PHYSICAL REVIEW D VOLUME 8, NUMBER 9 1 NOVEMBER 1973

Multiplicity Distributions in Multiperipheral Models with Isospin

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(Received 25 April 1973)

We develop a systematic approach to the study of multiplicity distributions in a general class of multiperipheral models (MPM) with isospin. Three features characterize these models: (1) the exchanged particles have definite isospin; (2) the production cross sections factorize into isospin- and dynamics-dependent terms; and (3) the high-energy behavior of the total cross section is governed by a leading Regge pole. Our approach is based on the construction of the general form of the generating function for the multiplicity distribution. Using this technique we are able to study independently the separate effects of the isospin constraints and of the underlying, isospin-independent dynamics. We find that, although a number of general features of the multiplicity distributions are sensitive only to the specific isospin exchange, the details of the multiplicity distributions depend crucially on both isospin and dynamics. In particular, when nontrivial dynamical correlations are present, the behavior of the multiplicity moments can be altered significantly from that expected by isospin considerations alone.

I. INTRODUCTION

Study of the full multiplicity distribution in high-energy collisions has long been hampered by the lack of data on neutral particles in the final state. To investigate the isospin properties of production mechanisms, for example, a knowledge of the "prong" cross sections only—or, equivalently, $P(n_c)$, where n_c is the number of charged pairs—is of limited use, since many of the features of these distributions reflect model-independent consequences of charge conservation.

Recently, however, groups at Serpukhov,[1] CERN,[2] and the National Accelerator Laboratory[3] (NAL) have all released new high-energy data involving neutral particles. Although these results are of necessity still limited both in scope and in accuracy, their influence in stimulating discussion of the isospin properties of multiplicity distributions has been profound. The predictions of many simple theoretical models have been studied[4-8] and, to the extent that it is possible, compared with the data.

Among the models analyzed have been several simple versions of the multiperipheral model (MPM).[4,5,7] These versions have differed in the nature of the particles exchanged along the multiperipheral chain, but they have all shared the common assumption that the isospin-independent dynamics leads to a Poisson distribution in the total number of particles, $n \equiv 2n_- + n_0$. Further, there has been no obvious way of relaxing this "Poisson dynamics" assumption and thus of calculating multiplicity distributions in more general versions of the MPM.

Although the Poisson limit is a valid starting point, it is well known that any realistic MPM will contain—independent of any isospin considerations—nonvanishing dynamical correlations, which are incompatible with the simple Poisson distribution.[9] Thus the extent to which one can classify the consequences of Poisson dynamics as MPM predictions is unclear.

To clarify this question we develop in the present article a systematic approach to the study of multiplicity distributions in general MPM with isospin.

In Sec. II we review those properties of the MPM that are relevant to evaluating the multiplicity distribution in the presence of both isospin effects and dynamical correlations. We determine the general form of the generating function for the multiplicity distribution and then, using this generating function, examine certain features of the multiplicity distribution which depend on the isospin structure of the particular MPM, but which are insensitive to the details of the dynamical correlations. For definiteness, we choose two specific isospin-exchange mechanisms as illustrations: the "H model,"[5] in which $I=\frac{1}{2}$ objects are exchanged and $I=1$ objects produced; and the "I model,"[5] in which $I=1$ objects are exchanged and produced.

In Secs. III and IV, to investigate the interplay of isospin and dynamical effects, we examine several different models for the underlying dynamics in conjunction with both the H- and I-isospin mechanisms. Of particular interest is the result that dynamical correlations can indeed alter radically the structure of the multiplicity distribution predicted by considering only the Poisson dynamics limit.

We conclude in Sec. V with a summary of our findings and a discussion of their implications for attempts to use current experimental multiplicity distributions to discriminate among various models of multiparticle production.

II. GENERAL STRUCTURE OF THE MULTIPERIPHERAL MODEL WITH ISOSPIN

A. Differential Cross Sections

We begin by specifying the class of models which we wish to investigate. Consider a multiperipheral mechanism for the production process (Fig. 1)

$$a+b\to a'+b'+1+2+\ldots+n, \qquad (2.1)$$

where a and b represent the initial particles, a' and b' represent the leading final particles, and $(1, 2, \ldots, n)$ represent the additional particles produced. For simplicity, we assume that all additional particles are pions; the momentum and the charge (or I_3) of the final pions are denoted by q_m and i_m. We make the following assumptions: (1) The exchanged particles have definite isospin I. All the isospin dependence of the amplitude is given by the products of the isotopic-invariant couplings at each vertex. (2) The differential cross sections are given approximately by the diagonal terms (as shown in Fig. 2). (3) The isotopic-spin-independent part of the cross sections generates a leading Regge pole in the total cross section.

For this class of MPM, the properly normalized n-particle exclusive cross sections, after integration over the transverse momenta, are of the form

$$\frac{1}{\sigma_0}(d\sigma^{i_1\cdots i_n})_{\alpha\beta}=\lambda^n(S^{i_1}\cdots S^{i_n})_{\alpha\beta}$$
$$\times h_n(q_1,\ldots,q_n)\,d\Phi_1\ldots d\Phi_n\ . \qquad (2.2)$$

This result is derived in the Appendix. Here σ_0 is a normalization constant, λ describes the coupling strength, S^i (i represents the charge or I_3 of a final particle) is a constant matrix whose explicit form depends on the exchange mechanism, the subscripts α and β denote the isospin indices of the first and the last exchanged particles, $d\Phi = dq^+/q^+$ is the longitudinal phase space, and $h_n(q_1, \ldots, q_n)$ is a numerical function depending only on the momenta of the final (additional) particles but not their isospin. In other words, the factor $S^{i_1}S^{i_2}\ldots S^{i_n}$ contains all the isospin dependence of the cross sections, and h_n contains all the detailed dynamics.

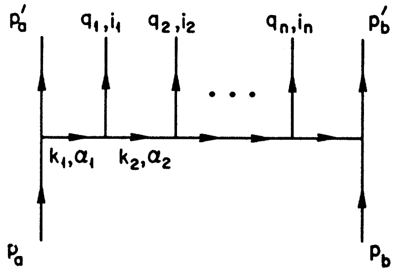

FIG. 1. A typical multiperipheral production amplitude.

The assumption of a leading Regge pole is realized in most simple MPM's including the multi-Regge models. As illustrated in Ref. 10, a sufficient condition for generating a leading Regge pole in the total cross section is the existence of factorization in the exclusive cross sections. In terms of h_n, this factorization requires that, for $(y_1, y_2, \ldots, y_m) \gg (y_{m+1}, \ldots, y_n)$,

$$h_n(q_1,\ldots,q_n)\to h_m(q_1,\ldots,q_m)$$
$$\times h_{n-m}(q_{m+1},\ldots,q_n), \qquad (2.3)$$

where $y_i=\ln(q_i^+/m)\equiv\ln[(q_i^0+q_i^3)/m]$ is the rapidity of the ith particle. The validity of this factorization property is not necessary for the general discussion given in this section. However, in the explicit models studied in Secs. III and IV, we adopt this simple mechanism of generating Regge poles. Physically, this factorization property implies that the exclusive function h_n has only short-range correlations. We refer the readers to Ref. 10 for the various consequences and limitations of this factorization postulate.

In this paper, we shall study models with only pions in the final state and with a fixed-isospin (I) exchange. In particular, we study models with exchanged isospin $I=\frac{1}{2}$ (called H model[5]) and with $I=1$ (called I model[5]).

In the Appendix, we find that for the H model, the matrices S in (2.2) are given by

$$S^+=\begin{pmatrix}0&2\\0&0\end{pmatrix},\quad S^-=\begin{pmatrix}0&0\\2&0\end{pmatrix},\quad S^0=\begin{pmatrix}1&0\\0&1\end{pmatrix}, \qquad (2.4)$$

and for the I model

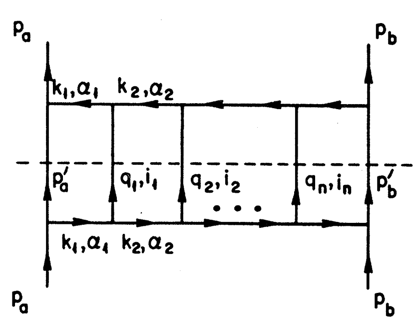

FIG. 2. Only the diagonal "uncrossed ladder" terms are included in the differential cross sections.

$$S^+=\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix},\quad S^-=\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix},\quad S^0=\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}. \tag{2.5}$$

In (2.4) and (2.5), the superscripts ±, 0 denote the charge of the final pions $\pi^\pm$, π^0. The dimension of the matrices S is $(2I+1)\times(2I+1)$, corresponding to the dimension of the multiplets of exchanged particles.

