(1962); L. Katz, G. Ricca, T. E. Drake, and H. S. Kaplan, Phys. Lett. 28B, 114 (1968).

[17]J. Goldemberg and C. Schaerf, Phys. Lett. 20, 193 (1966).

[18]See, for example, C. W. Kim and M. Ram, Phys. Rev. 162, 1584 (1967).

[19]The state $|\alpha\rangle$ need not correspond to a real particle. It is possible to approximate the $J^P=1^+$ contribution to Eq. (41) by a pole of appropriate mass. See Ref. 18.

[20]We note that in the impulse approximation only the nucleon form factors are used. For the nucleons

$$\frac{F_M(q^2,n\leftrightarrow p)}{F_M(0;n\leftrightarrow p)}\approx\frac{F_A(q^2,n\leftrightarrow p)}{F_A(0;n\leftrightarrow p)}$$

$$\approx(1-q^2/M_A{}^2)^{-2},$$

with $M_A{}^2\cong 1.1$ BeV^2. If, assuming a dipole fit, we write $[F_V(q^2;n\leftrightarrow p)+F_M(q^2;n\leftrightarrow p)]/[F_V(0;n\leftrightarrow p)+F_M(0;n\leftrightarrow p)]\approx(1-q^2/M^2)^{-2}$, we find that $M^2\cong 0.71$ BeV^2, so that

$$\frac{F_M(q^2;n\leftrightarrow p)}{F_M(0;n\leftrightarrow p)}\approx\frac{[F_V(q^2;n\leftrightarrow p)+F_M(q^2;n\leftrightarrow p)]}{[F_V(0,n\leftrightarrow p)+F_M(0;n\leftrightarrow p)]}.$$

Thus Eq. (58) may be a reasonably good assumption.

[21]J. Frazier and C. W. Kim, Phys. Rev. 177, 2560 (1969).

[22]Y. Nambu, Phys. Rev. Lett. 4, 380 (1960).

[23]We also note that the results for Γ_d and Γ_q depend on M, the mass used in F_A [see Eq. (39)]. Our value of $M=224\pm25$ MeV is in reasonable agreement with an effective value of $M=293$ MeV obtained from an impulse-approximation calculation, i.e.,

$$1/(1+m_\mu{}^2/M^2)^2$$
$$=\frac{\left\langle\psi_f\left|\sum_{i=1}^{2}F_A(-m_\mu{}^2,n\leftrightarrow p)e^{i\vec{q}\cdot\vec{r}^{(i)}}\tau^{-(i)}\right|\psi_i\right\rangle\Big|_{(\vec{q}^2=-m_\mu{}^2)}}{\left\langle\psi_f\left|\sum_{i=2}^{2}F_A(0,n\leftrightarrow p)e^{i\vec{q}\cdot\vec{r}^{(i)}}\tau^{-(i)}\right|\psi_i\right\rangle\Big|_{\vec{q}^2=0}}$$

[24]I-T. Wang et al. [Phys. Rev. 139, B1528 (1965)] give result (74a); A. Placci, E. Zavattini, A. Bertin, and A. Vitale [Phys. Rev. Lett. 25, 475 (1970)] give result (74b).

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Prediction of the Width and Slope of the Decay $X^0\to\eta\pi\pi$ by Finite Dispersion Relations*

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A model of the $X^0\to\eta\pi\pi$ decay amplitude is constructed using finite dispersion relations. The predicted decay width and energy dependence of the Dalitz plot are consistent with the latest data.

I. INTRODUCTION

Our understanding of scattering amplitudes has been greatly extended over the past few years through the application of Cauchy's theorem over (either effectively or explicitly) finite contours. In particular, finite-energy sum rules (FESR) have led to interesting relations between the high-energy (Regge) and low-energy (resonance) forms of scattering amplitudes.[1,2] In addition, the utility of finite dispersion relations (FDR), as a means of exploiting a knowledge of the Regge and resonance parts of an amplitude to determine its low-energy behavior, has come to be recognized.[3-6]

One of the most promising applications of FDR is to three-body decays, where the decay amplitude is related by crossing to the corresponding two-body scattering amplitude. This approach was used by Aviv and Nussinov[3] to describe the decay $\omega\to 2\pi\gamma$, with encouraging results. Later applications of FDR to $\eta\to\pi^0\gamma\gamma$ (Ref. 4) and $\eta\to\pi\pi\gamma$ (Ref. 5) have yielded results in good agreement with experiment.

