Low-Energy Theorem for $\gamma \to 3\pi$ and a Unitarity Bound on the Decay Rate for $K \to \pi\gamma\gamma$

J. Smith*
Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, New York 11790

Z. E. S. Uy
Department of Physics, University of the Philippines, Quezon City, Philippines
(Received 7 May 1973)

Using the model-independent amplitude for $\pi\pi \to \pi\gamma\gamma$ predicted by the theory of anomalies, we derive unitarity bounds for the branching ratios $\Gamma(K^+ \to \pi^+ 2\gamma)/\Gamma(K^+ \to \pi^+\pi^0)$ and $\Gamma(K_S^0 \to \pi^0 2\gamma)/\Gamma(K_S^0 \to \pi^+\pi^-)$.

I. INTRODUCTION

The theory of anomalies in $\pi^0 \to 2\gamma$ decay[1] has so far escaped any real experimental test. The basic question of whether the $\pi^0 \to 2\gamma$ decay is suppressed or not to order m_π cannot be solved by examination of this one decay rate. However, a low-energy theorem, derived independently by Adler *et al.*,[2] Terentiev,[3] and Wong[4] relates the $\pi^0 \to 2\gamma$ coupling constant to the $\gamma \to 3\pi$ amplitude (evaluated at unphysical soft-pion limits). This relation, namely,

$$eF^{3\pi} = F^{\pi} f_\pi^{-2}, \tag{1.1}$$

where $F^{3\pi}$, F^{π}, and f_π are coupling constants describing the $\gamma \to 3\pi$ amplitude, the $\pi^0 \to 2\gamma$ amplitude, and the $\pi \to \mu\nu$ amplitude, respectively, allows a new avenue of approach to the subject. Indeed, if Eq. (1.1) is valid, then it allows one to conclude that the $\pi^0 \to 2\gamma$ decay proceeds through the anomaly and renormalized perturbation theory is true to any finite order. The importance of this statement has been stressed by Aviv and Zee.[5] In a separate paper Zee[6] examined several reactions with the view of checking Eq. (1.1). Unfortunately, the reactions he considered have background problems.

To see this in more detail we must examine the definitions of the quantities involved. Let us write the general amplitudes for $\pi^0 \to 2\gamma$ and $\gamma \to \pi^+\pi^-\pi^0$ as

$$M(\pi^0 \to \gamma(k_1) + \gamma(k_2)) = iF^{\pi}(k_1^2, k_2^2, (k_1+k_2)^2) \times \epsilon_{\alpha\beta\gamma\delta}\, k_1^\alpha k_2^\beta \epsilon_1^\gamma \epsilon_2^\delta \tag{1.2}$$

and

$$M(\gamma(k) \to \pi^0(p_0) + \pi^+(p_1) + \pi^-(p_2)) = -iF^{3\pi}((p_0+p_1)^2, (p_1+p_2)^2, p_0^2, p_1^2, p_2^2, (p_0+p_1+p_2)^2)\epsilon_{\alpha\beta\gamma\delta}\,\epsilon^\alpha p_0^\beta p_1^\gamma p_2^\delta, \tag{1.3}$$

where ϵ^α is the photon polarization vector and the external lines are off their physical mass shells. The result in Eq. (1.1) relates $F^{\pi} = F^{\pi}(0,0,0)$ to $F^{3\pi} = F^{3\pi}(0,0,0,0,0,0,)$. However, the usual assumption of partial conservation of axial-vector current (PCAC) tells us that the experimental quantities $F^{\pi}(0,0,m_\pi^2)$ and $F^{3\pi}((p_0+p_1)^2, (p_1+p_2)^2, m_\pi^2, m_\pi^2, m_\pi^2, (p_0+p_1+p_2)^2)$ can be replaced by F^{π} and $F^{3\pi}$ so long as $(p_0+p_1)^2$, $(p_1+p_2)^2$, and $(p_0+p_1+p_2)^2$ are small. It is the last restriction which is difficult to satisfy. For instance, the reaction $e^+e^- \to \gamma \to 3\pi$ (Ref. 6) only measures $F^{3\pi}$ where $(p_0+p_1+p_2)^2 > 9m_\pi^2$. Also, this amplitude is allowed to proceed via ω-meson exchange with a much larger rate than that expected from the $F^{3\pi}$ contribution. Another reaction which directly tests Eq. (1.1) is $\pi + Z \to 2\pi + Z$,[6] but this reaction is dominated by ρ-meson production.

Alternative ways of checking Eq. (1.1) use the fact that the $2\gamma \to \pi^+\pi^-\pi^0$ and $2\gamma \to \pi^0\pi^0\pi^0$ amplitudes contain the amplitude $F^{3\pi}$. In fact, the most general amplitude for $2\gamma \to 3\pi$, when expanded up to second order in the momenta, is completely determined by gauge invariance, current algebra, and PCAC, and the fact that the electromagnetic current commutes with the neutral axial charge at equal times. Hence the amplitude depends upon $F^{3\pi}$ and one other parameter describing the nature of the chiral symmetry breaking. Possible reactions for checking Eq. (1.1) can now be extended to $e^+e^- \to e^+e^- 3\pi$ (via two-photon exchange[7,8]) and $\gamma + Z \to 3\pi + Z$.[9] Unfortunately, the cross sections are so low that experimental detection is not possible at the present time. There are also background problems due to η-meson production ($2\gamma \to \eta$) followed by the decay $\eta \to 3\pi$.

We have therefore tried to find a check of Eq. (1.1) by more indirect means. The decays $K^+ \to \pi^+\gamma\gamma$ (see Refs. 10–12) and $K_S^0 \to \pi^0\gamma\gamma$ (see Ref. 13) are possible candidates. At the present time these decay modes have not been observed. Theoretical estimates are usually based on pole-model calculations or current-algebra models.[14] Rigorous unitarity bounds for these amplitudes based on real intermediate states would be very useful. Possible real intermediate states contain two pions or three pions. Sehgal[15] recently examined a model with a three-pion intermediate state for the imaginary part of the decay amplitude describing $K_L \to \pi^0\gamma\gamma$, i.e., $K_L \to \pi^0\pi^+\pi^- \to \pi^0\gamma\gamma$, where the two charged pions annihilate each other and produce two real photons. The bound then depends upon a model for the reaction $\pi^+\pi^- \to \gamma\gamma$, and Sehgal used perturbation theory. However, the bound he derived is only rigorous in a small portion of the Dalitz plot. Sehgal also considered the decay $K^+ \to \pi^+\gamma\gamma$, assuming the parity-conserving decay $K^+ \to \pi^+\pi^+\pi^-$ is constant and two of the final pions interact to produce two real photons. Again, this bound is model-dependent and is only rigorous in a small part of the Dalitz plot.