B. Multiplicity Distribution

1. Generating Function

To study the multiplicity distribution it is convenient to construct the following generating function[11,12]:

$$G(x, Y)=\sum \lambda^n (x_+S^+ + x_-S^- + x_0S^0)^n \int d\Phi_1 \cdots d\Phi_n h_n \tag{2.6}$$

$$=\left(\frac{\sigma_T}{\sigma_0}\right)\sum_{n_+,n_-,n_0}(x_+)^{n_+}(x_-)^{n_-}(x_0)^{n_0}P(n_+, n_-, n_0), \tag{2.7}$$

where x's are q-independent parameters, $d\Phi = dq^+/q^+$, and $Y=\ln(s/m^2)$ is the total length of the phase space in terms of rapidity. The total cross section σ_T, is given by

$$\frac{\sigma_T}{\sigma_0}\equiv G(1, Y)=\sum \lambda^n (S^+ + S^- + S^0)^n \int d\Phi_1 \cdots d\Phi_n h_n \ . \tag{2.8}$$

$P(n_+, n_-, n_0)$ is the multiplicity distribution function for n_+ positive pions, n_- negative pions, and n_0 neutral pions, and is normalized by

$$\sum_{n_+,n_-,n_0} P(n_+, n_-, n_0)=1 \ . \tag{2.9}$$

Knowing $G(x, Y)$, one can compute $P(n_+, n_-, n_0)$.

Since only the matrix

$$S(x)\equiv x_+S^+ + x_-S^- + x_0S^0 \tag{2.10}$$

enters into (2.6) and (2.7), potential difficulties due to the noncommutating of the S_i never arise. Thus, we can compute $G(x, Y)$ as if there were no S factors at all. We introduce

$$\bar{G}(z, Y)=\sum_n z^n \int d\Phi_1 \cdots d\Phi_n h_n \ . \tag{2.11}$$

From the assumption (3) of the existence of the leading Regge pole, we have at large s

$$\bar{G}(z, Y)_{\text{large } s} \overset{=}{} \beta(z)\exp[\alpha(z)\, Y], \tag{2.12}$$

with

$$Y=\ln(s/m^2)\ , \tag{2.13}$$

where $\alpha(z)$ is the trajectory function. Hence the large-s behavior of $G(x, Y)$ is given by the matrix equation

$$G(x, Y)=\beta(\lambda S(x))\exp[\alpha(\lambda S(x))\, Y]\ . \tag{2.14}$$

The corrections to (2.12) and (2.14) arise from secondary Regge trajectories produced by the dynamics. These can be incorporated directly in any specific model.

In H model, we have

$$S(x)=\begin{pmatrix}x_0 & 2x_-\\ 2x_- & x_0\end{pmatrix}. \tag{2.15}$$

We choose x_0 to be real and $x_-=x_+^*$ so that the matrix $S(x)$ is always Hermitian. The matrix $S(x)$ can be diagonalized, giving the two eigenvalues

$$\mu_0=x_0+2(x_+x_-)^{1/2} \tag{2.16a}$$

and

$$\mu_1=x_0-2(x_+x_-)^{1/2}\ . \tag{2.16b}$$

The projection operators $P_i (i=0,1)$ can be computed straightforwardly and are given in the Appendix.

Hence the contribution of the leading Regge pole to the generating function in H model can be written as

$$G(x, Y)=\beta(\lambda\mu_0)\exp[\lambda\alpha(\mu_0)\, Y]P_0 + \beta(\lambda\mu_1)\exp[\lambda\alpha(\mu_1)\, Y]P_1. \tag{2.17}$$

In I model, we can make similar decomposition. The (isospin) matrix is

$$S(x)\equiv\begin{pmatrix}x_0 & x_+ & 0\\ x_- & 0 & x_+\\ 0 & x_- & x_-\end{pmatrix}\ , \tag{2.18}$$

and its eigenvalues are

$$\mu_0=\tfrac{1}{2}[x_0+(x_0^2+8x_+x_-)^{1/2}], \tag{2.19a}$$

$$\mu_1=x_0\ , \tag{2.19b}$$

and

$$\mu_2=\tfrac{1}{2}[x_0-(x_0^2+8x_+x_-)^{1/2}]\ , \tag{2.19c}$$

respectively. The corresponding projection operators P_0, P_1, and P_2 are also given in the Appendix.

In terms of μ_i and P_i the generating function in I model can be expressed as

$$G(x, Y)=\sum_{i=0}^{2}\beta(\lambda\mu_i)\exp[\alpha(\lambda, \mu_i)\, Y]P_i\ . \tag{2.20}$$

Equation (2.20) can be generalized trivially to a system with arbitrary isospin exchange.

According to a well-known theorem of statistical mechanics,[13] $\alpha(\lambda)$ is nondecreasing function of λ.

From (2.16) and (2.19) we have

$$\mu_0 > \mu_1 > \mu_2 \ , \tag{2.21}$$

and hence $\alpha(\mu_0)$ is the highest-lying trajectory function. Thus at high energy, the contributions of $\alpha(\mu_1)$ and $\alpha(\mu_2)$, as well as those of the secondary dynamical trajectories, can be ignored, and we obtain

$$G(x, Y) = \beta(\lambda\mu_0)\exp[\alpha(\lambda\mu_0)\, Y]\, P_0 \ . \tag{2.22}$$

2. Relations Among Leading Contributions to Multiplicity Distributions for Various Charge States

In this subsection, we ignore the contribution from all lower trajectories and concentrate on the leading Regge-pole contribution (2.22). The fact that (2.22) is a function of $x^{\pm}$, x^0 only through $\mu_0(x)$ implies some nontrivial relations among various multiplicity correlation functions. In our model, these relations depend only on the isospin of the exchanged particles, and are independent of the explicit form of h_n. Thus, these relations can be used to differentiate the H model from the I model independent of the dynamics.

Those features of the multiplicity distributions which are both theoretically interesting and experimentally accessible are the average multiplicity $\langle n_i\rangle$ and the second moments of the multiplicity distributions,

$$f_2^{ij} \equiv \langle n_i n_j\rangle - \langle n_i\rangle\langle n_j\rangle - \delta_{ij}\langle n_i\rangle \ . \tag{2.23}$$

In the above expression n_+ (n_-) and n_0 are the numbers of positive (negative) and neutral pions, and

$$\langle A\rangle \equiv \sum_{\{n\}} A(\{n\})\, P(\{n\}), \qquad \{n\} = (n_+, n_-, n_0) \ . \tag{2.24}$$

These moments of multiplicity distribution are related to the trajectory function $\alpha(\lambda\mu_0(x))$, to leading order in Y, through

$$\langle n_i\rangle = Y \frac{\partial}{\partial x_i}\, \alpha(\lambda\mu_0(x))\Big|_{x=1} \tag{2.25}$$

and

$$f_2^{ij} = Y \frac{\partial^2}{\partial x_i\,\partial x_j}\, \alpha(\lambda\mu_0(x))\Big|_{x=1} \ . \tag{2.26}$$

Thus the $\langle n_i\rangle$ and f_2^{ij}'s are interrelated by the "chain-rule" of partial differentiation.

In H model, we have $\mu_0 = x_0 + 2(x_+x_-)^{1/2}$. Thus we obtain after differentiations

$$\langle n_0\rangle = \langle n_+\rangle - \langle n_-\rangle = \tfrac{1}{3}\langle n\rangle \ , \tag{2.27}$$

$$f_2^{00} = \tfrac{1}{9}\Delta - \tfrac{1}{9}\langle n\rangle \ , \tag{2.28a}$$

$$f_2^{0+} = f_2^{0-} = \tfrac{1}{9}\Delta - \tfrac{1}{9}\langle n\rangle \ , \tag{2.28b}$$

$$f_2^{++} = f_2^{--} = \tfrac{1}{9}\Delta - \tfrac{5}{18}\langle n\rangle \ , \tag{2.28c}$$

$$f_2^{+-} = \tfrac{1}{9}\Delta + \tfrac{1}{18}\langle n\rangle , \tag{2.28d}$$

where $n = n_+ + n_- + n_0$ is the total number of pions,

$$\langle n\rangle = \mu_0 \frac{d}{d\mu_0}\, \alpha(\lambda\mu_0)\Big|_{x=1} \tag{2.29}$$

is the average pion multiplicity, and

$$\Delta = D^2 \equiv \langle n^2\rangle - \langle n\rangle^2 = \left(\mu_0 \frac{d}{d\mu_0}\right)^2 \alpha(\lambda\mu_0)\Big|_{x=1} \tag{2.30}$$

is the standard deviation. In I model, we have (2.27) and, instead of (2.28),

$$f_2^{00} = \tfrac{1}{9}\Delta - \tfrac{1}{27}\langle n\rangle \ , \tag{2.31a}$$

$$f_2^{0+} = f_2^{0-} = \tfrac{1}{9}\Delta - \tfrac{4}{27}\langle n\rangle \ , \tag{2.31b}$$

$$f_2^{++} = f_2^{--} = \tfrac{1}{9}\Delta - \tfrac{7}{27}\langle n\rangle \ , \tag{2.31c}$$

$$f_2^{+-} = \tfrac{1}{9}\Delta + \tfrac{2}{27}\langle n\rangle \ . \tag{2.31d}$$

Equation (2.27) is a direct consequence of isospin conservation and is independent of the dynamical production mechanism. Since this relation does not test any of our basic assumptions we are not interested in it. Equations (2.28) and (2.31), on the other hand, depend on the multiperipheral mechanism and the isospin of the exchanged particle (but are independent of the detailed dynamical function h_n). Hence, they are of considerable physical interest. It is important to point out that, in both the H and I models and as a direct consequence of (2.28) and (2.31), a Poisson distribution in the total pion number n necessarily implies non-Poisson distributions in the charged pions ($n_{\pm}$).

