Muon-Capture Rates in Deuterium and the Weak Form Factors in the Timelike Region*

S. L. Mintz

Center for Theoretical Studies, University of Miami, Coral Gables, Florida 33124†
and The Johns Hopkins University, Baltimore, Maryland 21218

(Received 28 November 1972; revised manuscript received 14 May 1973)

The rates for muon capture in deuterium from the doublet and quartet states are calculated by the use of the elementary-particle treatment. The matrix elements of the vector and axial-vector currents are each constructed in terms of two independent form factors. The form factors describing the matrix element of the vector current are obtained in the spacelike region from electrodisintegration and photodisintegration data by the use of the conserved-vector-current hypothesis. The form factors describing the matrix element of the axial-vector current are obtained in the spacelike region by the use of the hypothesis of partial conservation of axial-vector current and arguments based on the impulse approximation. The capture rates are found to be sensitive to the behavior of the form factors near $q^2 = -m_\mu^2$. The capture rates from the doublet and quartet states are found to be $\Gamma_d = 374 \pm 18$ $\sec^{-1}$ and $\Gamma_q = 6.07 \pm 0.04$ $\sec^{-1}$. It is shown that 25.6% of the contribution to Γ_d comes from the timelike region and that an analytic continuation of the dipole fit for the form factors to the timelike region from the spacelike region is consistent with experimental data.

I. INTRODUCTION

In the past few years a number of papers[1] treating muon capture in various nuclei by means of an elementary-particle-model approach[2] have been published. The processes treated have all been of the form $\mu^- + i \to f + \nu_\mu$, where i and f, the initial and final nuclei, are single particles. A process of this kind takes place at a fixed, spacelike value of q^2 where

$$q_\mu = (P_f - P_i)_\mu \tag{1}$$

is the momentum transfer. The hadronic part of the matrix element for this type of process is proportional to $\langle f| J_\mu^\dagger(0) | i\rangle$. This matrix element is in turn described by form factors which are scalar functions of q^2.

In the elementary-particle approach, the form factors describing the matrix element of the weak vector current are obtained from the electromagnetic form factors via the conserved-vector-current (CVC) hypothesis. The axial-vector-current form factors are usually obtained from beta-decay data by making use of the hypothesis of partial conservation of the axial-vector current (PCAC) and a result derived via the impulse approximation.

The advantage of the elementary-particle approach over the conventional impulse-approximation treatment for this type of problem is that the elementary-particle approach avoids the use of nuclear wave functions. The cross sections calculated by means of an impulse-approximation treatment sometimes depend sensitively on these wave functions which are, in general, not well known.

In this paper we calculate, using an elementary-particle approach, the rates for muon capture[3] in deuterium, $\mu^- + d \to n + n + \nu_\mu$, from the doublet and quartet states. This process differs from the one we have just mentioned because two hadrons are contained in the final state instead of one hadron. This fact leads to a number of complications which will be discussed below.

A study of muon capture in deuterium is important for a number of reasons. Since the deuteron ($J^P = 1^+$) is the complex nucleus with the smallest nucleon number, it is important that theoretical calculations and experimental results be in agreement for this case. Information about the n-n scattering length can be extracted from this reaction. In addition, calculation of the capture rate for this process involves values of q^2 in the timelike region, so the behavior of the form factors in this region may be studied.

In Sec. II of this paper we obtain the general form for the matrix elements of the vector and axial-vector currents by the use of the Lehmann-Symanzik-Zimmermann (LSZ) formalism. General properties of the form factors are also discussed. In Sec. III we obtain an expression for the matrix element of the vector current. The form factors are determined in the spacelike region from electrodisintegration and photodisintegration data via the CVC hypothesis. In Sec. IV we obtain an expression for the matrix element of the axial-vector current. The form factors are obtained in the spacelike region by the use of results based on the impulse approximation and by the use of the PCAC hypothesis. In Sec. V we obtain the muon-capture rates from the doublet and quartet states,

Γ_d and Γ_q, respectively. In Sec. VI we compare these results with experimentally determined values and discuss the analytic continuation of the form factors to the timelike region from the spacelike region.

II. GENERAL FORMULATION

The transition matrix element for the process $\mu^- + d \to n + n + \nu_\mu$, may be written as follows:

$$M = [(\tfrac{1}{2})^{1/2} G] \cos\theta_C \langle nn | J_\lambda^\dagger(0) | d \rangle \bar{u}_\nu (1-\gamma_5) \gamma^\lambda u_\mu \tag{2}$$

to the lowest order in G $(=1.02\times 10^5/m_p^2)$, the weak coupling constant, where θ_C is the Cabibbo angle $(\cos\theta_C = 0.98)$,

$$J_\lambda(x) = V_\lambda(x) - A_\lambda(x) \tag{3}$$

is the hadronic part of the weak current, $V_\lambda(x)$ and $A_\lambda(x)$ are the vector and axial-vector parts of $J_\lambda(x)$, and m_p is the proton mass. We shall be primarily concerned with the matrix element

$$\langle nn | J_\lambda^\dagger(0) | d \rangle = \langle nn | V_\lambda^\dagger(0) | d \rangle - \langle nn | A_\lambda^\dagger(0) | d \rangle .$$