The bounds we derive for the decay modes are *model-independent* to the extent that the low-energy theorem given in Eq. (1.1) is satisfied. We use two pion dominance of the unitarity conditions for $K^+ \to \pi^+\gamma\gamma$ and $K_S^0 \to \pi^0\gamma\gamma$, i.e., the decays proceed in the chain $K \to \pi\pi \to \pi\gamma\gamma$. The coupling constants for the parity violating transitions $K \to \pi\pi$ are well-known. In fact, if we compute the branching ratios $\Gamma(K^+ \to \pi^+\gamma\gamma)/\Gamma(K^+ \to \pi^+\pi^0)$ and $\Gamma(K_S^0 \to \pi^0\gamma\gamma)/\Gamma(K_S^0 \to \pi^+\pi^-)$, these coupling constants cancel. The amplitude for $\pi\pi \to \pi\gamma\gamma$ is completely determined via the relation (1.1) and we obtain a model-independent unitarity limit calculated from the absorptive part of the $K \to \pi\gamma\gamma$ amplitude. Unfortunately, we cannot give any reliable estimate for the contribution from the real part of the $K \to \pi\gamma\gamma$ amplitude. We are well aware that the real part could be large. If experiments are performed to push the present branching ratios down and they find the decay modes at a level significantly above our bound, then relation (1.1) is not being tested. On the other hand, if the decays are not seen and the unitarity limit is violated, then Eq. (1.1) could be in trouble. We would like to add that the soft-pion extrapolation required in our calculation is reasonable. The pions are on their physical mass shells and the values of the other invariants are bounded by Dalitz plot limits and the mass of the kaon. Both decay modes have background problems from the decays $K^+ \to \pi^+\pi^0$ and $K_S^0 \to \pi^0\pi^0$, where a real π^0 decays immediately into two photons. However, for $K^+ \to \pi^+\pi^0$ this gives a sharp spike in the decay π^+ momentum spectrum at 205 MeV/c and we are forced to stay away from this particular pole. In reality, this is advantageous because the chiral-symmetry-breaking parameter occurs only in the π^0 pole term in the $\pi\pi \to \pi\gamma\gamma$ amplitude. This term is small away from the pole so we lose nothing by dropping it. Therefore the bound only depends on the value of $F^{3\pi}$.

In Sec. II we discuss in detail our model for the amplitude describing the $K^+ \to \pi^+\gamma\gamma$ decay and derive the analytic expression for the bound. Section III has a corresponding discussion for the decay $K_S^0 \to \pi^0\gamma\gamma$. In Sec. IV we summarize our conclusions. Some of the integrals required in the calculation are given in the Appendix.

II. UNITARITY BOUND FOR THE DECAY $K^+ \to \pi^+\gamma\gamma$

The decay $K^+ \to \pi^+\gamma\gamma$ has an imaginary part given by the sequence $K^+ \to \pi^+\pi^0 \to \pi^+\gamma\gamma$ which involves the parity violating $K^+ \to \pi^+\pi^0$ amplitude. Another contribution to the imaginary part comes from $K^+ \to \pi^+\pi^+\pi^- \to \pi^+\gamma\gamma$ and this involves the parity conserving transition $K^+ \to \pi^+\pi^+\pi^-$. We concentrate here on the two pion intermediate state because we can derive a rigorous bound using this contribution. The 3π state is suppressed by lack of phase space. The remaining part of the decay amplitude then involves the transition $\pi^+\pi^0 \to \pi^+\gamma\gamma$. There are five terms which contribute to this transition amplitude. They are depicted in Fig. 1. The "seagull" term is necessary to maintain gauge invariance. The other four terms involve bremsstrahlung radiation from the

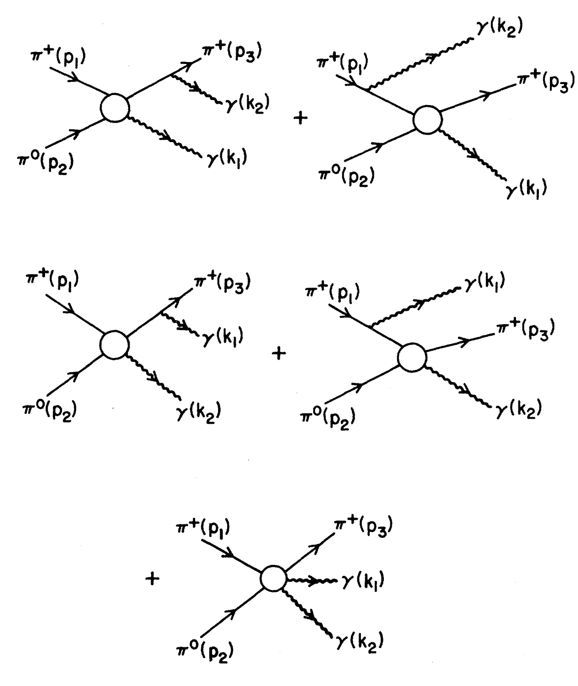

FIG. 1. Feynman diagrams describing the amplitude $\langle\pi^+\pi^0|M|\pi^+\gamma\gamma\rangle$.

$\pi\pi\to\pi\gamma$ amplitude. Note that there should be no infrared divergence in our answer for the $K\to\pi\gamma\gamma$ amplitude because angular momentum conservation forces the amplitude for $K\to\pi\gamma$ to vanish identically. Hence, we expect the $K\to\pi\gamma\gamma$ amplitude to be finite as the energy of each photon tends independently to zero (i.e., in two corners of the Dalitz plot). There will also be terms of higher order in the photon energy which vanish in these corners. Generally speaking, these terms are two orders of magnitude smaller than the finite ones.