In view of the above successes, we were led to apply FDR to the decay $X^0(957)\to\eta\pi\pi$, where the X^0 is assumed to have $J^P=0^-$. Whereas previous attempts[7-15] to describe $X^0\to\eta\pi\pi$ could predict only the width *or* the slope of the decay distribution, we have attempted to predict *both* the width *and* the slope. Our results are consistent with the latest data.

In Sec. II we give the details of our model for the $X^0\to\eta\pi\pi$ decay amplitude. Our results are presented in Sec. III, where they are compared with the experimental data. Section IV contains a discussion of our predictions in which comparison is made with other theoretical work on $X^0\to\eta\pi\pi$.

II. DETERMINATION OF THE AMPLITUDE

We begin by considering the two-body scattering process

$$X(p)+\pi(-q_1)\to\eta(k)+\pi(q_2)\tag{2.1}$$

(see Fig. 1). The respective momenta of the parti-

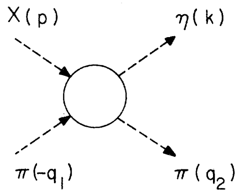

FIG. 1. The scattering process $\pi + X \to \pi + \eta$.

cles are indicated in parentheses and satisfy $p = k + q_1 + q_2$. (The pions can, of course, be charged or neutral.) The amplitude for this process $A(\nu, t)$ is even under crossing, $\nu \leftrightarrow -\nu$: $A(\nu, t) = A(-\nu, t)$. Here

$$\nu = \tfrac{1}{2}(s - u),$$
$$s = (p - q_1)^2,$$
$$t = (q_1 + q_2)^2,$$

and

$$u = (p - q_2)^2 .$$

$A(\nu, t)$ is assumed to satisfy the following FDR:

$$\begin{aligned} A(\nu, t) &= \frac{1}{2\pi i}\oint d\nu' \frac{A(\nu', t)}{\nu' - \nu} \\ &= \frac{2}{\pi}\int_0^N d\nu' \nu' \frac{\operatorname{Im} A(\nu', t)}{\nu'^2 - \nu^2} + \frac{1}{2\pi i} \\ &\qquad \times \int_{C_N} d\nu' \frac{A(\nu', t)}{\nu' - \nu} \end{aligned} \tag{2.2}$$

along the contour shown in Fig. 2. The first term arises from the portion of the contour along the usual cuts on the positive and negative Reν axis, while the last integral is the contribution from semicircles with radii $|\nu'| = N$ in the upper and lower half-planes, denoted collectively by C_N. N is given by

$$N = s_{\max} - \tfrac{1}{2}(m_X^2 + m_\eta^2 + 2m_\pi^2) + \tfrac{1}{2}t . \tag{2.3}$$

Our choice of $s_{\max}$ will be given below. We now make the assumption that $A(\nu, t)$, as determined from the FDR, correctly describes the decay

$$X(p) \to \eta(k) + \pi(q_1) + \pi(q_2) \tag{2.4}$$

throughout the allowed phase space.[16]

The determination of $A(\nu, t)$ depends on the evaluation of the two terms in Eq. (1). It is reasonable to expect that the first term will be dominated by $\eta\pi$ resonances. On the other hand, we have the perhaps unreasonable hope that the second term may be evaluated by making use of the Regge asymptotic form of $A(\nu, t)$ along C_N, even when $s_{\max}$ in Eq. (2.3) lies in the intermediate-energy region. This approximation has been vindicated in past calculations and can be justified theoretically on the basis of duality.[2] We denote the two terms in Eq. (2.2) by $A_{\text{res}}(\nu, t)$ and $A_{\text{Reg}}(\nu, t)$, respectively, and so we have

$$A(\nu, t) = A_{\text{res}}(\nu, t) + A_{\text{Reg}}(\nu, t). \tag{2.5}$$

We first discuss the determination of A_{res}.