This matrix element is constructed in terms of the quantities N and $N\gamma_5$, where N stands for the vectors Q_μ, P_μ, ξ_μ, and d_μ, the Dirac matrices γ_μ and $\sigma_{\mu\nu}$, the totally antisymmetric tensor $\epsilon_{\mu\nu\rho\lambda}$, etc., where $p_{1\mu}$ and $p_{2\mu}$ are the four-momenta of the two neutrons, d_μ is the deuteron four-momentum, ξ_μ is the deuteron polarization vector, and

$$Q_\mu \equiv p_{1\mu} + p_{2\mu}, \quad P_\mu \equiv p_{1\mu} - p_{2\mu}, \quad q_\mu \equiv Q_\mu - d_\mu .$$

Letting $J_\mu^\dagger$ stand for either $V_\mu^\dagger$ or $A_\mu^\dagger$ (or $J_\mu^\dagger$), we use the LSZ formalism[4] to obtain

$$\langle n_1 n_2 | J_\mu^\dagger(0) | d \rangle = \eta \bar{u}_\alpha(p_1, s_1) \bar{u}_\beta(p_2, s_2) \times (C_\mu^\nu(p_1, p_2, d))_{\alpha\beta} \xi_\nu(d), \tag{4}$$

where $\eta = [m^2/(E_1 E_2)]^{1/2} (2\pi)^{-1/2} (2d_0)^{-1/2}$; E_1, E_2, and d_0 are the energies of the two neutrons and the deuteron, respectively, s_1 and s_2 are the spins of the two neutrons, and m is the nucleon mass. The matrix $(C_\mu^\nu(p_1, p_2, d))_{\alpha\beta}$ is a four-by-four matrix (α, β run from 0 to 3) with the following property:

$$(C_\mu^\nu(p_1, p_2, d))_{\alpha\beta} = -(C_\mu^\nu(p_2, p_1, d))_{\beta\alpha} . \tag{5}$$

Equation (5) expresses the fact that there are two identical fermions (two neutrons) in the final state. Because of the property of C_μ^ν expressed by Eq. (5), it is convenient in constructing it to make use of the set of Dirac matrices

$$C, \ \gamma_5 C, \ \gamma_\mu C, \ \gamma_\mu \gamma_5 C, \ \sigma_{\mu\nu} C \tag{6}$$

(in place of the standard set $1, \gamma_5, \gamma_\mu, \gamma_5, \sigma_{\mu\nu}$), where

$$C = i\gamma^{(2)}\gamma^{(0)} \tag{7}$$

is the charge-conjugation matrix.[5] The members of the set given by Eq. (6) are either symmetric or antisymmetric under the interchange of their matrix indices (α, β), with $(\gamma_\mu C)_{\alpha\beta}$ and $(\sigma_{\mu\nu} C)_{\alpha\beta}$ being symmetric, and $C_{\alpha\beta}$, $(\gamma_5 C)_{\alpha\beta}$, and $(\gamma_\mu\gamma_5 C)_{\alpha\beta}$ being antisymmetric.

We use the following relations[6] to rewrite Eq. (4):

$$(C\gamma^0)_{\alpha\beta} u_\beta^\dagger(p, s) = v_\alpha(p, s) e^{i\delta(p,s)} , \tag{8}$$

$$(C\gamma^0)_{\alpha\beta} v_\beta^\dagger(p, s) = u_\alpha(p, s) e^{i\delta(p,s)} . \tag{9}$$

The phase factor $e^{i\delta(p,s)}$ in the above relations will always appear linearly in the matrix elements considered here and will disappear in the calculation of any cross section, capture rate, etc. Hence we shall ignore it.

Using Eqs. (8) and (9), we rewrite Eq. (4) as follows:

$$\begin{aligned}\langle n_1 n_2 | J_\mu^\dagger(0) | d \rangle &= \eta \bar{u}_\alpha(p_1, s_1) \bar{u}_\beta(p_2, s_2) (C_\mu^\nu(p_1, p_2, d))_{\alpha\beta} \xi_\nu \\ &= \eta \bar{u}_\alpha(p_1, s_1) \bar{u}_\beta(p_2, s_2) (M_\mu^\nu(p_1, p_2, d) C)_{\alpha\beta} \xi_\nu(d) \\ &= \eta \bar{u}_\alpha(p_1, s_1) M_\mu^\nu(p_1, p_2, d) v_\beta(p_2, s_2) \xi_\nu \equiv \bar{u} M_\mu^\nu v \xi_\nu ,\end{aligned} \tag{10}$$

where

$$(C_\mu^\nu(p_1, p_2, d))_{\alpha\beta} = (M_\mu^\nu(p_1, p_2, d) C)_{\alpha\beta} . \tag{11}$$

We shall generally find it convenient to write the current matrix elements in the form given by Eq. (10). In the following sections we shall determine the matrix M_μ^ν.

We close this section by observing that three independent scalar variables (instead of one, q^2, in the case of the process $i + \mu^- \to f + \nu_\mu$) are needed to construct the form factors which describe M_μ^ν. The three variables which we choose are q^2, Q^2, and $P\cdot d$.