From (2.28d) and (2.31d), we find that f_2^{+-} is always positive. This result is also clearly independent of the dynamics; it follows directly from the change conservation condition, $n_+ = n_-$[14]; namely, using $n_+ = n_-$, we obtain

$$f_2^{+-} \equiv \langle n_+ n_-\rangle - \langle n_+\rangle\langle n_-\rangle = \langle {n_+}^2\rangle - \langle n_+\rangle^2 > 0 \ . \tag{2.32}$$

Incidentally, (2.32) reveals that f_2^{+-} and f_2^{++} are linearly related.

C. Multiplicity Distributions near the Means

In very high-energy collisions, one expects to see a large number of pions in the final states. For typical MPM the number distribution is Poisson-like; that is, it predicts $f_n \propto \ln s$. Thus, at very high energy, the majority of the events will have multiplicity very close to the mean with a width $\sim\sqrt{n}$. For this reason, we study here the multiplicity distributions near the mean. As one might guess from the statistical mechanics,[15] the distribution function near the mean is of the generalized Gaussian form, and depends on the dynamics only through the mean $\langle n_i\rangle$ and the deviations Δ_{ij}(or f_2^{ij}). In the following, let us apply the general technique of the statistical mechanics to our problem.

Knowing $G(i, Y)$, we can construct $P(n_+, n_-, n_0)$ either by explicit Taylor's expansion or by Cauchy's theorem[16]:

$$P_{\alpha\beta}(n_+, n_-, n_0)=\frac{1}{(2\pi i)^3}\oint \frac{dx_+}{x_+^{n_++1}}\frac{dx_-}{x_-^{n_-+1}}\frac{dx_0}{x_0^{n_0+1}}\times\left[\frac{G(x, Y)_{\alpha\beta}}{G(1, Y)_{\alpha\beta}}\right] . \quad (2.33)$$

For a neutral initial state, charge conservation implies[12,17]

$$G(x, Y)=G(x_+x_-, x_0, Y) . \quad (2.34)$$

Using the result that

$$\oint dx_+\oint dx_-=\oint\frac{dx_+}{x_+}\oint d(x_+x_-), \quad (2.35)$$

we have

$$P(n_+, n_-, n_0)=\delta_{n_+,n_-}\frac{1}{(2\pi i)^2}\oint\frac{dx_0}{x_0^{n_0+1}}\oint\frac{dx_c}{x_c^{n_++1}}\times\left[\frac{G(x_0, x_c, Y)}{G(1, Y)}\right], \quad (2.36)$$

with

$$x_c=x_+x_- . \quad (2.37)$$

Since the dependence on the charges of the first- and the last-exchanged particles (i.e. indices α and β) disappears in the high-energy limit, we have suppressed it in (2.36). To obtain the exact form of $P(n_+, n_-, n_0)$, we need to know the function $G(x, Y)$ exactly. However, at large n, if we are only interested in the multiplicity distribution around the mean—i.e. around $n_i\sim\langle n_i\rangle$—we can carry out the integrals in (2.36) by the method of steepest descents. In this calculation, only $\alpha'|_{x=1}$ and $\alpha''|_{x=1}$, or, equivalently, only $\langle n\rangle$ and Δ are required. Since this method is standard in statistical mechanics, we simply summarize the results. We obtain, in H model,

$$P(n_+, n_-, n_0)=\delta_{n_+,n_-}\frac{3}{\pi(2\Delta\langle n\rangle)^{1/2}}\exp\left\{-\left(\frac{1}{2\Delta}+\frac{1}{\langle n\rangle}\right)(n_0-\langle n_0\rangle)^2 -2\left(\frac{1}{\Delta}-\frac{1}{\langle n\rangle}\right)(n_0-\langle n_0\rangle)(n_+-\langle n_+\rangle)-\left(\frac{2}{\Delta}+\frac{1}{\langle n\rangle}\right)(n_+-\langle n_+\rangle)^2\right\} , \quad (2.38a)$$

$$=\frac{\delta_{n_+n_-}}{\pi(2\Delta\langle n\rangle)^{1/2}}\exp\left[-\frac{1}{2\Delta}(n-\langle n\rangle)^2\right]\exp\left[-\frac{1}{\langle n\rangle}(n_+-n_0)^2\right] ; \quad (2.38b)$$

and in I model, we obtain

$$P(n_+, n_-, n_0)=\frac{\delta_{n_+,n_-}}{\pi}\left(\frac{27}{8\Delta\langle n\rangle}\right)^{1/2}\exp\left\{-\left(\frac{1}{2\Delta}+\frac{3}{4\langle n\rangle}\right)(n_0-\langle n_0\rangle)^2 -2\left(\frac{1}{\Delta}-\frac{3}{4\langle n\rangle}\right)(n_0-\langle n_0\rangle)(n_+-\langle n_+\rangle)-\left(\frac{2}{\Delta}+\frac{3}{4\langle n\rangle}\right)(n_+-\langle n_+\rangle)^2\right\}, \quad (2.39a)$$

$$=\frac{\delta_{n_+,n_-}}{\pi}\left(\frac{27}{8\Delta\langle n\rangle}\right)^{1/2}\exp\left[-\frac{1}{2\Delta}(n-\langle n\rangle)^2\right]\exp\left[-\frac{3}{4\langle n\rangle}(n_+-n_0)^2\right], \quad (2.39b)$$

where $\langle n\rangle$ and Δ are given in (2.29, 2.30). As a consistency check, one can recover (2.28) and (2.31) from (2.38) and (2.39) easily.

Gaussian distributions of the form (2.38b), (2.39b) were proposed recently by Horn and Schwimmer[6] to describe the qualitative features of the correlations between neutral and charged pions. The first Gaussian describes the distribution of the total number of pions around the mean, and the second Gaussian describes the distributuion of n_+, n_0 around $n_+=n_0$. In our models, the Gaussian structure emerges naturally in the expansion around the mean. In particular, we find that the width in (n_+-n_0) distribution is of $O(\sqrt{n})$, comparable to $D=\sqrt{\Delta}$.

Given (2.38) and (2.39), we can study various correlation effects among the multiplicities. In particular, we can study the average n_0 multiplicity for a fixed n_+ and vice versa. We shall refer to these quantities as "associated means" and denote them by

$$(n_0)_{n_+}\equiv\sum_{n_0} n_0P(n_0, n_+)\Big/\sum_{n_0}P(n_0, n_+) , \quad (2.40a)$$

$$(n_+)_{n_0} \equiv \sum_{n_+} n_+ P(n_0, n_+) \Big/ \sum_{n_+} P(n_0, n_+) \ . \qquad (2.40b)$$

In (2.40) we have suppressed the trivial δ_{n_+, n_-} factor. Present crude experimental results suggest a linear dependence of $(n_0)_{n_+}$ on n_+.[1-3,18] As we shall see, this linear dependence emerges naturally around the mean.