As the data on $\eta\pi$ resonances are rather meager at the present time, we will assume, with Young and Lassila,[5] that the integral along the cut is saturated by the contribution from the A_2 meson, provided that $s_{\max}$ is not too large. We take[5] $s_{\max}$ to lie midway between m_A^2 and $\frac{7}{3}m_A^2$, the position of the Regge recurrence of the A_2 with $J^P = 4^+$. Hence,

$$s_{\max} = \tfrac{5}{3}m_A^2 = 2.86 \text{ GeV}^2 . \tag{2.6}$$

The couplings of the A_2 to $\eta\pi$ and $X^0\pi$ are defined as

$$\mathcal{L}_{A_2\eta\pi} = g_{A_2\eta\pi}\vec{A}_2^{\mu\nu}\cdot \eta\overleftrightarrow{\partial}_\mu\overleftrightarrow{\partial}_\nu\vec{\pi} \tag{2.7}$$

and

$$\mathcal{L}_{A_2X^0\pi} = g_{A_2X\pi}\vec{A}_2^{\mu\nu}\cdot X^0\overleftrightarrow{\partial}_\mu\overleftrightarrow{\partial}_\nu\vec{\pi} . \tag{2.8}$$

A_{res} is found to be

$$A_{\text{res}}(\nu, t) = g_{A_2X\pi}g_{A_2\eta\pi}\frac{4t^2 + Bt + C}{\nu - \frac{1}{2}t + D} + (\nu \to -\nu), \tag{2.9}$$

where

$$B = 4(m_A^2 - \sigma) + 4(m_X^2 - m_\pi^2)(m_\eta^2 - m_\pi^2)/m_A^2, \tag{2.10}$$

$$\begin{aligned} C = {} & (m_A^2 - \sigma)^2 \\ & + 2(m_A^2 - \sigma)(m_X^2 - m_\pi^2)(m_\eta^2 - m_\pi^2)/m_A^2 \\ & + \tfrac{2}{3}(m_X^2 - m_\pi^2)^2(m_\eta^2 - m_\pi^2)^2/m_A^4 \\ & - \tfrac{1}{3}(m_A^2 - 2m_X^2 - 2m_\pi^2)(m_A^2 - 2m_\eta^2 - 2m_\pi^2) \\ & + [(2m_X^2 + 2m_\pi^2 - m_A^2)(m_\eta^2 - m_\pi^2)^2 \\ & \quad + (2m_\eta^2 + 2m_\pi^2 - m_A^2)(m_X^2 - m_\pi^2)^2]/3m_A^2 , \end{aligned} \tag{2.11}$$

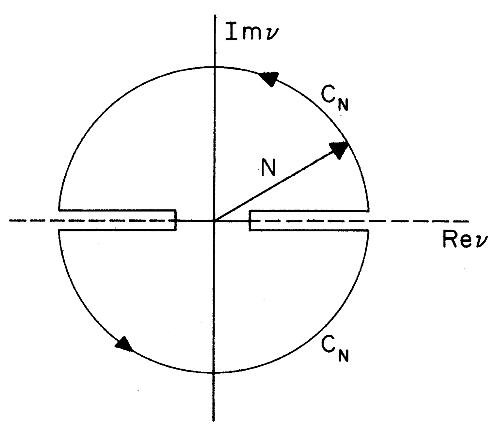

FIG. 2. ν-plane contour over which the FDR in Eq. (2.2) and the FESR in Eq. (2.21) are taken.

$$D=\tfrac{1}{2}\sigma-m_A^{\,2}, \tag{2.12}$$

and

$$\sigma=m_X^{\,2}+m_\eta^{\,2}+2m_\pi^{\,2}. \tag{2.13}$$

The A_2 contribution is proportional to $g_{A_2X\pi}g_{A_2\eta\pi}$ which can be obtained, up to an unimportant sign, from the respective widths[17] of $A_2\to X\pi$ and $A_2\to\eta\pi$ and Eq. (2.7). There is more experimental uncertainty in the width of $A_2\to X\pi$; however the mean of the allowed values of this width,[17] $\simeq 1$ MeV, agrees with the prediction of Glashow and Socolow.[18] As our inputs, we choose

$$\Gamma(A_2\to X^0\pi)=1 \text{ MeV} \tag{2.14}$$

and

$$\Gamma(A_2\to\eta\pi)=16 \text{ MeV}, \tag{2.15}$$

which lead to

$$|g_{A_2X\pi}g_{A_2\eta\pi}|=9.95 \text{ GeV}^{-2}. \tag{2.16}$$

We next turn to the determination of A_{Reg}. In general we expect the asymptotic form of $A(\nu,t)$ to be governed by ordinary $(P',\epsilon,\ldots)$ Regge exchange plus Pomeranchuk exchange. Thus we have