The scalars Q^2 and q^2 are symmetric under the interchange of $p_{1\mu}$ and $p_{2\mu}$, the neutron momenta, but $P\cdot d$ is antisymmetric. Thus, if a form factor describing M_μ^ν is to be antisymmetric, it must be of the form

$$F(Q^2, q^2, P\cdot d) = \frac{P\cdot d}{M_d^2} G_1(Q^2, q^2) + \frac{(P\cdot d)^3}{M_d^6} G_2(Q^2, q^2) + \cdots , \tag{12}$$

where we have constructed dimensionless coefficients of $P\cdot d$ (i.e., G_1, G_2, etc.) by dividing by powers of M_d, the deuteron mass, which is a mass characteristic of this calculation. The quantity $P\cdot d/M_d{}^2$ is small:

$$\left|\frac{P\cdot d}{M_d{}^2}\right|=\left|\frac{(E_1-E_2)M_d}{M_d{}^2}-\frac{P\cdot d}{M_d{}^2}\right|$$

$$\approx\left|\frac{p_1{}^2-p_2{}^2-P\cdot d}{M_d{}^2}\right|\ll 1, \tag{13}$$

since p_1, p_2, and d are nonrelativistic (i.e., $|p_1|/E_1\ll 1$, etc.). We shall assume that the form factors are smoothly varying functions and that form factors of the type given by Eq. (12) will be sufficiently small in the region of interest to us to be ignored. Thus all form factors used to describe M_μ^ν will be symmetric under the interchange of $p_{1\mu}$ and $p_{2\mu}$.

III. THE MATRIX ELEMENT OF THE VECTOR CURRENT

In this section we construct the matrix element of the vector current $\langle nn|V_\mu^\dagger(0)|d\rangle$. Because each neutron has two possible helicity states, the deuteron has three possible helicity states, and the vector current has four components, in general a maximum of $2\times2\times4\times3=48$ independent amplitudes (i.e., form factors) are necessary to construct $\langle nn|V_\mu^\dagger(0)|d\rangle$. Arguments based on parity can be used to reduce this number to 24. [Exactly the same arguments may be used in the axial-vector-current case so that a maximum of 24 form factors are needed to describe $\langle nn|A_\mu^\dagger(0)|d\rangle$.]

From the assumption[7] that V_μ is conserved (in fact, that it is the conserved isotopic-spin current), i.e.,

$$\partial_\mu V^\mu(x)=0 \tag{14}$$

(hereafter called the CVC hypothesis), we obtain the equation

$$\langle nn|\partial_\mu V^{\mu\dagger}(x)|d\rangle|_0=0 \tag{15}$$

Since $\partial^\mu V_\mu^\dagger(x)$ is a scalar under rotations, Eq. (15) corresponds to $2\times2\times1\times3=12$ conditions. Arguments based on parity again halve this number, leaving 6 conditions. Thus a maximal number[8] of 18 form factors is needed to construct $\langle nn|V_\mu^\dagger(0)|d\rangle$.

However, an impulse-approximation calculation of $\langle nn|V_\mu^\dagger(0)|d\rangle$ requires only two form factors and the impulse approximation is accurate[9] to within 10% (if accurate wave functions are available). We shall, therefore, construct the matrix element $\langle nn|V_\mu^\dagger(0)|d\rangle$ in terms of two form factors.

From the behavior of the vector current $V_\mu^\dagger(x)$ under a parity transformation (denoting the parity operator by $\mathcal{P}$), we have

$$\begin{aligned}\langle n_1,\vec{p}_1;n_2,\vec{p}_2|V_j^\dagger(0)|d,\vec{d}\rangle &=\langle n_1,\vec{p}_1;n_2,\vec{p}_2|\mathcal{P}^{-1}\mathcal{P}V_j^\dagger(0)\mathcal{P}^{-1}\mathcal{P}|d,\vec{d}\rangle\\ &=-\langle n_1,-\vec{p}_1;n_2-\vec{p}_2|V_j^\dagger(0)|d,-\vec{d}\rangle,\quad j=1,2,3\end{aligned} \tag{16}$$

$$\begin{aligned}\langle n_1,\vec{p}_1;n_2,\vec{p}_2|V_0^\dagger(0)|d,\vec{d}\rangle &=\langle n_1,\vec{p}_1;n_2,\vec{p}_2|\mathcal{P}^{-1}\mathcal{P}V_0^\dagger(0)\mathcal{P}^{-1}\mathcal{P}|d,\vec{d}\rangle\\ &=\langle n_1,-\vec{p}_1;n_2,-\vec{p}_2|V_0^\dagger(0)|d,-\vec{d}\rangle.\end{aligned}$$

Writing $\langle nn|V_0^\dagger(0)|d\rangle$ in the form given by Eq. (10):

$$\langle n_1n_2|V_\mu^\dagger(0)|d\rangle=\eta\bar{u}(p_1)M_\mu^\nu(p_1,p_2,d)v(p_2)\xi_\nu(d), \tag{17}$$

we obtain from Eq. (16) that

$$\begin{aligned}M_j^i(\vec{p}_1,\vec{p}_2,\vec{d})&=\gamma_0M_j^i(-\vec{p}_1,-\vec{p}_2,-\vec{d})\gamma_0,\\ M_j^0(\vec{p}_1,\vec{p}_2,\vec{d})&=-\gamma_0M_j^0(-\vec{p}_1,-\vec{p}_2,-\vec{d})\gamma_0,\\ M_0^i(\vec{p}_1,\vec{p}_2,\vec{d})&=-\gamma_0M_0^i(-\vec{p}_1,-\vec{p}_2,-\vec{d})\gamma_0,\\ M_0^0(\vec{p}_1,\vec{p}_2,\vec{d})&=\gamma_0M_0^0(-\vec{p}_1,-\vec{p}_2,-\vec{d})\gamma_0.\end{aligned} \tag{18}$$

In addition $M_\mu^\nu C$ must satisfy Eq. (5).