To compute the associated means, we substitute (2.38) and (2.39) into (2.40), and replace the discrete sum over n_i by an integral. We obtain in H model,

$$(n_0)_{n_+} = \langle n_0 \rangle + \frac{\Delta - \langle n \rangle}{\Delta + \frac{1}{2}\langle n \rangle} (n_+ - \langle n_+ \rangle) \ , \qquad (2.41a)$$

$$(n_+)_{n_0} = \langle n_+ \rangle + \frac{\Delta - \langle n \rangle}{\Delta + 2\langle n \rangle} (n_0 - \langle n_0 \rangle) \ ; \qquad (2.41b)$$

and in I model,

$$(n_0)_{n_+} = \langle n_0 \rangle + \frac{\Delta - \frac{4}{3}\langle n \rangle}{\Delta + \frac{2}{3}\langle n \rangle} (n_+ - \langle n_+ \rangle) \ , \qquad (2.42a)$$

$$(n_+)_{n_0} = \langle n_+ \rangle + \frac{\Delta - \frac{4}{3}\langle n \rangle}{\Delta + \frac{8}{3}\langle n \rangle} (n_0 - \langle n_0 \rangle) \ . \qquad (2.42b)$$

Both equations can be cast into the form

$$(n_0)_{n_+} = \langle n_0 \rangle + \frac{f_2^{0+}}{f_2^{++} + \langle n_+ \rangle} (n_+ - \langle n_+ \rangle) \ , \qquad (2.43a)$$

$$(n_+)_{n_0} = \langle n_+ \rangle + \frac{f_2^{0+}}{f_2^{00} + \langle n_0 \rangle} (n_0 - \langle n_0 \rangle) \ . \qquad (2.43b)$$

From this analysis we see that the linear $n_i - \langle n_i \rangle$ dependence near the mean arises naturally in any MPM in which $f_2^{0+} \neq 0$. Since both isospin and dynamical correlations can contribute to f_2^{0+}, we cannot make a general statement about this parameter from isospin considerations alone. The results of Secs. III and IV, in which explicit models for the dynamics are considered, will illustrate this point.

III. TWO-BODY, NEAREST-NEIGHBOR DYNAMICS

A. Motivation

To obtain more detailed results on multiplicity distributions in the MPM we must choose an explicit form for the dynamical function $h_N(q_1, \ldots, q_N)$ in (2.2). In selecting a specific model for h_N, we are guided by several recent analyses[9,10,19,20] which have established that the two-body, nearest-neighbor approximation,[21] in which

$$h_N(q_1^+, \ldots, q_N^+) = h(q_1^+, q_2^+)\, h(q_2^+, q_3^+) \ldots h(q_{N-1}^+, q_N^+) \qquad (3.1)$$

retains all the general features of multiperipheralism and simplifies calculations significantly. More specifically, these results suggest that, in many cases of interest, when h_N is considered as a function of q_i^+ only—that is, when all transverse momenta are integrated over—$h(q_1^+, q_2^+)$ can be parametrized as

$$\begin{aligned} h(q_1^+, q_2^+) &= 1 + a\,(q_2^+/q_1^+)^b, \text{ for } q_2^+ < q_1^+ \\ &= 1 + a\exp(-b\,|\,y_1 - y_2\,|) \ , \end{aligned} \qquad (3.2)$$

where $y_i = \ln(q_i^+/m)$ is the rapidity of the ith particle. In (3.2) different values of a and b are known[9] to correspond to particular types of multiperipheral models; thus, for example, the choice ($a = -1$, $b = 1$) produces results similar to the "ϕ^3" ladder model,[10] whereas ($a = 1$, $b = 1$) represents a Chew-Pignotti model with $\alpha_{\text{in}} = \frac{1}{4}$. Notice that the values of the parameters a and b are restricted. Since the differential cross sections must be everywhere positive, $a \geq -1$. Further, since the correlations must decrease with increasing rapidity separation, $b > 0$.

Given the explicit form of $h(q_1^+, q_2^+)$ in (3.2) it is straightforward to determine the asymptotic form behavior of the isospin-independent part of the total cross section:

$$\sigma_T(s, z) \equiv \sum z^n \int_m^{s/m} \frac{dq_1^+}{q_1^+} \int_m^{q_1^+} \frac{dq_2^+}{q_2^+} \cdots \int_m^{q_{n-1}^+} \frac{dq_n^+}{q_n^+} \times h(q_1^+, q_2^+) \ldots h(q_{n-1}^+, q_n^+) \ . \qquad (3.3)$$

As $s \to \infty$, σ_T becomes

$$\sigma_T(s, z) = \beta_+(z)\, e^{\alpha_+(z)Y} + \beta_-(z)\, e^{\alpha_-(z)Y} \ , \qquad (3.4)$$

where the Regge poles $\alpha_\pm(z)$ are determined by the eigenvalue equation[9,10,19,21]

$$1 = z\int_0^1 dw\, w^{\alpha(z)-1}(1 + a w^b) \ . \qquad (3.5)$$

Using (3.5) we find

$$\alpha_\pm(z) = \tfrac{1}{2}\{(1+a)z - b \pm [((1+a)z - b)^2 + 4bz]^{1/2}\} \qquad (3.6)$$

and

$$\beta_\pm(z) = \left[1 + a\left(\frac{\alpha_\pm}{\alpha_\pm + b}\right)^2\right]^{-1} \ . \qquad (3.7)$$

Combining (3.4) with (2.6) and (2.14), we see that in the presence of isospin the generating function for the multiplicity distribution in the approximation of two-body, nearest-neighbor interactions can be written as

$$G(x_+, x_-, x_0) = \sum_{J=0}^{2I} \sum_{\tau=\pm} P_J \beta_\tau(\lambda\mu_J)\, e^{\alpha_\tau(\lambda\mu_J)Y} . \qquad (3.8)$$

Here I is the isospin of the exchanged particle; μ_J is the jth eigenvalue—$\mu_0 > \mu_1 > \ldots > \mu_J$, at $x_i = 1$—of the matrix S introduced in Sec. II, P_J is the corresponding projection operator; and the sum over J runs over, in essence, the values of

t-channel isospin allowed by the given fundamental exchange. Recall that, since the P_J are matrices, $G(x_+, x_-, x_0)$ is a $(2I+1)\times(2I+1)$ matrix. For simplicity, we shall consider only certain of these matrix elements. Specifically, we shall study $G_{+-}(x_+, x_-, x_0) = G_{-+}(x_+, x_-, x_0)$ for the H model and $G_{00}(x_+, x_-, x_0)$ for the I model.[22] The analysis of other elements by our technique is straightforward.

B. Multiplicity Moments

From (3.8) we can calculate all multiplicity moments for arbitrary $a\,(\geqslant -1)$ and $b\,(>0)$. To illustrate the effect of dynamical correlations on these moments, however, it is sufficient to compare the two simple cases mentioned above—namely, $a=1$, $b=1$, and $a=-1$, $b=1$—with the results that follow when there are no dynamical correlations ($a=0$). In Tables I and II we present the leading terms in the first few moments in these three cases for the H and I models, respectively.

To interpret these results we recall the connection between the function $h(q_1^+, q_2^+)$ and the equivalent gas analogy[23] potential between two adjacent particles in rapidity space. Standard results of statistical mechanics yield the correspondence:

$$h(|y_1 - y_2|) = e^{-V(|y_1 - y_2|)}\ , \tag{3.9}$$

which implies that

$$V(|y_1 - y_2|) = -\ln[1 + a\exp(-b|y_1 - y_2|)]\,. \tag{3.10}$$

Clearly for a less than (greater than) zero, the potential is repulsive (attractive). The cases $a=-1$, and $a\to+\infty$, respectively, represent the limits of infinitely repulsive and attractive potentials.

Since an attractive (repulsive) potential would produce positive (negative) inclusive two-particle correlations and therefore—in the absence of isospin—a positive (negative) f_2, the most interesting entries in Tables I and II involve terms in which isospin and dynamical effects produce opposing trends. In these cases we observe from the Tables that—for the values of a and b represented there—the isospin structure shows a strong tendency to persist. For example, the correlations between (π^0, π^0)—described by f_2^{00}—in the I model, which are positive in the limit $a=0$, remain positive even when $a=-1$, despite the strongly repulsive nature of the dynamical correlations. Although only the case $b=1$ is presented in the Table, this result is easily seen to be independent of b. In the H model, on the other hand, f_2^{00}, which is zero in the limit $a=0$, becomes positive (negative) when the dynamical correlations are attractive (repulsive). Thus, although the dynamical correlation in these models produce significant quantitative changes in the f_2^{ij}, they produce qualitative changes only in cases in which isospin effects are small or absent. Notice, however, that among these cases are just those of greatest current interest. We see, for instance, that in the H model, dynamical correlations can make f_2^{0+} either positive or negative. This implies that, in the H model, even the simplest nontrivial MPM can accommodate either an increasing $\langle n_0\rangle_{n_-}$ or a decreasing $\langle n_0\rangle_{n_-}$, depending on the underlying isospin-independent

TABLE I. The leading terms of the first few moments of the multiplicity distributions for a multiperipheral model with $I=\frac{1}{2}$ exchange with initial state $(I_3)_a=\frac{1}{2}=-(I_3)_b$. The column labeled $a=0$ corresponds to no dynamical correlations. That labeled $a=-1$, $b=1$ indicates the results of negative dynamical correlations, such as found in a ϕ^3 ladder model. The column headed $a=1$, $b=1$ reflects the positive dynamical correlations found in a Chew-Pignotti-type multi-Regge model with $\alpha=\frac{1}{4}$. Recall that $Y=\ln(s/m^2)$.