$$A(\nu,t)\underset{\nu\to\infty}{\sim}\frac{\pi\beta_{P'}}{\Gamma(\alpha_{P'}(t))\sin\pi\alpha_{P'}(t)}[\nu^{\alpha_{P'}(t)}+(-\nu)^{\alpha_{P'}(t)}]+\frac{\pi\beta_\epsilon}{\Gamma(\alpha_\epsilon(t)+1)\sin\pi\alpha_\epsilon(t)}[\nu^{\alpha_\epsilon(t)}+(-\nu)^{\alpha_\epsilon(t)}]+(\text{Pomeranchuk}), \tag{2.17}$$

where $\beta_{P'}$ and β_ϵ are taken to be constants.[19] However, we use only the first two terms in evaluating the integral along C_N, in keeping with the expectation that the Pomeranchuk contribution in the decay region will be small.[3,20]

Substituting Eq. (2.17) into the second term of Eq. (2.2) we obtain

$$A_{\text{Reg}}(\nu,t)=2\frac{\beta_{P'}}{\Gamma(\alpha_{P'}(t))}\sum_{n=0}^{\infty}\frac{N^{\alpha_{P'}(t)}}{\alpha_{P'}(t)-2n}\left(\frac{\nu}{N}\right)^{2n}+2\frac{\beta_\epsilon}{\Gamma(\alpha_\epsilon(t)+1)}\sum_{n=0}^{\infty}\frac{N^{\alpha_\epsilon(t)}}{\alpha_\epsilon(t)-2n}\left(\frac{\nu}{N}\right)^{2n}. \tag{2.18}$$

The P' Regge trajectory is taken to be

$$\alpha_{P'}(t)=0.5+t. \tag{2.19}$$

As for the ϵ trajectory, we will assume initially that it is likewise purely real, i.e., that the width of the ϵ resonance may be neglected in the calculation since the ϵ lies well outside the decay region. Thus we choose

$$\alpha_\epsilon(t)=-0.8+t, \tag{2.20}$$

corresponding to a $J^P=0^+$ ϵ resonance at ~900 MeV.[21,22] The effect of other possible forms for $\alpha_\epsilon(t)$ will be discussed in Sec. III.

We can determine $\beta_{P'}$ and β_ϵ in terms of $g_{A_2X\pi}g_{A_2\eta\pi}$ by means of the FESR

$$0=\frac{2}{\pi}\int_0^N d\nu\,\nu\,\text{Im}A(\nu,t)+\frac{1}{2\pi i}\int_{C_N}d\nu\,\nu A(\nu,t) \tag{2.21}$$

taken along the contour shown in Fig. 2. Saturation of the first integral with the A_2 and substitution of Eq. (2.17) into the second integral leads to

$$g_{A_2X\pi}g_{A_2\eta\pi}(\tfrac{1}{2}t-D)(4t^2+Bt+C)=\frac{\beta_{P'}N^{\alpha_{P'}(t)+2}}{\Gamma(\alpha_{P'}(t))[\alpha_{P'}(t)+2]}+\frac{\beta_\epsilon N^{\alpha_\epsilon(t)+2}}{\Gamma(\alpha_\epsilon(t)+1)[\alpha_\epsilon(t)+2]}, \tag{2.22}$$

where we have assumed the validity of the Freund-Harari conjecture,[23] according to which only the non-Pomeranchuk Regge exchanges are related to the resonances via the FESR. The first two terms in an expansion of Eq. (2.22) about $t=0$ lead to two simultaneous equations in $\beta_{P'}$ and β_ϵ, which can be solved in terms of $g_{A_2X\pi}g_{A_2\eta\pi}$ and $s_{\max}$.[3] Note that a rather small extrapolation is involved in going from $t=0$ to the decay region, in which the maximum value of t is $\simeq 0.17$ GeV^2.

III. RESULTS

We have thus arrived at an expression for $A(\nu,t)$ which contains no free parameters and which may now be used to predict the decay width and slope of $X^0\to\eta\pi\pi$. We find, with the values of the A_2 partial widths, the trajectory intercepts and $s_{\max}$ given above, that the total decay width Γ (into both charged and neutral pions) is

$$\Gamma\equiv\Gamma_{\text{tot}}(X^0\to\eta\pi\pi)=2.85 \text{ MeV}. \tag{3.1}$$

If the radiative decay $X^0\to\pi\pi\gamma$ accounts for a third of the total width[17,24] then we would have

$$\Gamma(X^0\to\text{all})\simeq 4.25 \text{ MeV}. \tag{3.2}$$

This is consistent with the preferred[17] upper bound of 10 MeV.