The matrix M_μ^ν is constructed from terms of the form Gm_μ^ν, where G is a form factor and m_μ^ν has exactly the same properties as M_μ^ν. Using the method of exhaustion, we list in Table I the set of matrices Gm_μ^ν which at first sight might be independent and which might be of order lower than p^2/m^2, where p is the magnitude of the nucleon 3-momentum and m is the nucleon mass. (We expect[10] all form factors to be of the same order of magnitude when normalized by the insertion of powers of suitable masses, but we do not know which masses are the correct ones to use. However, we normalize using the nucleon mass and ignore all terms Gm_μ^ν where $|\bar{u}m_\mu^\nu v|$ is of order p^2/m^2 or higher. This procedure yields a current which agrees in form to the lowest order with that obtained by use of the impulse approximation.)

The spinor identities

$$\begin{aligned}&\bar{u}(p_1)(P_\mu+i\sigma_{\mu\nu}Q^\nu)v(p_2)=0,\\ &2m\bar{u}(p_1)\gamma_\mu\gamma_5v(p_2)=\bar{u}(p_1)(Q_\mu+i\sigma_{\mu\nu}P^\nu)\gamma_5v(p_2)\end{aligned} \tag{19}$$

are used to eliminate terms G_{12}, G_{13}, G_{14}, G_{15}, and G_{16} (we identify terms in Table I by the form factors associated with them, i.e., term G_1 is $G_1\xi_\mu$). The CVC condition Eq. (15) together with Eq. (17) implies that

$$iq^\mu\bar{u}(p_1,s_1)M_\mu^\nu(p_1,p_2,d)v(p_2,s_2)\xi_\nu(d)=0. \tag{20}$$

TABLE I. List of matrices out of which the matrix element of the vector current is constructed. The matrices are listed in rows adjoining the terms by which they are multiplied (or contracted).

$\bar{u}v$	$G_1\xi_\mu$	$(G_2/M_d^2)Q_\mu \xi\cdot Q$	$(G_3/M_d^2)d_\mu \xi\cdot Q$	$(G_{16}/M_d^2)P_\mu \xi\cdot P$
$\bar{u}\gamma_5 v$	$(G_0/M_d^2)\epsilon^{\mu\eta\rho\sigma}\xi_\eta Q_\rho d_\sigma$			
$\bar{u}\gamma_\mu v$	$(G_4/M_d)\xi\cdot P$	$(G_5/M_d)\xi^\mu P_\nu$		
$\bar{u}\gamma_\mu\gamma_5 v$	$(G_6/M_d)\epsilon^{\mu\rho\sigma\lambda}\xi_\rho Q_\sigma$	$(G_7/M_d)\epsilon^{\mu\rho\sigma\lambda}\xi_\rho d_\sigma$	$(G_8/M_d^3)\epsilon^{\mu\rho\sigma\lambda}\xi_\nu Q_\rho d_\sigma Q_\lambda$	$(G_8'/M_d^3)\epsilon^{\mu\nu\rho\sigma}\xi_\nu Q_\rho d_\sigma d^\lambda$
$\bar{u}\sigma_{\mu\nu}v$	$(G_9/M_d^2)d^\nu \xi\cdot P$	$(G_{10}/M_d^2)Q^\nu \xi\cdot P$	$(G_{11}/M_d^2)\xi^\mu d^\nu P^\rho$	$(G_{12}/M_d^2)\xi^\mu Q^\nu P^\lambda$
		$(G_{13}/M_d^2)\xi^\mu P^\nu Q^\rho$	$(G_{14}/M_d^2)P^\nu \xi\cdot Q$	$(G_{15}/M_d^2)Q^\mu P^\nu \xi^\rho$

We contract the remaining 12 terms in Table I with q_μ and list the result in Table II.