Fig. 1 does not contain all the possible diagrams one can write down. For instance, there is the diagram in Fig. 2. However, it makes a large contribution only in the region where the π^0 is on mass shell and that region must be excluded because of background from the real decay $K^+\to\pi^+\pi^0\to\pi^+2\gamma$. The diagram in Fig. 2 is model-dependent because it depends upon a chiral-symmetry-breaking parameter. Its inclusion (with some appropriate cut to avoid the π^0 pole) would make essentially no difference to the bound we are about to derive.

The amplitude for $\pi^+(p_1)+\pi^0(p_2)\to\pi^+(p_3)+\gamma(k_1)+\gamma(k_2)$ can therefore be written in the form

$$\langle\pi^+\pi^0|M|\pi^+\gamma\gamma\rangle=eF^{3\pi}\epsilon_{\alpha\beta\gamma\delta}\left[\frac{\epsilon_2\cdot p_3}{k_2\cdot p_3}k_1^\alpha\epsilon_1^\beta(p_3+k_2)^\gamma p_1^\delta-\frac{\epsilon_2\cdot p_1}{k_2\cdot p_1}k_1^\alpha\epsilon_1^\beta p_3^\gamma(p_1-k_2)^\delta+\frac{\epsilon_1\cdot p_3}{k_1\cdot p_3}k_2^\alpha\epsilon_2^\beta(p_3+k_1)^\gamma p_1^\delta\right.$$
$$\left.-\frac{\epsilon_1\cdot p_1}{k_1\cdot p_1}k_2^\alpha\epsilon_2^\beta p_3^\gamma(p_1-k_1)^\delta+(k_1-k_2)^\alpha\epsilon_1^\beta\epsilon_2^\gamma(p_3+k_1+k_2-p_1)^\delta\right], \tag{2.1}$$

where $F^{3\pi}=F^{3\pi}(M^2,(p_1-p_3)^2,m_\pi^2,m_\pi^2,m_\pi^2,(p_1+p_2-p_3)^2)$. The invariants are constrained to be in the physical region of the Dalitz plot for the decay, so the extrapolation to the amplitude $F^{3\pi}(0,0,0,0,0,0)$ is as reasonable as any of the other extrapolations made in applying soft-pion theorems to K decays. We are thus justified in using Eq. (1.1) to give the value for $eF^{3\pi}$. Note that the amplitude is gauge-invariant in both photons. The rest of the calculation is straightforward. The unitarity condition for the $K^+(P)\to\pi^+2\gamma$ amplitude[16] gives

$$\text{Abs}\,\langle K^+|M|\pi^+\gamma\gamma\rangle=-\frac{1}{16\pi}\left\{\frac{2}{\pi}\int\frac{d^3p_1}{2E_1}\int\frac{d^3p_2}{2E_2}\,\delta^4(P-p_1-p_2)\langle K^+|M|\pi^+\pi^0\rangle\langle\pi^+\pi^0|M|\pi^+\gamma\gamma\rangle^*\right\}. \tag{2.2}$$

The $K\to\pi\pi$ amplitude $\langle\pi^+\pi^0|M|K^+\rangle$ is a constant, so we remove it from the integral and call it λ. The decay rate for $K^+\to\pi^+\pi^0$ is therefore given by ($\beta^2=1-4m_\pi^2/M^2$)

$$\Gamma(K^+\to\pi^+\pi^0)=\frac{\lambda^2\beta}{16\pi M}. \tag{2.3}$$

The evaluation of the integrations in Eq. (2.2) can be done in the c.m. frame. However, we prefer to have results which are explicitly covariant (and of course gauge-invariant) so we use the integrals tabulated in the Appendix. After some algebra we find

$$\text{Abs}\,\langle K^+|M|\pi^+\gamma\gamma\rangle=\left(-\frac{\lambda eF^{3\pi}}{64\pi}\right)\epsilon_{\alpha\beta\gamma\delta}\times\left[Kk_1^\alpha k_2^\beta\epsilon_1^\gamma\epsilon_2^\delta-I_2k_1^\alpha\epsilon_1^\beta\epsilon_2^\gamma P^\delta-I_1k_2^\alpha\epsilon_2^\beta\epsilon_1^\gamma P^\delta+I_2\frac{P\cdot\epsilon_2}{P\cdot k_2}k_1^\alpha\epsilon_1^\beta k_2^\gamma P^\delta+I_1\frac{P\cdot\epsilon_1}{P\cdot k_1}k_2^\alpha\epsilon_2^\beta k_1^\gamma P^\delta\right], \tag{2.4}$$

where we have used the following notation:

$$K=\frac{T}{P\cdot k_1}+\frac{T}{P\cdot k_2},$$
$$I_1=\frac{T-2\beta P\cdot k_1}{P\cdot k_1},$$
$$I_2=\frac{T-2\beta P\cdot k_2}{P\cdot k_2},$$
$$T=s\beta-2m_\pi^2V,$$
$$V=\ln\left(\frac{1+\beta}{1-\beta}\right),$$

and $s=P^2=M^2$. The equation is still explicitly gauge-invariant. However, it is also divergent as the photon momenta tend to zero. The divergence comes from the terms in $T/P\cdot k_1$ and $T/P\cdot k_2$ in I_1 and I_2, respectively. Also, it does not have the correct structure. In general, there are two amplitudes which contribute to the parity-violating

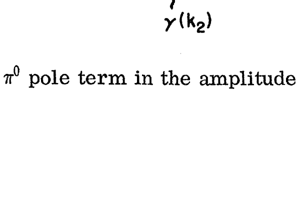

FIG. 2. The π^0 pole term in the amplitude $\langle\pi^+\pi^0|M|\pi^+\gamma\gamma\rangle$.

transition $K^+\to\pi^+\gamma\gamma$. These amplitudes must be gauge-invariant in both photons, i.e.,

$$\begin{aligned}\langle K^+|M|\pi^+\gamma\gamma\rangle =& A\epsilon_{\alpha\beta\gamma\delta}k_1^\alpha k_2^\beta \epsilon_1^\gamma \epsilon_2^\delta\\ &+B\epsilon_{\alpha\beta\gamma\delta}[k_1^\alpha\epsilon_1^\beta(P\cdot k_2\epsilon_2^\gamma - P\cdot\epsilon_2 k_2^\gamma)P^\delta\\ &+k_2^\alpha\epsilon_2^\beta(P\cdot k_1\epsilon_1^\gamma - P\cdot\epsilon_1 k_1^\gamma)P^\delta],\end{aligned}\tag{2.5}$$

where A and B are functions of the Dalitz-plot variables. Both amplitudes have absorptive parts and Eq. (2.4) cannot be written in this form. Obviously we must add some other contributions to Eq. (2.4) to cancel the infrared divergence and make our final answer compatible with Eq. (2.5).