Next we consider the decay distribution of the η. To a good approximation, $A(\nu,t)$ can be expanded in the decay region as

$$A(\nu,t)\simeq A(0,t)\simeq\text{const}(1+ay), \tag{3.3}$$

where

$$y=\frac{m_x-Q}{m_\pi Q}T_\eta-1, \tag{3.4}$$

with

$$Q = m_x - m_\eta - 2m_\pi ,$$

and T_η is the kinetic energy of the η in the rest-frame of the X^0. The parameter a is referred to as the slope. We predict that

$$a = -0.15 . \quad (3.5)$$

The decay distribution for $X^0 \to \eta\pi\pi$ has been measured by a number of groups.[25-29] The early experiment of London *et al.*[25] and the more recent work of Dufey *et al.*[26] and Aguilar-Benitez *et al.*[29] support a value of a in the range $-0.2 \gtrsim a \gtrsim -0.5$. On the other hand, the recent analysis by Rittenberg[27] and the experiment of Danburg *et al.*[28] indicate a much smaller value in the range $0.0 \gtrsim a \gtrsim -0.16$.

These determinations of a cover a fair range of values. Nevertheless, several conclusions can be drawn from the data. In the first place, a is certainly negative. In addition, $|a|$ is probably fairly small. Our estimate, based on the latest data, especially those of Danburg *et al.*[28] which have ~500 events (corresponding to η decay into neutrals) indicating $a \simeq -0.05$, is that a lies in the range $-0.05 \gtrsim a \gtrsim -0.25$.

If $A(\nu, t)$ is expanded to order y^2 so that

$$A(\nu, t) \cong \text{const} \times (1 + ay + by^2) ,$$

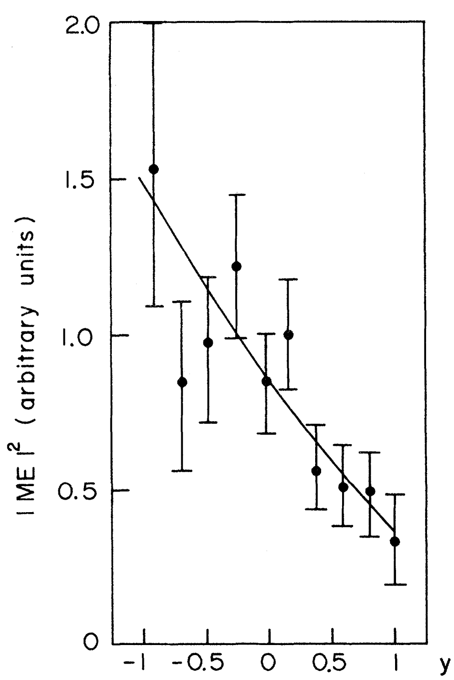

FIG. 3. Decay spectrum for $X \to \eta\pi\pi$. The modulus-squared of the matrix element ME is plotted, in arbitrary units, against the variable y defined in Eq. (3.4). [ME $\propto A(0, t)$]. The data are from Ref. 26.

TABLE I. Predictions of $\Gamma = \Gamma(X^0 \to \eta\pi\pi)$ and a [defined in Eq. (3.3)] for selected values of the parameters different from the preferred choices of $s_{\max} = 2.86$ GeV^2, $\alpha_{P'}(0) = 0.5$, and $\alpha_\epsilon(0) = -0.8$. α' represents the common slope of the P' and ϵ Regge trajectories. The predictions opposite a parameter in the table correspond to changes in that one parameter, all other parameters having their preferred values.

Parameter	Value	Γ (MeV)	a
$s_{\max}$	2.62 GeV^2	3.9	−0.16
	3.12 GeV^2	2.1	−0.14
α'	0.9 GeV^{-2}	3.7	−0.13
$\alpha_{P'}(0)$	0.3	13.5	−0.11
	0.7	1.8	−0.17
$\alpha_\epsilon(0)$	−0.6	42	−0.10
	−0.9	0.08	−0.23

then we predict

$$b = +0.01 .$$

This may be compared with the result of Dufey *et al.*[26] that

$$b = 0.03 \pm 0.04 .$$

In Fig. 3 we compare the decay distribution predicted by our model with the data of Dufey *et al.*[26] The complete y dependence of our amplitude has been used (with $\nu = 0$). It can be seen that the model is quite compatible with the data, even though the data favor a larger value of $|a|$ than does the model.