We wish to extract all the information provided by the CVC condition [Eq. (15)]. To do this, we multiply Eq. (20) by $\bar{v}(p_2, s_2)Au(p_1, s_1)$, where A is a Dirac matrix which we shall choose to be 1, γ_5, or $\gamma_\lambda\gamma_5$. We can then sum over the spins s_1 and s_2 to obtain a trace relation:

$$0=\sum_{s_1 s_2} iq^\mu \bar{u}(p_1,s_1)M^\nu_\mu v(p_2,s_2)\,\xi_\nu(d)\bar{v}(p_2,s_1)\times Au(p_1,s_1)$$

$$=\mathrm{Tr}\left\{iq^\mu\frac{(\not p_1+m)}{2m}M^\nu_\mu\frac{(\not p_2-m)}{2m}A\right\}\xi_\nu(d)\,. \tag{21}$$

This relation with the choices of A given above (where we make use of the free index λ when $A=\gamma_\lambda\gamma_5$ by contracting it successively with P^λ, d^λ, and ξ^λ), together with arguments based on magnitude can be used to eliminate all terms except G_0, G_6, and G_7, and to show that $G_6=-G_7$. We, therefore, obtain

$$\langle nn|V^\dagger_\mu(0)|d\rangle=\eta\bar{u}(p_1)\left(\frac{F_1}{M_d^2}\epsilon_{\mu\nu\rho\sigma}\xi^\nu Q^\rho d^\sigma+\frac{F_2}{M_d}\gamma^\nu\epsilon_{\nu\rho\sigma\mu}\xi^\rho q^\sigma\right)\gamma_5 v(p_2)\,, \tag{22}$$

where the form factors have been relabeled by setting $G_0=F_1$ and $G_6=F_2$.

To check the form of this matrix element, we compare it with that obtained from an impulse-approximation treatment. We require that the form of Eq. (22) agree with the impulse-approximation treatment in the nonrelativistic limit. The impulse approximation for the matrix element of the vector current in the absence of final-state interactions is written[11] as

$$\langle nn|V^\dagger_\mu(0)|d\rangle=\left\langle\psi_{nn}\left|\sum_{i=1}^{2}\left[F_V(q^2;n\to p)\gamma^{(i)}_\mu+i\sigma^{(i)}_{\mu\nu}q^\nu\frac{F_M}{2m}(q^2;\to p)\right]\tau^{-(i)}e^{i\vec{q}\cdot\vec{r}^{(i)}}\right|\psi_d\right\rangle, \tag{23}$$

where $F_V(q^2;n\to p)$ and $F_M(q^2;n\to p)$ are the weak vector and the weak magnetic form factors for the nucleon case, respectively, ψ_{nn} is the wave function of the outgoing neutrons, and ψ_d is the deuteron wave function. The two forms for $\langle nn|V^\dagger_\mu(0)|d\rangle$, Eqs. (22) and (23), are most easily compared by considering specific cases which lead to lowest-order contributions. Thus for the case $\langle nn;S=0,S_z=0|V^{(1)\dagger}(0)|d;S=1,S_z=1\rangle$, one finds from Eq. (22) that

$$\langle nn;S=0,S_z=0|V^{(1)\dagger}(0)|d;S=1,S_z=1\rangle=\frac{(F_1-F_2)}{M_d}\frac{q^{(3)}}{\sqrt{2}}\,, \tag{24}$$

and from Eq. (23) that

$$\langle nn;S=0,S_z=0|V^{(1)\dagger}|d;S=1,S_z=1\rangle=2R(\vec{q})\,[F_V(q^2;n\to p)+F_M(q^2;n\to p)]\,q^{(3)}/2m\,, \tag{25}$$

TABLE II. List of the remaining matrices in Table I contracted with q_μ. These are the matrices from which the matrix element of the divergence of the vector current is constructed.

No.	Term	No.	Term
1	$G_1\xi\cdot Q$	7	$(G_7/M_d)\gamma_\mu\gamma_5\epsilon^{\mu\alpha\beta\gamma}\xi_\alpha d_\beta Q_\gamma$
2	$(G_2/M_d^2)(Q^2-Q\cdot d)\xi\cdot Q$	8	$(G_8/M_d^3)\gamma_\mu\gamma_5\epsilon^{\mu\alpha\beta\gamma}\xi_\alpha d_\beta Q_\gamma(Q^2-Q\cdot d)$
3	$(G_3/M_d^2)(Q\cdot d-d^2)\xi\cdot Q$	9	$(G_8'/M_d^3)\gamma_\mu\gamma_5\epsilon^{\mu\alpha\beta\gamma}\xi_\alpha d_\beta Q_\gamma(Q^2-Q\cdot d)$
4	$(G_4/M_d)(-\not d)\xi\cdot P$	10	$(G_9/M_d^2)\sigma_{\mu\nu}Q^\mu d^\nu\xi\cdot P$
5	$(G_5/M_d)\xi(-P\cdot d)$	11	$(G_{10}/M_d^2)\sigma_{\mu\nu}Q^\mu d^\nu\xi\cdot P$
6	$(G_6/M_d)\gamma_\mu\gamma_5\epsilon^{\mu\alpha\beta\gamma}\xi_\alpha d_\beta Q_\gamma$	12	$(G_{11}/M_d^2)\sigma_{\mu\nu}d^\mu\xi^\nu P\cdot d$

where $R(\vec{q})$ is a function defined by

$$R(\vec{q}) \equiv \int \psi^*_{nn} e^{i\vec{q}\cdot\vec{r}} \psi_d \, d\tau \,. \qquad (26)$$

The integration in Eq. (26) is over the nucleon space variables in relative coordinates. We shall not be concerned with the exact form of $R(\vec{q})$. Values for it may be found by using specific forms[12] for ψ_{nn} and ψ_d. Other cases are listed in Table III. Thus, in the nonrelativistic limit Eqs. (22) and (23) have the same form.