	$a=0$	$a=-1,\ b=1$	$a=1,\ b=1$
$\langle n_1\rangle$	λY	$\dfrac{\lambda Y}{(1+12\lambda)^{1/2}}$	$\lambda Y\left[1+\dfrac{6\lambda}{(1+36\lambda^2)^{1/2}}\right]$
f_2^{++}	$-\frac{1}{2}\lambda Y$	$-\frac{1}{2}\lambda Y\left[\dfrac{1+16\lambda}{(1+12\lambda)^{3/2}}\right]$	$-\frac{1}{2}\lambda Y\left[\frac{1}{2}+\dfrac{\lambda+108\lambda^3}{(1+36\lambda^2)^{3/2}}\right]$
f_2^{+-}	$+\frac{1}{2}\lambda Y$	$+\frac{1}{2}\lambda Y\left[\dfrac{1+8\lambda}{(1+12\lambda)^{3/2}}\right]$	$\frac{1}{2}\lambda Y\left[\frac{1}{2}+\dfrac{5\lambda+108\lambda^3}{(1+36\lambda^2)^{3/2}}\right]$
f_2^{+0}	0	$\dfrac{-2\lambda^2 Y}{(1+12\lambda)^{3/2}}$	$\dfrac{2\lambda^2 Y}{(1+36\lambda^2)^{3/2}}$
f_2^{00}	0	$\dfrac{-2\lambda^2 Y}{(1+12\lambda)^{3/2}}$	$\dfrac{2\lambda^2 Y}{(1+36\lambda^2)^{3/2}}$

TABLE II. The leading terms of the first few moments of the multiplicity distributions for a multiperipheral model with $I=1$ exchange, with initial state $(I_3)_a=0=(I_3)_b$. The column labeled $a=0$ corresponds to no dynamical correlations. That labeled $a=-1$, $b=1$ indicates the results of negative dynamical correlations, such as found in a ϕ^3 ladder model. The column headed $a=1$, $b=1$ reflects the positive dynamical correlation found in a Chew-Pignotti-type multi-Regge model with $\alpha=\frac{1}{4}$. Recall that $Y=\ln(s/m^2)$.

	$a=0$	$a=-1,\ b=1$	$a=1,\ b=1$
$\langle n_i\rangle$	$(\frac{2}{3})\lambda Y$	$\dfrac{(\frac{2}{3})\lambda Y}{(1+8\lambda)^{1/2}}$	$\dfrac{(\frac{2}{3})\lambda Y}{3}\left[1+\dfrac{4\lambda}{(1+16\lambda^2)^{1/2}}\right]$
f_2^{++}	$-\frac{8}{27}\lambda Y$	$\dfrac{-8\lambda Y(1+11\lambda)}{27(1+8\lambda)^{3/2}}$	$-\dfrac{8\lambda Y}{27}\left[1+\dfrac{\lambda+64\lambda^3}{(1+16\lambda^2)^{3/2}}\right]$
f_2^{+-}	$\frac{10}{27}\lambda Y$	$\dfrac{\lambda Y(10+56\lambda)}{27(1+8\lambda)^{3/2}}$	$\dfrac{10\lambda Y}{27}\left[1+\dfrac{32(\lambda+10\lambda^3)}{(1+16\lambda^2)^{3/2}}\right]$
f_2^{+0}	$-\frac{2}{27}\lambda Y$	$-\dfrac{2\lambda Y(1+20\lambda)}{27(1+8\lambda)^{3/2}}$	$-\dfrac{2\lambda Y}{27}\left[1-\dfrac{8(\lambda-8\lambda^3)}{(1+16\lambda^2)^{3/2}}\right]$
f_2^{00}	$\frac{4}{27}\lambda Y$	$\dfrac{4\lambda Y(1+2\lambda)}{27(1+8\lambda)^{3/2}}$	$\dfrac{4\lambda Y}{27}\left[1+\dfrac{2(5\lambda+32\lambda^3)}{(1+16\lambda^2)^{3/2}}\right]$

dynamics. For the I model, f_2^{0+}, which is negative if there exist no dynamical correlations, remains negative even in the presence of moderately attractive exclusive correlations. However, the form of f_2^{0+} in this case suggests that a more attractive dynamics could make f_2^{0+} positive. In Sec. IV we shall see that this is indeed the case.

The simple structure of $\alpha(z)$ in (3.6) allows us to go beyond the results discussed for specific a and b and to draw some general conclusions about the relative importance of dynamical and isospin correlations for potentials of the form of (3.10). First, we see that as $\lambda\to 0$ (and hence as $z\to 0$), the isospin correlations must dominate. To demonstrate this, we observe that

$$f_2^{ij}=\lambda^2\left(\frac{d^2\alpha}{dz^2}\right)\bigg|_{z=\lambda\mu_0}\frac{\partial\mu_0}{\partial x_i}\frac{\partial\mu_0}{\partial x_j}+\lambda\left(\frac{d\alpha}{dz}\right)\bigg|_{z=\lambda\mu_0}\frac{\partial^2\mu_0}{\partial x_i\,\partial x_j}\ . \tag{3.11}$$

Notice that in the case of Poisson dynamics, $\alpha(z)$ is linear in z, and consequently the first term vanishes. Thus the second term in (3.11) represents the correlations induced by isospin constraints alone: that is, the residual correlations which remain in the absence of any dynamical correlations. Since $\alpha(z)$ and all its derivatives are finite as $z\to 0$, in this limit these isospin correlations are indeed dominant. Similarly, with α as given by (3.6), for $\lambda\to\infty$ and hence for $z\to\infty$—provided $a\neq -1$—the isospin structure again emerges. This is clear from

$$\alpha(z)\underset{z\to\infty}{\simeq}(1+a)z-\frac{ab}{1+a}+O\left(\frac{1}{z}\right) \tag{3.12}$$

and (3.11). Further, the isospin structure also dominates in the limits $a\to+\infty$ for fixed b and $b\to\infty$ for a fixed a.

From the explicit forms of $G(x_+, x_-, x_0)$ implied by (3.8), one can calculate directly the probability distribution $P(n_+, n_-, n_0)$ for any of the specific models. The complexity of these distributions, however, renders them somewhat obscure and not particularly useful as illustrative examples. Thus to extend our discussion to quantities like $P(n_+, n_-, n_0)$ and $\langle n_0\rangle_{n_-}$, we shall simplify the underlying dynamics further. In Sec. IV we describe the resulting models.

IV. δ-FUNCTION DYNAMICS

A. Multiplicity Moments

A further simplification in the dynamics results if we consider decreasing the exclusive correlation length—the parameter b^{-1} of Sec. III—to zero in a manner such that the integrated effects of the correlations do not vanish. Intuitively, we anticipate that the limiting form of $h(y_1-y_2)$ should then be

$$h(y_1-y_2)=1+c\,\delta(y_1-y_2)\ . \tag{4.1}$$

By (3.5) this form of h implies that the isospin-independent part of the total cross section in this model is dominated by a single Regge trajectory,[24]

$$\alpha(\lambda)=\frac{\lambda}{1-c\lambda}\ . \tag{4.2}$$

In addition, we find $\beta(\lambda)=1$, independent of c.

An intuitive picture of the δ-function model follows from considering its relation to the models discussed in Sec. III. If we take the limit $a, b\to\infty$ with $a/b\equiv c$ fixed—subject to $c\lambda<1$—in (3.6), we recover immediately the form of $\alpha(\lambda)$ in (4.2). Further, from the restriction $a\geq -1$, we see that this limit can be achieved only for $a>0$. Hence the δ-function model is physical only for $0\leq c<1/\lambda$. For this range of c, the model does indeed represent the limit of zero exclusive—and inclusive[25]—correlation lengths in the nearest-neighbor models.

As there is only one Regge trajectory, the generating functions for the multiplicity distributions in the H and I models are simply

$$G(x_+, x_-, x_0)=\sum_{J=0}^{2J+1}P_J\exp\left(\frac{\lambda\mu_J}{1-c\lambda\mu_J}Y\right). \tag{4.3}$$

Given the forms of P_J and μ_J from Sec. II, it is straightforward to calculate the first few moments of the multiplicity distributions in both the H and I models. In Table III, we summarize the results of these calculations. As in our previous discussion, we treat explicitly only those matrix ele-

TABLE III. The leading terms in the first few moments of the multiplicity distribution for the δ-function dynamics model. The first column lists the results of a model with $I=\frac{1}{2}$ exchange. The second illustrates the $I=1$ exchange case. For the H model, $0\leq c<1/(3\lambda)$. For the I model, $0\leq c<1/(2\lambda)$.