The extra diagrams we have to consider must contain the $K\pi\pi$ coupling constant and $F^{3\pi}$. Hence, they can only have the decay $K\to\pi\pi\gamma$ followed by $\pi\pi\to\pi\gamma$, and are shown in Fig. 3. We include bremsstrahlung radiation from the $K\pi\pi$ vertex and, for completeness, two structure-dependent terms corresponding to electric and magnetic radiation. These diagrams only have absorptive parts for $(p_1+p_2)^2=(P-k)^2\geq 4m_\pi{}^2$, where k is either k_1 or k_2. In the Dalitz-plot variables this means that the energies of the photons satisfy the inequalities $\omega_1\leq(M^2-4m_\pi{}^2)/2M$ and $\omega_2\leq(M^2-4m_\pi{}^2)/2M$. If we write the matrix element

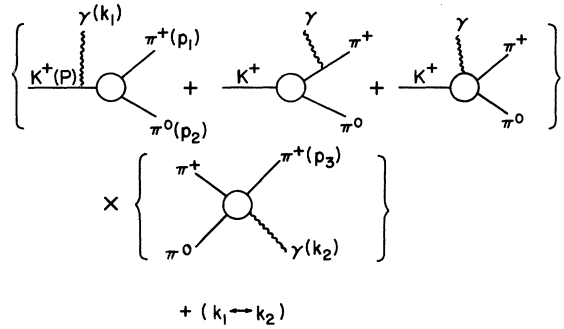

FIG. 3. Feynman diagrams describing the absorptive part of the amplitude for $K^+\to\pi^+\gamma\gamma$ in the sequence $K^+\to\pi^+\pi^0\gamma\to\pi^+\gamma\gamma$.

for $K^+(P)\to\pi^+(p_1)+\pi^0(p_2)+\gamma(k_1)$ as

$$\begin{aligned}\langle K^+|M|\pi^+\pi^0\gamma\rangle = e\lambda\Big[&\left(\frac{p_1\cdot\epsilon_1}{p_1\cdot k_1}-\frac{P\cdot\epsilon_1}{P\cdot k_1}\right)\\ &+\frac{\mathcal{E}}{M^4}(\epsilon_1^\mu k_1^\nu-\epsilon_1^\nu k_1^\mu)p_{1\mu}P_\nu\\ &+\frac{\mathfrak{M}}{M^4}\epsilon_{\mu\nu\sigma\tau}\epsilon_1^\mu k_1^\nu p_1^\sigma P^\nu\Big],\end{aligned}\tag{2.6}$$

then $\mathcal{E}$ and $\mathfrak{M}$ are dimensionless electric and magnetic structure terms with phases given by the $\pi\pi$-scattering phase shifts. Hence

$$\mathrm{Abs}\langle K^+|M|\pi^+\gamma\gamma\rangle = -\frac{1}{16\pi}\left[\frac{2}{\pi}\int\frac{d^3p_1}{2E_1}\int\frac{d^2p_2}{2E_2}\delta^4(P-p_1-p_2)\langle K^+|M|\pi^+\pi^0\gamma\rangle\langle\pi^+\pi^0|M|\pi^+\gamma\rangle^*\right].\tag{2.7}$$

If we now substitute Eq. (2.6) into Eq. (2.7) and take both photons into account, then we can use the integrals listed in the Appendix. During this calculation we drop the term containing $\mathfrak{M}$ because it contributes to the opposite parity transition $K^+\to\pi^+\gamma\gamma$ (from the product of two ϵ terms). The answer is

$$\begin{aligned}\mathrm{Abs}\langle K^+|M|\pi^+\gamma\gamma\rangle = -\frac{e\lambda F^{3\pi}}{64\pi}\epsilon_{\alpha\beta\gamma\delta}\Bigg\{&\Bigg[\left(\frac{T}{P\cdot k_1}+\frac{T}{P\cdot k_2}\right)k_1^\alpha k_2^\beta\epsilon_1^\gamma\epsilon_2^\delta-\frac{T}{(P\cdot k_1)^2}(P\cdot\epsilon_1k_1^\gamma-P\cdot k_1\epsilon_1^\gamma)k_2^\alpha\epsilon_2^\beta P^\delta\\ &-\frac{T}{(P\cdot k_2)^2}(P\cdot\epsilon_2k_2^\gamma-P\cdot k_2\epsilon_2^\gamma)k_1^\alpha\epsilon_1^\beta P^\delta\Bigg]\\ &+\frac{\mathcal{E}}{M^4}\left(-\frac{\beta^3s}{3}\right)[(P\cdot k_1+P\cdot k_2)k_1^\alpha k_2^\beta\epsilon_1^\gamma\epsilon_2^\delta+(P\cdot\epsilon_1k_1^\gamma-P\cdot k_1\epsilon_1^\gamma)k_2^\alpha\epsilon_2^\beta P^\delta\\ &+(P\cdot\epsilon_2k_2^\gamma-P\cdot k_2\epsilon_2^\gamma)k_1^\alpha\epsilon_1^\beta P^\delta]\Bigg\}.\end{aligned}\tag{2.8}$$