The form factors F_1 and F_2 can be related to the electromagnetic form factors by application of the relations of the CVC hypothesis.[7]

$$\begin{aligned}[I^-, J_\mu^{(3)}(0)] &= [I^-, J_\mu^{\text{em}}(0)] \\ &= V_\mu^\dagger(0)\,, \end{aligned} \qquad (27)$$

where J_μ^{em} is the electromagnetic current density, $J_\mu^{(3)}$ is the isovector part of J_μ^{em}, and I^- is the isospin-lowering operator. Taking the matrix element of Eq. (27) between $|d\rangle$ and $|nn\rangle$, we see that

$$\langle nn | V_\mu^\dagger(0) | d\rangle = \sqrt{2}\, \langle np | J_\mu^{\text{em}}(0) | d\rangle\,, \qquad (28)$$

where we have used the fact that the deuteron is an isoscalar and that $|nn\rangle$ is in an isospin state $I=1$, $I_3=-1$ ($\langle nn| I^- = \sqrt{2}\,\langle np|$). In Eq. (28) the state $|np\rangle$ is an $I=1$, $I_3=0$ state.

The matrix element $\langle np | J_\mu^{\text{em}}(0) | d\rangle$ may be written from Eqs. (28) and (22) as

$$\langle np | J_\mu^{\text{em}}(0) | d\rangle = \eta \bar{u}(p_1)\left(\frac{F_a}{M_d^2}\epsilon_{\mu\nu\rho\sigma}\xi^\nu Q^\rho d^\sigma + \frac{F_b}{M_d}\gamma^\nu \epsilon_{\nu\rho\sigma\mu}\xi^\rho q^\sigma\right)\gamma_5 v(p_2)\,, \qquad (29)$$

so that

$$\sqrt{2}\,F_a = F_1 \text{ and } \sqrt{2}\,F_b = F_2\,. \qquad (30)$$

Ideally, F_a and F_b would be determined from electron scattering, but in the region of our interest $|q^2| \le m_\mu^2$, where m_μ is the muon mass, only a few experiments have been performed. However, we note that in the impulse approximation [see, for example, Eq. (25)] the form factors factorize, i.e.,

$$F^I = F(q^2, n \leftrightarrow p) R(\vec{q})\,, \qquad (31)$$

where the superscript I denotes the impulse approximation. The result Eq. (31) is true for the axial-vector form factors as well (which can be seen from Table III) because to the lowest order, the integrations which occur in the axial-vector case are the same as those which occur in the vector case. Thus $R(\vec{q})$ again appears as a factor and the result Eq. (31) holds for the axial-vector case. We shall, therefore, assume that the form factors F_1, F_2, F_a, and F_b (and the axial-vector form factors F_A and F_P which will be defined in Sec. IV) factorize similarly, i.e.,

$$F_i(q^2, Q^2, P\cdot d) \cong F(Q^2, P\cdot d) f_i(q^2)\,, \quad i = 1, 2, a, b, A, P \qquad (32)$$

where $F(Q^2, P\cdot d)$ is analogous to $R(\vec{q})$ in Eq. (31). We can now use photodisintegration data to obtain $F(Q^2, P\cdot d)$ since photodisintegration takes place at $q^2=0$ and $f_i(q^2=0)$ is just a constant. Electrodisintegration data can then be used to determine $f_i(q^2)$.

The matrix element squared for photodisintegration, averaged over the spins of the initial particles and summed over the spins of the final particles, is

$$|M|^2 \simeq \frac{e^2}{6}\frac{k^2}{m}(F_a - F_b)^2\,, \qquad (33)$$

which yields the following differential cross section:

$$\begin{aligned}\frac{d\sigma}{d\Omega}(\theta, k) &= \frac{p_1 k e^2}{24(2\pi)^2 M_d^2}(F_a - F_b)^2 \\ &= \frac{p_1 k e^2}{24(2\pi)^2 M_d^2} F^2(Q^2, P\cdot d) \\ &\quad \times [f_a(q^2=0) - f_b(q^2=0)]\,, \end{aligned} \qquad (34)$$

TABLE III. The matrix elements $\langle n_1 n_2; S=0, S_z=0 | V^{i\dagger}(0) | d; S=1, S_z=1\rangle$ and $\langle n_1 n_2; S=0, S_z=0 | A_\mu^\dagger(0) | d; S=1\rangle$ are tabulated for $i=1,2,3$. The superscripts I and EPT stand for impulse approximation and elementary particle treatment, respectively. The q^2 dependence of the form factors has been suppressed.

