multiperipheral mechanism. In our model the simplistic assumption of producing only one pion from each chain leads to average multiplicities of the form:

$$\langle n\rangle=a+bY^3\,. \tag{59}$$

Finally it can be shown that the sum of the quasielastic partial cross sections, i.e., the cross sections for producing n pions only from the clusters, equals one half the total cross section, while

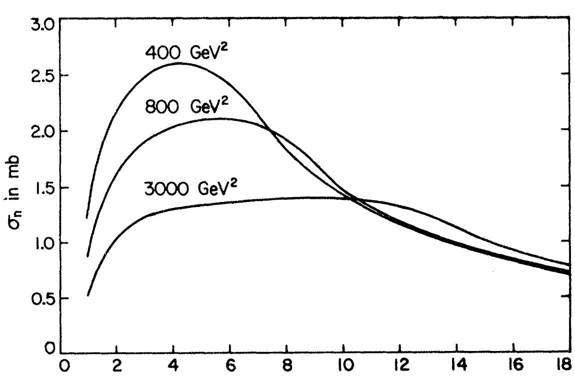

FIG. 4. Inelastic partial cross sections as a function of the number of pions produced, computed from Eq. (58), for three different values of the energy.

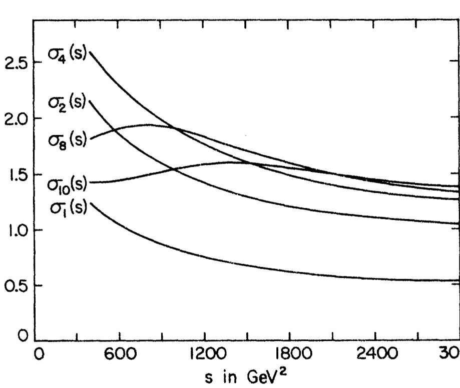

FIG. 5. Behavior of some of the inelastic partial cross sections as a function of energy.

the other half comes from the cases where at least one pion is produced from the chains.

We have presented here a very simple way to introduce a diffractive production mechanism into unitary multiperipheral amplitudes at high energies. The model so constructed is completely determined by the definition of two functions; the basic amplitude $W(Y, \vec{B}; y, \vec{b}; m_1', m_2', m_1, m_2)$ and the coupling function $g(m^2)$ for the production of an excited state of the external particles. The first function may be obtained by the requirement of self-consistency of elastic amplitudes, but of course it will always depend on the kind of multiperipheral mechanism that is being adopted. The second function, $g(m^2)$, requires a detailed knowledge of the dynamics of excitation of the external hadrons. However, here again unitarity and the consistency requirement limit the asymptotic behavior within well-defined bounds. It is interesting to notice that the integral over the square of the coupling function plays a prominent role in determining the asymptotic behavior of total cross sections, and through unitarity also in determining the nature of the Pomeranchuk singularity. In our model the amplitude leading to constant total cross sections has the appealing feature of producing a singularity structure in the J plane that collapses into a simple pole at $J=1$ when $t=0$. On the other hand, the amplitude leading to total cross sections saturating the Froissart bound gives an elastic cross section which is a constant fraction of the total cross section, a behavior that seems to be appearing in the latest experiments.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge a number of interesting suggestions from and discussions with J. Botke and R. Sugar.

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*Work supported by the National Science Foundation.

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[3]J. Fulco and R. Sugar, Phys. Rev. D 4, 1919 (1971); R. Aviv, R. Sugar, and R. Blankenbecler, Phys. Rev. D 5, 3252 (1972) (hereafter called ASB); G. Calucci, R. Jengo, and C. Rebbi, Nuovo Cimento 4A, 330 (1971); 6A, 601 (1971).

[4]For simplicity we have considered the case where only one pion is produced from each of the exchanged chains; generalization to more pions will be presented elsewhere.

[5]Here and in what follows a tilded quantity is always defined without the appropriate coupling functions for the *external* masses.

[6]See, for example, R. Sugar in *Lectures in Theoretical Physics*, edited by A. O. Barut and W. E. Brittin (Colorado Associated Univ. Press, Boulder, 1972), Vol. XIV A.

[7]To ensure the validity of the eikonal approximation, we require W to vanish unless y lies in the range $(1-\epsilon)y_2 \leq y \leq (1-\epsilon)y_1$, where ϵ is arbitrarily small.

[8]R. C. Hwa, Phys. Rev. Lett. 26, 1143 (1971); M. Jacob and R. Slansky, Phys. Rev. D 5, 1847 (1972).

[9]This seems to be the behavior indicated by some preliminary new results at ISR (CERN Intersecting Storage Rings).