An indication of the sensitivity of our results to changes in the various parameters of the model is given in Table I. As can be seen immediately from Table I, the predicted width and slope change very little when fairly sizable changes are made in $s_{\max}$ and α'. This is also true for changes in $\alpha_{P'}(0)$, since we have allowed a quite exaggerated variation in this parameter. In fact, the best fits[30] to high-energy elastic scattering data invariably have $\alpha_{P'}(0) \cong \frac{1}{2}$.

On the other hand, one is left with the impression from Table I that our results depend strongly on the value of $\alpha_\epsilon(0)$. This impression may be somewhat misleading for the following reasons. In the first place, while there is indeed variation in the slope, the predicted values of a lie well within the acceptable range. Secondly, the unacceptably large values of Γ obtained by decreasing $|\alpha_\epsilon(0)|$ arise from the neglect of the possibly considerable width of the ϵ resonance.[31] In order to estimate the effect of a finite width ϵ on our results, we have repeated the calculation with the ϵ trajectory of Eq. (2.20) replaced by

TABLE II. Predictions based on the use of the complex ϵ Regge trajectory given in Eq. (3.6) in place of the real trajectory of Eq. (2.20).

α_0	Γ (MeV)	a
−0.5	21	−0.22
−0.6	4.8	−0.23
−0.7	1.5	−0.24
−0.8	1.1	−0.28

$$\alpha_\epsilon(t) = \alpha_0 + t + i0.28(t - 4m_\pi^2)^{1/2}, \qquad (3.6)$$

which corresponds to $\Gamma(\epsilon \to 2\pi) \simeq 270$ MeV. The results for a range of $\alpha_0\ [\simeq \alpha_\epsilon(0)]$ are shown in Table II. The predicted widths, with the exception[32] of that corresponding to $\alpha_0 = -0.5$, are all acceptable. The magnitudes of the slopes have increased[33] over the zero-width ϵ case but are compatible with most of the data. In fact, it is encouraging to note from Tables I and II that the slope obtained in the present work is quite insensitive to reasonable changes in *any* of the parameters.

Finally, we would like to make several comments on the dependence of our results on the product $g_{A_2X\pi}g_{A_2\eta\pi}$, which can be seen from Eqs. (2.9) and (2.22). Of course the prediction of a is completely independent of the coupling constants. However Γ is proportional to $\Gamma(A_2 \to X\pi)\Gamma(A_2 \to \eta\pi)$. While the value of $\Gamma(A_2 \to \eta\pi)$ is reasonably well established,[17] there is uncertainty in the value of $\Gamma(A_2 \to X\pi)$. Nevertheless, we want to reemphasize that the number we have used for this width is not only the mean experimental value, but also the one predicted by the successful model of Glashow and Socolow.[18]

IV. DISCUSSION

There have been a number of previous predictions[7-15] of the width *or* the slope of $X^0 \to \eta\pi\pi$. These have been based on crossing-symmetric models,[7-9] on chiral Lagrangians,[9-13] and on current algebra.[14,15] Until the present calculation however, no model has yielded reasonable values for both the width and the slope, although several approaches[12,13] have led to plausible relations between the two.

The decay width has not proved excessively difficult to predict. Calculations based on a variety of different techniques[8,9,11,14] have all produced values in the range 1 MeV $\leq \Gamma \leq$ 10 MeV. An exception is the prediction of Weisz *et al.*[15] of the rather small value, $\Gamma = 0.2$ MeV. Their result was obtained by determining the decay amplitude from the σ term for $\pi X^0 \to \pi\eta$ and using[26] $a = -0.28$. From Table I it can be seen that this connection between slope and width roughly corresponds in our model to the case $\alpha_\epsilon(0) = -0.9$. We are now investigating the current-algebra properties of our model.