In the soft-photon regions, the infrared-divergent terms in this equation exactly cancel the infrared-divergent terms in Eq. (2.4) so that the amplitude is finite as ω_1 and ω_2 tend to zero. The additional terms arising from the electric multipole amplitude are higher order in the photon momenta. They vanish in the limit when ω_1 and ω_2 tend to zero, and only make a small contribution when ω_1 and ω_2 are large. If we now use Eq. (2.8) over the whole Dalitz plot, we can add the two results to write the absorptive part as in Eq. (2.4) and with new values for K, I_1, and I_2, namely,

$$\begin{aligned}K&=2\left(\frac{T}{P\cdot k_1}+\frac{T}{P\cdot k_2}\right)-\frac{\beta^3\mathcal{E}(P\cdot k_1+P\cdot k_2)}{3M^2},\\ I_1&=-2\beta-\frac{\mathcal{E}P\cdot k_1\beta^3}{3M^2},\\ I_2&=-2\beta-\frac{\mathcal{E}P\cdot k_2\beta^3}{3M^2}.\end{aligned}\tag{2.9}$$

This answer can be rewritten in the form of Eq. (2.5). Although we are only allowed to add the answers in the region of the Dalitz plot where $\omega_1 \leq (M^2-4m_\pi^2)/2M$ and $\omega_2 \leq (M^2-4m_\pi^2)/2M$, Eq. (2.8) gives a very small contribution if we simply extend it outside that region. Remember that the maximum values of the photon energies are ω_1 or $\omega_2 = (M^2-m_\pi^2)/2M$. When the photon energy is large, the powers of k in the denominator make the terms independent of $\mathcal{E}$ very small. Experimentally, nothing is known about the parameter $\mathcal{E}$. A recent experiment by Abrams *et al.*[17] found evidence for a structure-dependent term in the decay $K^+ \to \pi^+\pi^0\gamma$ which was compatible with magnetic rather than electric radiation. If we assume that $\mathcal{E}=1$, then the $\mathcal{E}$ term makes a 2% correction to decay rate calculated from the $\lambda F^{3\pi}$ term alone. We are therefore quite justified in dropping the term containing $\mathcal{E}$ (and also the term in $\mathfrak{M}$ which contributes to the opposite-parity amplitude in the decay).

If we now square the amplitude and calculate the decay rate from the absorptive part alone, we find

$$\Gamma(K^+ \to \pi^+\gamma\gamma) = \frac{M^5}{128\pi^3}\left(\frac{\lambda e F^{3\pi}}{64\pi}\right)^2 J, \tag{2.10}$$

where J is the dimensionless double integral:

$$J = \frac{1}{M^6}\int_{y_0}^{M-m_\pi} dy \int_{x_-}^{x_+} dx \left[2t^2K^2 - 4tuKI_1 - 4twKI_2 + 2t(2uw-ts)\left(\frac{I_1}{u}+\frac{I_2}{w}\right)K + 2u^2I_2^2 - 4(uw-ts)I_1I_2 + 2w^2I_1^2 - st(2uw-ts)\left(\frac{I_1^2}{u^2}+\frac{I_2^2}{w^2}\right)\right], \tag{2.11}$$

and we have introduced the following definitions:

$$u = P\cdot k_1 = M\omega_1,$$
$$w = P\cdot k_2 = M\omega_2,$$
$$t = k_1\cdot k_2,$$
$$y = M - E_\pi = \omega_1 + \omega_2,$$
$$y_0 = \frac{(M^2-m_\pi^2)}{2M},$$
$$x = \tfrac{1}{2}(\omega_1-\omega_2), \quad x_\pm = \pm\tfrac{1}{2}[(M-y)^2-m_\pi^2]^{1/2}.$$

We now assume the standard values of the coupling constants, namely, $F^\pi = \alpha/0.65\pi m_\pi$, $f_\pi = 0.68\, m_\pi$ and calculate the double integral numerically. It is convenient to first divide by the rate for $K^+ \to \pi^+\pi^0$ to eliminate the coupling constant λ, i.e.,

$$\frac{\Gamma(K^+\to\pi^+\gamma\gamma)}{\Gamma(K^+\to\pi^+\pi^0)} = \frac{M^6}{8\pi^2\beta}\left(\frac{F^\pi}{64\pi f_\pi^2}\right)^2 J = CJ. \tag{2.12}$$

Unfortunately, the large factor $(M/m_\pi)^6$ is more than compensated for by the factors of π in the denominator, so the numerical value of the coefficient is very small. We find that $C = 4.45\times10^{-8}$. If we now use Eq. (2.4) to calculate J, then the answer is infrared divergent in the photon energies. The pion spectrum is plotted in Fig. 4, where we see clearly the structure-dependent terms [i.e., the terms in Eq. (2.4) which are finite as ω_1 or ω_2 tends to zero] sitting on top of the infrared-divergent terms. If we now add our result in Eq. (2.8) to the Dalitz-plot regions where ω_1 and $\omega_2 \leq (M^2-4m_\pi^2)/2M$, then the infrared-divergent peak is canceled and the convergent part remains essentially unchanged. The pion spectrum for the final answer tends to zero at both ends of its range. Numerically we find that $J = 0.14$, so the final answer for the branching ratio

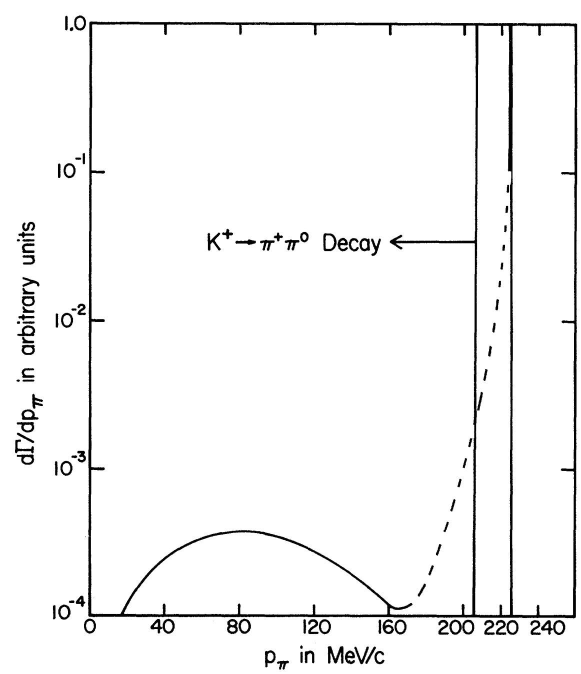

FIG. 4. The charged-pion spectrum in the decay $K^+ \to \pi^+\gamma\gamma$ calculated from the absorptive part of the amplitude with a $\pi^+\pi^0$ intermediate state. The dashed curve is the inner bremsstrahlung contribution which is canceled by the contribution from the $\pi^+\pi^0\gamma$ state.