i	$\langle nn \vert V^{i\dagger}(0) \vert d\rangle^I$	$\langle nn \vert A^{i\dagger}(0) \vert d\rangle^I$	$\langle nn \vert V^{i\dagger}(0) \vert d\rangle^{EPT}$	$\langle nn \vert A^{i\dagger}(0) \vert d\rangle^{EPT}$
1	$2R(\vec{q})q^{(3)}\dfrac{(F_V+F_M)}{2M}$	$-2R(\vec{q})F_A(n\leftrightarrow p)$	$N\dfrac{(F_0-F_6)}{M_d}\dfrac{i}{\sqrt{2}}q^{(3)}$	$N\dfrac{F_A(d\leftrightarrow nn)}{\sqrt{2}}$
2	$-2R(\vec{q})iq^{(3)}\dfrac{(F_V+F_M)}{2M}$	$-2iR(\vec{q})F_A(n\leftrightarrow p)$	$N\dfrac{(F_0-F_6)}{M_d}\dfrac{q^{(3)}}{\sqrt{2}}$	$\dfrac{F_A(d\leftrightarrow nn)}{\sqrt{2}}$
3	$2R(\vec{q})i(q^{(2)}-iq^{(1)})\dfrac{(F_V+F_M)}{2M}$	0	$-N(F_0-F_6)i\dfrac{(q^{(2)}-iq^{(1)})}{\sqrt{2}}$	0

where k is the magnitude of the photon momentum, p_1 is the magnitude of the proton momentum, and θ is the angle which the proton makes with the incident photon. All variables are in the c.m. frame. In calculating Eqs. (33) and (34) we have also assumed that F_a and F_b are relatively real as is true in the impulse approximation.

The electromagnetic processes involving the breakup of the deuteron, $\gamma+d\rightarrow n+p$ (photodisintegration) and $e+d\rightarrow n+p+e$ (electrodisintegration), involve the $I=0$, $I_3=0$ state of $|pn\rangle$ as well as the $I=1$, $I_3=0$ which we need. However, in the case of photodisintegration at the energies we shall use to obtain $F(Q^2, P\cdot d)$, i.e., $k\sim 20$ to 50 MeV, the ${}^1S$ and ${}^3P$ states which are $I=1$, $I_3=0$ predominate.[13] The situation for electrodisintegration will be discussed shortly.

From photodisintegration data[14] we find that $F^2(Q^2, P\cdot d)$ can be fitted to the following phenomenological form:

$$F^2(\theta)=K(\theta)\,a\,(2\pi)^2 M_d{}^2/(p_1 k e^2)\,, \tag{35}$$

where $K(\theta)$ is given by[15]

$$K(\theta)\equiv[f_1+f_2(1-\cos\theta)+f_3\sin^2\theta\cos\theta+f_4\sin^2\theta]\,, \tag{36}$$

with

$$\begin{aligned}
&f_1=6.4\,,\\
&f_2=6.4[1-f_4/(f_3+f_4)]\,,\\
&f_3=+1.122\frac{0.6988\times10^{-4}}{(k/M_d-0.01495)^2+0.507\times10^{-5}}\,,\\
&f_4=\frac{839.5}{1+1.27\times10^5(k/M_d)^2}\,,\\
&a=2.57\times10^{-9}/\text{MeV}^2\,.
\end{aligned} \tag{37}$$

In the above equations we have chosen a normalization for the form factors such that $f_a(q^2=0)-f_b(q^2=0)=1$.

Because in photodisintegration there are two bodies in the initial and final state and because the photon is real, i.e., it has four-momentum $(k_0, \vec{k})$ with $|\vec{k}|=k_0\equiv k$, only the variables k and θ are needed to describe the differential cross section for this process. In electrodisintegration (and muon capture) the exchanged quanta are not on the mass shell, and in addition there are three particles in the final state. To generalize Eq. (36) to the more complicated cases of electrodisintegration and muon capture, we note that $F(Q^2, P\cdot d)$ is a scalar quantity and hence can be evaluated in any convenient frame. We choose the center-of-mass frame of the exchanged quantum and the deuteron (see Fig. 1). In this frame the situation of a real photon colliding with a deuteron is analogous to that of the virtual photon colliding with the deuteron; the only difference is that for a virtual photon, there is no relation between its space momentum $\vec{q}$ and its fourth component of momentum q_0. Thus we do not know where to identify $|\vec{q}|$ with k and where to identify q_0 with k. In order to obtain reasonable values for $f_a(q^2=0)-f_b(q^2=0)$ (smoothly decreasing as $-q^2$ increases), however, we find that we must set $|\vec{q}|=k$ in $K(\theta)$ and $q_0=k$ in the denominator of Eq. (35).

The matrix element square for electrodisintegration summed over the spins of the final particles and averaged over the spins of the initial particles is given by

$$|M|\cong\frac{2}{3}\frac{q^2e^4}{m_e{}^2M_d{}^2q^4}(p_i\cdot p_f+E_i{}^2\sin^2\Theta)\,F^2(Q^2,P\cdot d)\times[f_a(q^2)-f_b(q^2)]^2\,, \tag{38}$$

where m_e is the electron mass, $p_{i\mu}$ and $p_{f\mu}$ are the initial and final momenta of the electrons, E_i is the initial energy of the electrons, and Θ is the angle that the direction of the outgoing electron makes with $\vec{q}=\vec{p}_i-\vec{p}_f$, the momentum transfer. Using experimental data[16] for the cross section $d\sigma/d\Omega_c\,dE_f$, we obtain values for $f_a(q^2)$