The $X^0 \to \eta\pi\pi$ decay distribution has been much more difficult to explain than the width. Two previous models[7,10] have yielded values of a which were negative but had magnitudes which, while not ruled out, seemed large compared with the latest data. Both the models of Moen and Moffat[7] and Schwinger[10] predict $a \lesssim -0.4$. While this is compatible with the analyses of London *et al*,[25] Dufey *et al.*,[26] and Aguilar-Benitez *et al.*,[29] it is certainly inconsistent with the work of Rittenberg[27] and Danburg *et al.*[28]

The calculations of Majumdar[12] and Schechter and Ueda,[13] while yielding relationships between Γ and a which are consistent with the present experimental data, are unable to predict either without a knowledge of the other, due to undetermined parameters. In addition, in Majumdar's model a is not a single-valued function of Γ.

In the foregoing we have presented a calculation, based on finite dispersion relations, in which *both* the width *and* the slope of the decay $X^0 \to \eta\pi\pi$ are predicted in good agreement with the data. We regard our prediction of the slope in particular as a major success of the FDR approach.

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[1]Among the earliest applications of FESR are K. Igi, Phys. Rev. Lett. 9, 76 (1962); A. Logunov, L. Soloviev, and A. N. Tavkhelidze, Phys. Lett. 24B, 181 (1967); R. Dolen, D. Horn, and C. Schmid, Phys. Rev. 166, 1768 (1968).

[2]For a recent review of FESR and some resulting developments such as duality, see J. D. Jackson, Rev. Mod. Phys. 42, 12 (1970).

[3]R. Aviv and S. Nussinov, Phys. Rev. D 2, 209 (1970).

[4]G. J. Gounaris and A. Verganelakis, Nucl. Phys. B34, 418 (1971).

[5]B.-L. Young and K. E. Lassila, Phys. Rev. Lett. 28, 1491 (1972); Phys. Rev. D 7, 2174 (1973).

[6]J. Baacke and J. Engels, Nucl. Phys. B51, 434 (1973);

J. Engels, *ibid.* B51, 269 (1973).

[7]I. O. Moen and J. W. Moffat, Nuovo Cimento Lett. 3, 473 (1970).

[8]J. Baacke, M. Jacob, and S. Pokorski, Nuovo Cimento 62A, 332 (1969).

[9]A. M. Harun-ar Rashid, Nuovo Cimento 64A, 985 (1969).

[10]J. Schwinger, Phys. Rev. 167, 1432 (1968).

[11]J. A. Cronin, Phys. Rev. 161, 1483 (1967).

[12]D. P. Majumdar, Phys. Rev. Lett. 21, 502 (1968).

[13]J. Schechter and Y. Ueda, Phys. Rev. D 3, 2874 (1971).

[14]V. S. Mathur, S. Okubo, and J. Subba Rao, Phys. Rev. D 1, 2058 (1970).

[15]P. Weisz, Riazuddin, and S. Oneda, Phys. Rev. D 5, 2264 (1972).

[16]As pointed out by Aviv and Nussinov (see Ref. 3) the amplitude obtained from the FDR may not have the correct analytic structure in the decay region. Following the practice of the previous applications, we will ignore this possible complication in what follows.

[17]Particle Data Group, Rev. Mod. Phys. 45, S1 (1973).

[18]S. Glashow and R. H. Socolow, Phys. Rev. Lett. 15, 329 (1965); the values of the partial widths used as input by Glashow and Socolow are still approximately valid and most of the predictions of their model are consistent with the latest data as given in Ref. 17.

[19]The only t dependence in the Regge residue functions is contained in the usual ghost-eliminating Γ functions (this is the t dependence assumed in Refs. 3–5). The scale factor has been taken to be 1 GeV^2.

[20]It was argued by M. Kugler [Phys. Lett. 31B, 379 (1970)] that a resonance approximation to reactions in which there are no exotic channels, as in the present case, may be inadequate to describe low-energy behavior. For $\pi X^0 \to \pi\eta$, however, because of the small X^0-η mixing, we expect the octet (P', ϵ) Regge exchange to dominate the singlet (Pomeranchuk) exchange. In addition, there is a crossing-symmetric model calculation of $\pi\pi$ and $K\pi$ scattering [J. W. Moffat and B. Weisman, Phys. Rev. D 6, 238 (1972)] which, in obtaining good fits to existing low- and high-energy data, finds a Pomeranchuk contribution which is much more strongly attenuated away from $t=0$ than are the ordinary (P', ρ) Regge contributions. If we were to use the Regge asymptotic behavior suggested by this model in our calculations, the P' and ϵ contributions would be virtually unchanged and would dominate the Pomeranchuk term over most of the phase space.