is

$$\frac{\Gamma(K^+\to\pi^+\gamma\gamma)}{\Gamma(K^+\to\pi^+\pi^0)}=6.2\times10^{-9}. \qquad (2.13)$$

It is unfortunate that this bound is so low. The numerical factors of α and π already cut the branching ratio down to the 10^{-8} level and there is no enhancement in the absorptive part of the matrix element. The bound is probably only of theoretical interest because one expects the real part of the amplitude (coming from pole terms) to give a large contribution. It would be very surprising if the real part turned out to be as small as the imaginary part. Our bound should be considered independently from the bound given by Sehgal because they are rigorous in different regions of the Dalitz plot. His is valid for $(P-p_3)^2=(k_1+k_2)^2\geq 4m_\pi^2$, so $E_\pi\leq(M^2-4m_\pi^2)/2M$, while the physically allowed range for E_π is $m_\pi\leq E_\pi\leq(M^2+m_\pi^2)/2M$. In other words, his bound is for low-energy pions (or high-energy photons), while our bound is valid over the whole Dalitz plot.

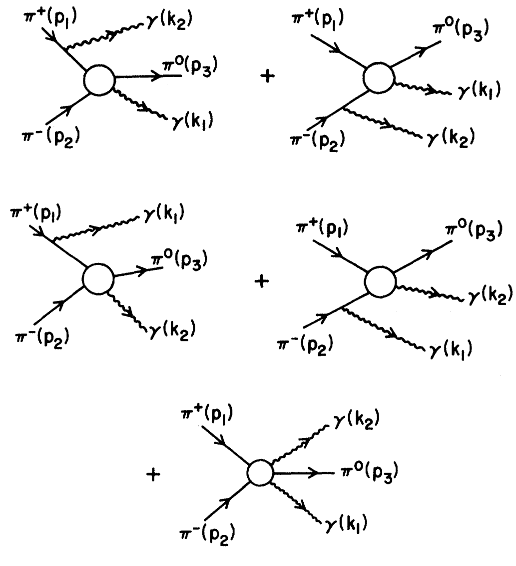

FIG. 5. Feynman diagrams describing the amplitude $\langle\pi^+\pi^-|M|\pi^0\gamma\gamma\rangle$.

III. UNITARITY BOUND FOR THE DECAY $K_S^0\to\pi^0\gamma\gamma$

We now repeat the analysis of Sec. II for the decay $K_S^0\to\pi^0\gamma\gamma$, under the assumption that CP is conserved (so $K_S^0\equiv K_1^0$). There are now two different intermediate states, namely, $\pi^+\pi^-$ and $\pi^0\pi^0$. When we neglect all contributions from π^0 pole terms, then only the $\pi^+\pi^-$ state is possible. The five Feynman diagrams which contribute to the amplitude for $\pi^+\pi^-\to\pi^0\gamma\gamma$ are shown in Fig. 5, and they give

$$\langle\pi^+\pi^-|M|\pi^0\gamma\gamma\rangle=eF^{3\pi}\epsilon_{\alpha\beta\gamma\delta}\left[\frac{p_1\cdot\epsilon_2}{p_1\cdot k_2}k_1^\alpha\epsilon_1^\beta(p_1-k_2)^\gamma p_2^\delta-\frac{p_2\cdot\epsilon_2}{p_2\cdot k_2}k_1^\alpha\epsilon_1^\beta p_1^\gamma(p_2-k_2)^\delta+\frac{p_1\cdot\epsilon_1}{p_1\cdot k_1}k_2^\alpha\epsilon_2^\beta(p_1-k_1)^\gamma p_2^\delta\right.$$
$$\left.-\frac{p_2\cdot\epsilon_1}{p_2\cdot k_1}k_2^\alpha\epsilon_2^\beta p_1^\gamma(p_2-k_1)^\delta+(k_1-k_2)^\alpha\epsilon_1^\beta\epsilon_2^\gamma p_3^\delta\right]. \qquad (3.1)$$

Straightforward application of the projection operator leads to an absorptive part almost identical to that for $K^+\to\pi^+\gamma\gamma$, i.e.,

$$\mathrm{Abs}\langle K_S^0|M|\pi^0\gamma\gamma\rangle=\left(\frac{-\lambda eF^{3\pi}}{32\pi}\right)\epsilon_{\alpha\beta\gamma\delta}\left[\hat{K}k_1^\alpha k_2^\beta\epsilon_1^\gamma\epsilon_2^\delta-I_2k_1^\alpha\epsilon_1^\beta\epsilon_2^\gamma P^\delta-I_2k_2^\alpha\epsilon_2^\beta\epsilon_1^\gamma P^\delta+I_2\frac{P\cdot\epsilon_2}{P\cdot k_1}k_1^\alpha\epsilon_1^\beta k_2^\gamma P^\delta\right.$$
$$\left.+I_1\frac{P\cdot\epsilon_1}{P\cdot k_1}k_2^\alpha\epsilon_2^\beta k_1^\gamma P^\delta\right], \qquad (3.2)$$

where $\hat{K}=I_1+I_2$ and I_1 and I_2 are the same as in Sec. II. The main difference between Eqs. (2.4) and (3.2) is the factor of 32π rather than 64π in the denominator. In Eq. (3.2) λ denotes the $K_S^0\to\pi^+\pi^-$ decay coupling constant. The rest of the analysis parallels that in Sec. II and we get an infrared divergence unless we add the additional graphs coming from the sequence $K_S^0\to\pi^+\pi^-\gamma\to\pi^0\gamma\gamma$. As in Sec. II these additional graphs can be calculated and yield