	H model	I model
$\langle n_i\rangle$	$\dfrac{\lambda Y}{(1-3\lambda c)^2}$	$\dfrac{(\frac{2}{3})\lambda Y}{(1-2\lambda c)^2}$
f_2^{++}	$\dfrac{\lambda Y(-1+7c\lambda)}{2(1-3c\lambda)^3}$	$\dfrac{-8\lambda Y(1-5c\lambda)}{27(1-2c\lambda)^3}$
f_2^{+-}	$\dfrac{\lambda Y(1+c\lambda)}{2(1-3c\lambda)^3}$	$\dfrac{2\lambda Y(5+2c\lambda)}{27(1-2c\lambda)^3}$
f_2^{+0}	$\dfrac{2c\lambda^2 Y}{(1-3c\lambda)^3}$	$\dfrac{-2\lambda Y(1-14c\lambda)}{27(1-2c\lambda)^3}$
f_2^{00}	$\dfrac{2c\lambda^2 Y}{(1-3c\lambda)^3}$	$\dfrac{4\lambda Y(1+4c\lambda)}{27(1-2c\lambda)^3}$

ments of G corresponding to chargeless initial states.

Since c must be positive for the δ-function model to be physically well defined the dynamical correlations are perforce positive. Hence the model can produce dramatic qualitative changes only in those f_2^{ij} which are negative in the limit of Poisson dynamics. In these cases, however, the effects of dynamical correlations are even more striking than in the models treated in Sec. III. We see, for example, that for both the H- and I-exchange mechanisms, the f_2^{++}—which in the previous models were always negative—can be made positive in the δ-function model by a suitable choice of c. Similarly, f_2^{0+} in the I model can be made positive here. Thus the results presented in Table III, coupled with those in Tables I and II, provide a vivid illustration of the importance of considering *both* dynamical and isospin effects in determining multiplicity moments in the MPM.

B. Multiplicity Distribution and the "Associated Means"

The full multiplicity distributions in both the H and I models with δ-function dynamics follow directly from an expansion of (4.3). Since we are interested chiefly in illustrating the interplay of dynamical and isospin effects, rather than in the detailed predictions of any specific model, we shall present only the results for the simpler H model. Further, we shall assume $c \ll 1$ and work to first order in c. Finally, we recall that we are considering only the matrix elements of $G(x_+, x_-, x_0)$ corresponding to a chargeless initial state. As discussed in Sec. II, this implies that G depends on x_+ and x_- only through the combination $x_c = x_+ \cdot x_-$.

With these simplifications we may write

$$G_{+-}(x_c, x_0) = (P_0)_{+-}\, e^{\alpha(\lambda\mu_0)Y} + (P_1)_{+-}\, e^{\alpha(\lambda\mu_1)Y} \tag{4.4}$$

$$\simeq e^{(x_0+2\sqrt{x_c})Y}\left[1 + c(x_0+2\sqrt{x_c})^2\lambda^2 Y\right] + e^{(x_0-2\sqrt{x_c})Y}\left[1 + c(x_0-2\sqrt{x_c})^2\lambda^2 Y\right] , \tag{4.5}$$

where (4.6) is valid to $O(c)$. Expanding (4.5) in powers of x_0 and x_c and recalling the normalization condition indicated by (2.7) and (2.9), we find that the probability for producing $n_0 \pi^0$'s and $n_+ \pi^+$'s—equivalently, n_+ charged pion pairs—is

$$P(n_+, n_0) = \frac{2}{N}\left[\frac{\lambda^{n_0+2n_+} 2^{2n_+} Y^{n_0+2n_+}}{n_0!\,(2n_+)!}\right] \times \left\{1 + \frac{c}{Y}\left[(n_0+2n_+)(n_0+2n_+-1)\right]\right\} , \quad n_- = n_+ , \tag{4.6}$$

where the normalization factor is

$$N = e^{3\lambda Y}(1 + 9c\lambda^2 Y) + e^{-\lambda Y}(1 + c\lambda^2 Y) . \tag{4.7}$$

In the limit $c = 0$ this reproduces the previously known multiplicity distribution of the H model with Poisson dynamics.[5]

The associated mean $\langle n_0\rangle_{n_c}$ follows directly from (4.6).[26] Using

$$\langle n_0\rangle_{n_c} P(n_+) = \sum_{n_0} n_0 P(n_+, n_0) , \tag{4.8}$$

where

$$P(n_+) = \sum_{n_0} P(n_+, n_0) , \tag{4.9}$$

we obtain

$$\langle n_0\rangle_{n_+} = \langle n_0\rangle + 4c\lambda(n_c - \langle n_+\rangle) + O(c^2) . \tag{4.10}$$

In this case, the associated mean shows a simple linear behavior in n_+[27]; since we require $c > 0$ for the consistency of the δ-function model, the trend of $\langle n_0\rangle_{n_c}$ is positive. This increasing trend is, of course, anticipated from the behavior of f_2^{0+} in this model. Conversely, for models with a negative f_2^{0+}, we would expect $\langle n_0\rangle_{n_c}$ to be, in general, a decreasing function of n_c.

As we have indicated, (4.10) is valid only to $O(c)$; higher-order terms will not necessarily possess the same simple functional form. Further, in the more complicated I model, the simple structure displayed by (4.10) is not present even to lowest order in c. Nonetheless (4.10) represents a most explicit indication of the potential significance of dynamical effects in multiplicity distributions.

V. DISCUSSION AND CONCLUSION

In simple multiperipheral models the differential cross section can be written as a product of two terms, one containing all the effects of the isospin structure of the multiple exchanges, and the second describing the underlying isospin-independent dynamics. Using this factorization we have been able to study in a direct manner the separate influences of isospin and of dynamics on the multiplicity distributions.

There are, roughly speaking, three broad categories into which the properties of the multiplicity distributions can be placed.

In the first category are those results, mostly obvious, which follow from charge conservation and are therefore totally model-independent. Examples of such properties are $\langle n_+\rangle = \langle n_-\rangle + Q_{\text{initial}}$ and the positivity of $f_2^{+-} \equiv \langle n^+ n^-\rangle - \langle n^+\rangle\langle n^-\rangle$.

In the second category are features which depend on the specific isospin-exchange structure but which are insensitive to the dynamics. For any dynamics, for example, there exist well-defined relations among total average multiplicity $\langle n\rangle$ and dispersion $\Delta \equiv D^2$, and the various multiplicities and dispersions for specific charge states. Further, the explicit numerical coefficients in these relations depend only on the isospin properties of the exchanged particles. Similarly, the coefficients in the forms of the multiplicity distributions around the mean are related to the isospin structure of the theory.

Finally, in the third category are those results which depend crucially on both isospin and dynamics. We have studied these features for two different isospin-exchange mechanisms in several dynamical models based on nearest-neighbor exclusive correlations. For dynamical correlations of weak to moderate strength, many of the multiplicity moments are qualitatively unchanged from the results predicted from considering solely isospin-induced correlations. However, the moments which the dynamical correlations do alter qualitatively—e.g., f_2^{0+}—seem to be those of greatest current interest. Further, for sufficiently strong, short-range exclusive correlations—exemplified in our calculations by the δ-function model of Sec. IV—virtually the entire multiplicity moment structure can be radically different from that predicted by isospin effects with Poisson dynamics.

One direct consequence of this last result should be emphasized. Recently several authors[4-8] have compared the predictions of multiperipheral and diffraction excitation models (DEM) for multiplicity moments and distributions in the presence of isospin. The trend of their conclusions has been that diffractive models can reconcile many different isospin mechanisms with the observed behavior of, say, $\langle n_0\rangle_{n_-}$, whereas in the MPM only a much more restricted class of isospin mechanisms is consistent with observations. In each of these discussions, however, only the Poisson dynamics limit of the MPM has been considered. For Poisson dynamics, the observed increase of $\langle n_0\rangle$ as a function of n_-, would seem to eliminate both the H and I models discussed in the text. But as our results explicitly demonstrate, if one includes nontrivial dynamics of an appropriate nature, both the H and I models can accommodate an increasing $\langle n_0\rangle_{n_-}$. That one should consider in this comparative context multiperipheral models with non-Poisson dynamics is clear for at least two reasons:

(1) Any realistic MPM contains significant dynamical correlations; and (2) Since the form of the partial cross sections typically assumed for the diffractive models—that is, $\sigma_n \sim 1/n^2$—implies the existence of very strong dynamical correlations, for fairness any comparison between MPM and DEM predictions should also admit dynamical correlations in the MPM.