[21]Although this choice is motivated by an earlier data compilation [Particle Data Group, Phys. Lett. 39B, 1 (1972)], we believe it may reasonably take into account the more complicated situation in the $\pi\pi$ $I=0$ S wave deduced by Protopopescu *et al.* (Ref. 22). Equation (2.20) represents a parametrization of the dominant ϵ singularity.

[22]S. D. Protopopescu *et al.*, Phys. Rev. D 7, 1279 (1973). The energy-dependent $\pi\pi$ phase-shift analysis carried out by Protopopescu *et al.* indicates the possible existence of two $I=0$, $J^P=0^+$ resonances. The stronger candidate is a narrow resonance (S^*) with a mass of ~ 1 GeV. Although less certain, the analysis also favors a broad resonance (ϵ) with a mass $\lesssim 700$ MeV and a width $\gtrsim 600$ MeV.

[23]P. G. O. Freund, Phys. Rev. Lett. 20, 235 (1968); H. Harari, *ibid.* 20, 1395 (1968).

[24]We are presently studying the application of FDR to the process $X^0 \to \pi\pi\gamma$.

[25]G. London *et al.*, Phys. Rev. 143, 1034 (1966). This analysis of $K^- p$ interactions at 2.24 GeV/c yielded data which are consistent with $a \simeq -0.4$ (see J. Schwinger, Ref. 10).

[26]J. P. Dufey *et al.*, Phys. Lett. 29B, 605 (1969). In this analysis of $\pi^- p \to n X^0$ at ~ 1.5 GeV/c the values $a=-0.25\pm 0.08$ and $b=0.03\pm 0.04$ were obtained.

[27]A. Rittenberg, Ph.D. thesis, UCRL No. 18863, 1969 (unpublished). From a study of the reaction $K^- p \to \Lambda X^0$ between 1.7 and 2.65 GeV/c, Rittenberg obtained a best fit to the Dalitz plot for $a=-0.11\pm 0.05$.

[28]J. S. Danburg *et al.*, in *Experimental Meson Spectroscopy—1972,* edited by A. H. Rosenfeld and K. W. Lai (American Institute of Physics, New York, 1972). In an analysis of $K^- p \to \Lambda X^0$ at 2.2 GeV/c, Danburg *et al.* find $a=-0.046^{+0.040}_{-0.039}$ for events in which the η decays into neutrals and $a=-0.08\pm 0.08$ for those events having charged η decays. They also find $\Gamma < 3.8$ MeV with a 90% confidence level.

[29]M. Aguilar-Benitez *et al.*, Phys. Rev. D 6, 29 (1972). This study of $K^- p \to \Lambda X^0$ at 3.9 and 4.6 GeV/c, which has fewer X^0 events than that of Ref. 28, yields $a=-0.34^{+0.17}_{-0.15}$.

[30]See, e.g., C. Meyers and P. L. Salin [Nucl. Phys. B19, 237 (1970)], who find $\alpha_{P'}(0)=0.45$ in a phenomenological analysis of KN and $\bar{K}N$ scattering and L. K. Chavda [Phys. Rev. 186, 1463 (1969)], who obtains $\alpha_{P'}(0)=0.54\pm 0.02$ in a fit to πN scattering data.

[31]The case $\alpha_\epsilon(0)=-0.9$ could correspond to the rather unlikely situation in which the ϵ of Protopopescu *et al.* (Ref. 22) is not a resonance, and their S^* represents the dominant effect in the $\pi\pi$ $I=0$ S wave. Another way of accommodating this situation in our model would be to leave $\alpha_\epsilon(0)=-0.8$ and set $\alpha' \approx 0.9\ \mathrm{GeV}^{-2}$. As can be seen from Table I, this alternative results in very minor changes in our results.

[32]If we make the appropriate changes in Eq. (3.6) ($\alpha_0 \approx -0.5$, $0.28 \to \sim 0.6$), so that it represents the ϵ of Ref. 22, we find $\Gamma \approx 1$ MeV.

[33]This increase will continue with increasing ϵ width for trajectories having the form of Eq. (3.6). Since the actual structure of the imaginary parts of Regge trajectories is unknown, the choice made here, while reasonable, should not be taken too seriously, especially for very broad resonances which magnify the effect of the t dependence.