$$\mathrm{Abs}\langle K_S^0|M|\pi^0\gamma\gamma\rangle=\left(-\frac{\lambda eF^{3\pi}}{32\pi}\right)\epsilon_{\alpha\beta\gamma\delta}\left\{\left[-\left(\frac{T}{P\cdot k_1}+\frac{T}{P\cdot k_2}\right)k_1^\alpha k_2^\beta\epsilon_1^\gamma\epsilon_2^\delta-\frac{T}{(P\cdot k_1)^2}(P\cdot\epsilon_1k_1^\gamma-P\cdot k_1\epsilon_1^\gamma)k_2^\alpha\epsilon_2^\beta P^\delta\right.\right.$$
$$\left.-\frac{T}{(P\cdot k_2)^2}(P\cdot\epsilon_2k_2^\gamma-P\cdot k_2\epsilon_2^\gamma)k_1^\alpha\epsilon_1^\beta P^\delta\right]$$
$$+\frac{\mathcal{E}}{M^4}\left(\frac{-s\beta^3}{6}\right)\left[(P\cdot k_1+P\cdot k_2)k_1^\alpha k_2^\beta\epsilon_1^\gamma\epsilon_2^\delta+(P\cdot\epsilon_1k_1^\gamma-P\cdot k_1\epsilon_1^\gamma)k_2^\alpha\epsilon_2^\beta P^\delta\right.$$
$$\left.\left.+(P\cdot\epsilon_2k_2^\gamma-P\cdot k_2\epsilon_2^\gamma)k_1^\alpha\epsilon_1^\beta P^\delta\right]\right\}. \qquad (3.3)$$

One sees that the infrared-divergent terms now cancel those in Eq. (3.2) and, if we apply Eq. (3.3) over the whole of the Dalitz plot, then we can write our final answer in the form of Eq. (2.4) with

$$K=-4\beta-\frac{\mathcal{E}\beta^3(P\cdot k_1+P\cdot k_2)}{6M^2},\quad I_1=-2\beta-\frac{\mathcal{E}\beta^3 P\cdot k_1}{6M^2},\quad I_2=-2\beta-\frac{\mathcal{E}\beta^3 P\cdot k_2}{6M^2}, \tag{3.4}$$

and a factor of 32π rather than 64π. Using the absorptive part alone, we can square the amplitude and write the formula for the branching ratio as

$$\frac{\Gamma(K_S^0\to\pi^0\gamma\gamma)}{\Gamma(K_S^0\to\pi^+\pi^-)}=\frac{M^6}{8\pi^2\beta}\left(\frac{F^\pi}{32\pi f_\pi{}^2}\right)^2\hat{J}, \tag{3.5}$$

where $\hat{J}$ is the same integral defined in Eq. (2.11) but with values of K, I_1, and I_2 as given by Eq. (3.4). The numerical value of the branching ratio is 4.6×10^{-9} when we set $\mathcal{E}=0$ and do the integrations over the Dalitz-plot regions correctly [i.e., use Eq. (3.3) only in the region $\omega_1\le(M^2-4m_\pi{}^2)/2M$, $\omega_2\le(M^2-4m_\pi{}^2)/2M$]. The final pion spectrum is almost identical to the structure-dependent part of the spectrum in Fig. 4.

IV. CONCLUSIONS

We have derived a rigorous unitarity bound for the branching ratios $\Gamma(K^+\to\pi^+\gamma\gamma)/\Gamma(K^+\to\pi^+\pi^0)$ and $\Gamma(K_S^0\to\pi^0\gamma\gamma)/\Gamma(K_S^0\to\pi^+\pi^-)$. The bound for the charged decay is the more interesting of the two. Our input consists of experimental data on the $K\to\pi\pi$ rates and the model-independent prediction of the $\pi\pi\to\pi\gamma\gamma$ amplitude from the theory of PCAC anomalies. This bound is a crucial test of Eq. (1.1) and, if it is violated, it may have serious theoretical consequences. One would first have to examine other possible real intermediate states to see if they could produce large cancellations. The various pole-model calculations (which have real amplitudes) all predict higher decay rates, but many of them are already eliminated by the latest experimental results.

ACKNOWLEDGMENTS

We would like to thank T. Kycia and members of his group for useful discussions.

APPENDIX

For the convenience of the reader we list the integrals required in deriving the unitarity bound. If we call the "projection operator"

$$\frac{2}{\pi}\int d^4p_1\,\delta(P^2-2P\cdot p_1)\delta(p_1{}^2-m^2)\theta(p_{10})\theta(P_0-p_{10})=\Pi,$$

then

$$\Pi 1=\beta,\quad \Pi p_{1\mu}=(\tfrac{1}{2}\beta)P_\mu,$$

$$\Pi\frac{1}{p_1\cdot k_1}=b(k_1),\quad \Pi p_1^\alpha p_1^\beta=A'g^{\alpha\beta}+B'P^\alpha P^\beta,$$

$$\Pi\frac{p_1^\delta}{p_1\cdot k_1}=\eta(k_1)k_1^\delta+\rho(k_1)P^\delta,$$

$$\Pi\frac{p_1^\alpha p_1^\beta}{p_1\cdot k_1}=A(k_1)g^{\alpha\beta}+B(k_1)(P^\alpha k_1^\beta+P^\beta k_1^\alpha)+D(k_1)P^\alpha P^\beta+E(k_1)k_1^\alpha k_1^\beta,$$

where

$$b(k_1)=\frac{1}{P\cdot k_1}\ln\left(\frac{1+\beta}{1-\beta}\right)=\frac{1}{P\cdot k_1}V,$$

$$A'=-\tfrac{1}{12}\beta(s-4m^2),\quad B'=\tfrac{1}{3}\beta(s-m^2),$$

$$\rho(k_1)=\frac{\beta}{P\cdot k_1},\quad \eta(k_1)=\frac{s(V-2\beta)}{2(P\cdot k_1)^2},$$

$$A(k_1)=\frac{2m^2V-s\beta}{4P\cdot k_1}=\frac{-T}{4P\cdot k_1},$$

$$B(k_1)=\frac{s\beta-2m^2V}{4(P\cdot k_1)^2}=\frac{T}{4(P\cdot k_1)^2},$$

$$D(k_1)=\frac{\beta}{2P\cdot k_1},\quad E(k_1)=\frac{s(sV+2m^2V-3\beta s)}{4(P\cdot k_1)^3}.$$

---

*Work supported in part by National Science Foundation Grant No. GP-32998X.