Thus we believe it is premature to exclude any of the simple isospin-exchange mechanisms in the MPM on the basis of current experimental results.[28] Only after more detailed observations, perhaps including both accurate determinations of the first few multiplicity moments for all charge states and inclusive correlation length studies, can one hope to disentangle the effects of isospin and dynamical correlations and then to discriminate meaningfully among specific models for the multiplicity distributions.

ACKNOWLEDGMENT

We thank Dr. Carl Kaysen and the faculty at the School of Natural Sciences, Institute for Advanced Study, for their hospitality.

APPENDIX

In this appendix, we study the general structure of the differential cross sections for the model introduced in Sec. II. The model describes the n-pion multiperipheral production processes with isospin (Figs. 1, 2)

$$a+b\rightarrow a'+b'+n \text{ pions}\,, \tag{A1}$$

and is defined by the three explicit assumptions given in Sec. II.

From assumption (1), we conclude that the scattering amplitude for process (A1) is of the form

$$M_{\alpha\beta}(1,2,\ldots,n)=(T^{i_1}\cdots T^{i_n})_{\alpha\beta}\,a_n(1,2,\ldots,n), \tag{A2}$$

where $(T^{i_n})_{\alpha_n,\alpha_{n+1}}$ $(i_n=\pm,0)$ denotes the isotopic-spin matrix and α_n is the I_3 eigenvalue of the nth-exchanged particle. Note that the T's in $(T^{i_1}\cdots T^{i_n})_{\alpha\beta}$ are multiplied together as matrices. The remaining function $a_n(1,2,\ldots,n)$ depends on the incident c.m. energy squared s and the momenta q_i of the final particles, but is independent of their isospins.

Keeping only the diagonal sum as in Fig. 2, we obtain the differential cross section

$$(d\sigma^{i_1\cdots i_n})_{\alpha\beta}\propto\frac{\lambda^n}{s}\,|T^{i_1}\cdots T^{i_n}|^2\,|a_n(1,2,\ldots,n)|^2 \times d^3\Phi_1\cdots d^3\Phi_n\,, \tag{A3}$$

where λ is the coupling strength, $1/s$ is the flux factor, and $d^3\Phi\equiv d^3k/k^0=dk^+d^2k/k^+$ is the invariant-phase-space element. Since we are interested at present only in the multiplicity distributions, we shall first carry out the transverse-momentum integrations. Then Eq. (A3) reduces to

$$\frac{1}{\sigma_0}(d\sigma^{i_1\cdots i_n})_{\alpha\beta}=\lambda^n\,|(T^{i_1}\cdots T^{i_n})_{\alpha\beta}|^2 \times h_n(1,2,\ldots,n)\,d\Phi_1\cdots d\Phi_n\,, \tag{A4}$$

where σ_0 is a normalization factor, h_n is obtained from $1/\sigma_0 s\,|a_n|^2$ by integrating out the transverse-momentum variables, and $d\Phi=dq^+/q^+$ is the longitudinal phase space. Note that h_n describes the differential cross section in a simple multiperipheral model (MPM) without isospin, as discussed in Ref. 10. It is important to notice that as far as the multiplicity and the longitudinal-momentum distributions are concerned, we can scale h_n and hence $d^n\sigma/\sigma_0$ by an arbitrary q-independent, but s-dependent factor σ_0. It is demonstrated in Ref. 10 that with a proper normalization factor σ_0, h_n in simple MPM without isospin can be chosen to obey a factorization property: namely, for $(q_1^+,\ldots,q_n^+)\gg(q_{m+1}^+,\ldots,q_n^+)$,

$$h_n(1,2,\ldots,n)\to h_m(1,2,\ldots,m) \times h_{n-m}(m+1,\ldots,n)\,. \tag{A5}$$

This factorization property has many interesting consequences. In particular, it leads naturally to our assumption 3, the Regge behavior of the asymptotic cross sections. In the explicit examples discussed in Secs. III and IV in the text, we adopt this simple mechanism of generating Regge behavior and restrict our attention to the class of h_n with this factorization property.

The expression $|(T^{i_1}\cdots T^{i_n})_{\alpha\beta}|^2$ can be simplified by noticing that, in the usual (I,I_3) representation, $T_{\alpha\beta}$ has at most one nonvanishing matrix element in each row and in each column. This property has the interesting consequence that

$$|(T^{i_1}\cdots T^{i_n})_{\alpha\beta}|^2\equiv\left|\sum_{\{\alpha_i\}}T^{i_1}_{\alpha\alpha_2}T^{i_2}_{\alpha_2\alpha_3}\cdots T^{i_n}_{\alpha_n\beta}\right|^2 =\sum_{\{\alpha_i\}}|T^{i_1}_{\alpha\alpha_2}|^2|T^{i_2}_{\alpha_2\alpha_3}|^2\cdots|T^{i_n}_{\alpha_n\beta}|^2. \tag{A6}$$

The final equality can be proved by induction. In terms of a new set of matrices

$$S^i_{\alpha\beta}\equiv|T^i_{\alpha\beta}|^2,\quad i=\pm1,0, \tag{A7}$$

we obtain

$$|(T^{i_1}\cdots T^{i_n})_{\alpha\beta}|^2=(S^{i_1}\cdots S^{i_n})_{\alpha\beta}\,, \tag{A8}$$

and consequently

$$\frac{1}{\sigma_0}d\sigma^{i_1\cdots i_n}=\lambda^n(S^{i_1}\cdots S^{i_n})_{\alpha\beta}\,h_n(1,2,\ldots,n) \times d\Phi_1\cdots d\Phi_n\,. \tag{A9}$$

Equation (A9) is the starting point of our main text.

Although (A8) follows from a trivial property of the isospin matrices, it greatly simplifies our calculation in the text. The readers are invited to check this relation explicitly in the simple cases.

For $I=\frac{1}{2}$ (H model), we choose

$$T^+=\begin{pmatrix}0&\sqrt{2}\\0&0\end{pmatrix},\quad T^-=\begin{pmatrix}0&0\\\sqrt{2}&0\end{pmatrix},\quad T^0=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \tag{A10}$$

and hence

$$S^+=\begin{pmatrix}0&2\\0&0\end{pmatrix},\quad S^-=\begin{pmatrix}0&0\\2&0\end{pmatrix},\quad S^0=\begin{pmatrix}1&0\\0&1\end{pmatrix}. \tag{A11}$$

For $I=1$ (I model), we have

$$T^+=\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix},\quad T^-=\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix},\quad T^0=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}, \tag{A12}$$

and hence

$$S^+=\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix},\quad S^-=\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix},\quad S^0=\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}. \tag{A13}$$

The explicit forms of the S's are essential in our calculations.

In the text, we refer to the eigenvalues and the projection operators of the following combination:

$$S(x)\equiv(x_+S^++x_-S^-+x_0S^0) =\sum_J\mu_JP_J\,, \tag{A14}$$

where μ_J are eigenvalues given in the text, P_J are projection operators satisfying

$$P_JP_{J'}=\delta_{JJ'}P_J\,, \tag{A15}$$

and J is the total isospin in t channel. The explicit forms of the projection operators are, in H model,

$$P_0=\begin{pmatrix}\frac{1}{2}&\frac{1}{2}(x_+/x_-)^{1/2}\\\frac{1}{2}(x_-/x_+)^{1/2}&\frac{1}{2}\end{pmatrix},\quad P_1=\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}(x_+/x_-)^{1/2}\\-\frac{1}{2}(x_-/x_+)^{1/2}&\frac{1}{2}\end{pmatrix}, \tag{A16}$$

and, in I model,

$$P_0 = \frac{1}{\mu_2(\mu_2-\mu_0)} \times \begin{pmatrix} x_+x_- & -x_+\mu_2 & x_+^2 \\ -x_-\mu_2 & \mu_2^2 & -x_+\mu_2 \\ x_-^2 & -x_-\mu_2 & x_+x_- \end{pmatrix}, \tag{A17a}$$

$$P_1 = \frac{1}{2x_+x_-}\begin{pmatrix} x_+x_- & 0 & -x_+^2 \\ 0 & 0 & 0 \\ -x_-^2 & 0 & x_+x_- \end{pmatrix}, \tag{A17b}$$

$$P_2 = \frac{1}{\mu_0(\mu_0-\mu_2)}\begin{pmatrix} x_+x_- & -x_+\mu_0 & x_+^2 \\ -x_-\mu_0 & \mu_0^2 & -x_+\mu_0 \\ x_-^2 & -x_-\mu_0 & x_+x_- \end{pmatrix}. \tag{A17c}$$

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*Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Grant 70-1866C.

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[1]O. Balea *et al.* [Nucl. Phys. B52, 414 (1973)] report data on neutral pions from 40-GeV/c $\pi^- p$ and $\pi^- n$ reactions.

[2]G. Flügge *et al.* [*Proceedings of the XVI International Conference on High Energy Physics, Chicago-Batavia, Ill., 1972,* edited by J. D. Jackson and A. Roberts (NAL, Batavia, Ill., 1973), Vol. 1] report data on neutral pions at ISR energies.

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[4]L. Caneschi and A. Schwimmer, Phys. Lett. 33B, 577 (1970).

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[8]K. Fialkowski, Rutherford Report No. RPP/T/34, 1972 (unpublished).

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[10]D. K. Campbell and S.-J. Chang, Phys. Rev. D 4, 1151 (1971); 4, 3658 (1971).

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[12]B. R. Webber [Nucl. Phys. B43, 541 (1972)] has studied the constraints imposed by charge conservation (and other additive quantum numbers) on the multiplicity generating function.

[13]In statistical mechanics this result is known as the theorem of Yang and Lee. See Phys. Rev. 87, 404 (1952) and K. Huang, *Statistical Mechanics* (Wiley, New York, 1963), Chap. 15.

[14]Notice that even if the initial state has charge $+Q$ the charge conservation condition $n_+ = n_- + Q$ still leads to (2.32).

[15]See, for example, K. Huang, *Statistical Mechanics* (Wiley, New York, 1963), Chap. 10.

[16]Using a similar approach, Weisberger has recently studied the prong distribution in high-energy collisions. See W. I. Weisberger, Phys. Rev. D 8, 1387 (1973).

[17]In the case of an initial state of charge $+Q$, one can show that for

$$G_{+Q}(x,y) \equiv x_+^{+Q} G(x_+x_-, x_0, y),$$

the identical analysis can be applied. The resulting multiplicity distribution is

$$P(n_+,n_-,n_0) = \delta_{n_+,n_-+Q}\left(\frac{1}{2\pi i}\right)^2 \times \oint \frac{dx_0}{x_0^{n_0+1}} \oint \frac{dx_c}{x_c^{n_-+1}} \left[\frac{G(x_0,x_c,Y)}{G(1,Y)}\right].$$

[18]The experimental data on $\langle n_0\rangle|_{n_-}$ are—because of the difficulties of detecting neutral particles—subject to large uncertainties. Thus the existence of any definite trend for $\langle n_0\rangle|_{n_-}$ has not been unambiguously established. At very high energy, however, it appears that $\langle n_0\rangle|_{n_-}$ is an increasing function of n_-. The existing high-energy data on this associated mean are contained in Refs. 1–3. For lower-energy data, see the following: (1) J. H. Campbell *et al.*, in *Proceedings of the XVI International Conference on High Energy Physics, Chicago-Batavia, Ill., 1972,* edited by J. D. Jackson and A. Roberts (NAL, Batavia, Ill., 1973), Vol. 1. These data are from pp interactions at 12.3 GeV/c. (2) H. Bøggild *et al.*, Nucl. Phys. B27, 285 (1971). These data are from pp interactions at 19 GeV/c. (3) J. W. Elbert *et al.*, Nucl. Phys. B19, 85 (1970). These data are from $\pi^- p$ interactions at 25 GeV/c. For phenomenological interpretations of these data, see in particular Refs. 7 and 8.

[19]T. D. Lee, Phys. Rev. D 6, 3617 (1972).

[20]R. C. Arnold, Argonne Report No. ANL/HEP 7241 (unpublished).

[21]For a review of the nearest-neighbor approximations in statistical mechanics, see R. P. Feynman, *Statistical Mechanics* (Benjamin, Reading, Mass., 1972), Chap. 4.

[22]Notice that if we considered other matrix elements of G, the average multiplicities of the various charge states would differ only by a finite amount. Hence the results in Tables I and II, which are to leading order in Y, are valid to this order for all matrix elements of G.

[23]R. P. Feynman, Phys. Rev. Lett. 23, 1415 (1969); in *High Energy Collisions,* edited by C. N. Yang *et al.* (Gordon and Breach, New York, 1969); K. Wilson, Cornell University Report No. CLNS-131, 1970 (unpublished).

[24]Notice that the trajectory function becomes ill defined for $c\lambda = 1$ and unphysical beyond this point. Since our primary interest in this model will be for small c, this peculiarity need not concern us.

[25]The trajectory function $\alpha_+(\lambda)$ approaches that given by (4.2) in this limit. That labeled $\alpha_-(\lambda)$ approaches $-\infty$ and can be ignored. Notice that this limiting procedure aids us in interpreting the model. Thus, for example, the two-particle inclusive correlations in the case of no isospin are

$$\tau_2(y_1, y_2) = \frac{2c\lambda^2}{(1-c\lambda)^3}\, dy_1 dy_2 \delta(y_1 - y_2).$$

This structure is not unexpected, since as remarked in the text, it represents a "short-range order" model in the limit that the inclusive correlation length $\xi = 0$.

[26]One can also use the equivalent technique of expanding $[x_0 \partial \ln G/\partial x_0|_{x_0=1}]$ directly in powers of x_+ to obtain $\langle n_0 \rangle_{n_+}$.

[27]The simple linear structure in the associated mean is, of course, present around the mean. This is shown in (2.41–43). However, (4.11) is valid, to $O(c)$ for *all* n_+.

[28]Since short-range order requires $f_2^{--} \propto \ln s$, in view of the observed rapid increase of f_2^{--}—perhaps as $(\ln s)^2$ or $\sqrt{s}$—the applicability of any simple MPM for describing the total multiplicity distribution is questionable. It remains possible that the MPM can describe a substantial fraction of high-energy production processes; the "two-component" theories, in which diffractive mechanisms control the low-multiplicity events, whereas a multiperipheral mechanism describes high-multiplicity events, provide specific realizations of this possibility. In any case, for purposes of phenomenology, the restriction that our comments apply only to the MPM must be kept in mind. For discussion of the "two-component" theories applied to the phenomenology of multiplicity distributions, see K. Fialkowski, Ref. 8.

PHYSICAL REVIEW D VOLUME 8, NUMBER 9 1 NOVEMBER 1973

Deformed Quark Currents and Anomalous Ward Identities. $\eta \to 3\pi$, $\pi^0 \to 2\gamma$, and e^+e^- Annihilation

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It is argued that the abstraction of the leading singular behavior, from the free-quark-model c-number disconnected matrix elements to the physical world, may be an unreliable assumption. A method is then devised to extract information from the vacuum expectation values of currents with a quarklike algebraic structure, without feeding the small-distance singularity structure as an input to the theory. Deformed quark currents are used, i.e., the physical currents are allowed to be linear combinations of differential forms on a basis set of U(12) local operators. The scheme is shown to provide an unambiguous framework in which to make the time-ordered product of vector currents covariant and to compute Schwinger terms, seagull terms, and anomalous Ward identities as explicit functions of the basic operator set. The results are applied to a derivation of low-energy theorems. In particular, a simple solution is found to the $\pi^0 \to 2\gamma$ and $\eta \to 3\pi$ current-algebra puzzles. With an additional postulate of algebraic uniqueness of the spectral functions, a sum rule is derived which relates the $\pi^0 \to 2\gamma$ decay width and the $e^+e^- \to$ hadrons cross sections. Its predictions concerning scale breaking in $e^+e^- \to$ hadrons are discussed.

I. INTRODUCTION

The formal structure of the connected light-cone commutators abstracted from the quark model has been extremely successful in explaining the scaling properties of the deep-inelastic scattering.

Assuming that the disconnected parts of the current commutators are also dictated by the quark model, the total cross section for e^+e^- annihilation into hadrons is predicted to scale as c/s for large s.[1] The asymptotic ratio $\sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ is then given by the sum of the squared quark charges.[2] Comparison with the triangle-graph[3] calculation of the $\pi^0 \to 2\gamma$ decay rate suggests a value of 2 for the ratio and a three-triplet model of current quarks.[4]

The experimental situation concerning these predictions is still inconclusive. The large e^+e^--annihilation cross sections obtained at Frascati are consistent with a pointlike structure, but then they might as well be explained by the opening of new quasi-two-body channels, or by the existence of the vector-meson daughter trajectories.

A question I want to raise here is whether we should expect properties abstracted from the disconnected parts of the quark-model commutators to be as reliable as those abstracted from the connected light-cone quark algebra. My guess is that we should not because the singularity structure of the disconnected parts has a greater chance of being interaction-dependent. The following ar-