[1]S. L. Adler, Phys. Rev. 177, 2426 (1969); J. S. Bell and R. Jackiw, Nuovo Cimento 60, 47 (1969). Excellent reviews of the theory of partial conservation of axial-vector current (PCAC) anomalies have been written by S. L. Adler and R. Jackiw. See S. L. Adler [*Lectures on Elementary Particles and Quantum Field Theory*, edited by S. Deser, M. Grisaru, and H. Pendleton (MIT, Cambridge, Mass., 1971)], and S. B. Treiman, R. Jackiw, and D. J. Gross, *Lectures on Current Algebra and Its Applications* (Princeton Univ. Press, Princeton, N. J., 1972), pp. 97-254 (by R. Jackiw).

[2]S. L. Adler, B. W. Lee, S. B. Treiman, and A. Zee, Phys. Rev. D 4, 3497 (1971).

[3]M. V. Terentiev, Zh. Eksp. Teor. Fiz. Pis'ma Red. 14, 140 (1971) [JETP Lett. 14, 94 (1971)].

[4]T. F. Wong, Phys. Rev. Lett. 27, 1617 (1971).

[5]R. Aviv and A. Zee, Phys. Rev. D 5, 2372 (1972).

[6]A. Zee, Phys. Rev. D 6, 900 (1972).

[7]M. Pratap, J. Smith, and Z. E. S. Uy, Phys. Rev. D 5, 269 (1972).

[8]R. Köberle, Phys. Lett. 38B, 169 (1972).

[9]J. Smith and N. Stanko, Phys. Rev. D 6, 1332 (1973).

[10]J. H. Klems, R. H. Hildebrand, and R. Stiening, Phys. Rev. Lett. 25, 473 (1970); Phys. Rev. D 4, 66 (1971).

[11]D. Ljung and D. Cline, Phys. Rev. D 8, 1307 (1973).

[12]R. J. Abrams, A. S. Carroll, T. F. Kycia, K. K. Li, D. N. Michael, P. M. Mockett, and R. Rubinstein, Phys. Rev. D (to be published).

[13]Nothing is known experimentally about this decay mode. As $K_L \to 2\pi$ violates CP invariance, the unitarity bound for $K_L \to \pi^0 2\gamma$ is so small that it is not of interest at present.

[14]The latest published paper is by M. Moshe and P. Singer, Phys. Rev. D 6, 1379 (1972). Most of the previous models have been ruled out by the experiments referred to in Refs. 10, 11, and 12.

[15]L. M. Sehgal, Phys. Rev. D 6, 367 (1972).

[16]Analogous calculations for $K \to 2\mu$ decay have been given by L. M. Sehgal [Nuovo Cimento 45, 785 (1966)], and by B. R. Martin, E. de Rafael, and J. Smith, [Phys. Rev. D 2, 179 (1970); 272(E) (1971)]. We have done the calculation in a longhand fashion which allows us to check gauge invariance at every stage.

[17]R. J. Abrams, A. S. Carroll, T. F. Kycia, K. K. Li, J. Menes, D. N. Michael, P. M. Mockett, and R. Rubinstein, Phys. Rev. Lett. 29, 1118 (1972).

PHYSICAL REVIEW D VOLUME 8, NUMBER 9 1 NOVEMBER 1973

CP Violation and Electric Dipole Moments in Gauge Theories of Weak Interactions*

A. Pais and J. R. Primack†

Rockefeller University, New York, New York 10021

(Received 14 May 1973)

We present some general considerations on computations of electric dipole moments (D) of leptons and of hadrons in gauge theories which incorporate CP violation. Technical aspects of the isolation of the CP-violating parts of the corresponding graphs are described. We emphasize the distinction (already familiar from g–2 constraints) between "mixed chirality" and "pure chirality" couplings. In the mixed-chirality case, D's can potentially appear to third order in the semiweak-coupling constant; in the pure-chirality case D is at least of fifth order. Estimates are given for prototypes of two classes of gauge models: (1) CP violation is implemented via the introduction of a small parameter. For the example considered, a model due to Mohapatra, D_{nucleon} and D_{lepton} are both of fifth order. (2) The violation is "maximal," as exemplified in the O(4) and O(4)×U(1) models. Here D_{muon} appears to third order in the (mixed-chirality) O(4) case. All other D's in the "maximal" examples are of fifth order. For the "maximal" model our estimates are below but may not be very far from the present experimental bounds. For the small-parameter model they are quite considerably smaller.

I. INTRODUCTION AND RESULTS

The experimental discovery[1] of CP violation in K_L decay implies the violation of T invariance if one accepts the CPT theorem. In addition, the K complex provides direct experimental tests of T invariance.[2-4] In fact, it is not necessary to assume CPT invariance at the outset in the experimental analysis of these decays since they may serve to test independently T invariance and CPT invariance.[5] Such tests have been performed[5] with the result that T violation has been established with 10 standard deviations, while there is no evidence (here or anywhere else) for CPT violation.[6]

As is well known, experimental attempts to observe T-violating effects outside the K complex have all turned out negative so far. This is particularly true in regard to searches for electric dipole moments (D) of nucleons and leptons. We recall that a nonvanishing D can only occur if both T and P violation are present, and that the possibility of the existence of D's for elementary particles and nuclei was first raised by Purcell and Ramsey[7] well before the discovery even of P violation. For spin-$\frac{1}{2}$ particles, the presence of a D implies the existence of an *effective* interaction of the form

$$iF_D(q^2)\bar{\psi}(p_1)\gamma_5\sigma_{\mu\nu}\psi(p_2)F_{\mu\nu}\ , \tag{1.1}$$

where $F_{\mu\nu}$ is the electromagnetic field and $q^2 = (p_1-p_2)^2$. D is defined by $D=F_D(0)$. Note that the possibility that $F_D(0)=0$ (though perhaps not natural) cannot be discarded out of hand.[8] That is to say, even in the presence of P and T violation the *static* quantity $F_D(0)$ could be zero.

It is the purpose of this paper to discuss the question of electric dipole moments for fermions in the context of the general strategy embodied in the gauge theories of weak and electromagnetic interactions.[9] In such theories currents appear which are associated with the generators of a compact Lie group. Like all relativistic local Lagrangian field theories, such theories are inherently CPT-invariant. They are P-violating by construction, since the particle states with