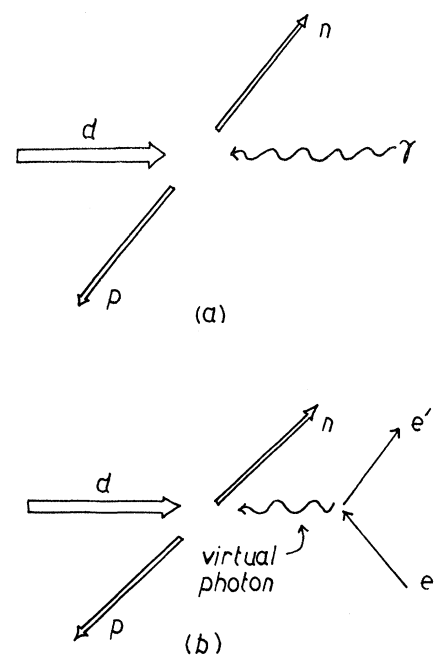

FIG. 1. Diagram showing the analogy between photodistintegration of the deuteron in the c.m. frame (a), and electrodisintegration of the deuteron in the center-of-mass frame of the deuteron and the virtual photon (b).

$-f_b(q^2)$ in the $-4000 \geq q^2 \geq -18\,000$ MeV^2 range. These results are shown in Fig. 2. We use a standard dipole fit to express the form factors and find a best fit to the data with

$$|f_a(q^2) - f_b(q^2)| = 1/(1 - q^2/M^2)^2\,, \qquad M = 224 \pm 25\,\text{MeV}\,, \quad q^2 \leq 0\,. \tag{39}$$

As has been noted, electrodisintegration involves the $I=0$, $I_3=0$ state of $|pn\rangle$ as well as the $I=1$, $I_3=0$ state which we need. However, in the data used here, the electron is scattered through 180° and it is known[17] for this case that the dominant state of the two nucleons is a ^1S state which is an $I=1$ state.

Thus we have determined $F(Q^2, P\cdot d)$ and the quantity $f_a(q^2)-f_b(q^2)$ in the spacelike region, $-q^2 \leq m_\mu^2$. Fortunately, in muon capture only the combination $f_1(q^2)-f_2(q^2)$ occurs (to the order in which we are working), so that we have effectively obtained $\langle nn|V_\mu^\dagger(0)|d\rangle$, the matrix element of the vector current.

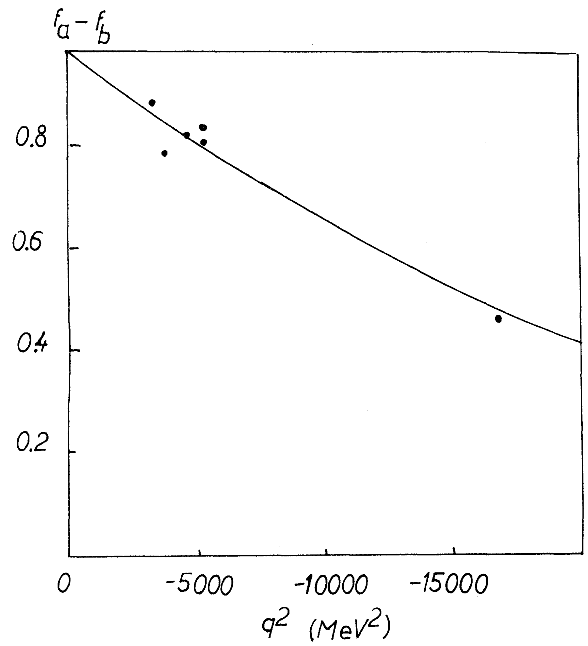

FIG. 2. Plot of the form factor $f_a(q^2)-f_b(q^2)$ as a function of q^2 in the spacelike region.

IV. THE MATRIX ELEMENT OF THE AXIAL-VECTOR CURRENT

We noted (in Sec. II) that a maximal number of 24 form factors was needed to describe $\langle nn|A_\mu^\dagger(0)|d\rangle$. However, in the impulse approximation only two form factors are required to describe $\langle nn|A_\mu^\dagger(0)|d\rangle$. We shall, therefore, construct $\langle nn|A_\mu^\dagger(0)|d\rangle$ in terms of two independent amplitudes. The axial current is not conserved so that arguments analogous to those used to obtain $\langle nn|V_\mu^\dagger(0)|d\rangle$ are not possible for the axial case; instead we shall use arguments based on dispersion relations, orders of magnitude, and the impulse approximation to obtain the matrix element of the axial-vector current.

We consider the crossed process $\mu^- + \bar{\nu}_\mu \to \bar{d} + n + n$ and use dispersion relations[18] to obtain the well-known form

$$\langle n_1 n_2 \bar{d}|A_\mu^\dagger(0)|0\rangle|_{q^2+i\epsilon} - \langle n_1 n_2 \bar{d}|A_\mu^\dagger(0)|0\rangle|_{q^2-i\epsilon} = \frac{2\pi i(2\pi)^3(i)^2\xi_\nu(\bar{d})}{(2\pi)^{3/2}(2\bar{d}_0)^{1/2}}\sum_i \delta^4(p_{1\mu}+p_{2\mu}+\bar{d}_\mu - p_{i\mu}) \times \langle n_1 n_2|j_d^{\nu\,\dagger}(0)|i\rangle\langle i|A_\mu^\dagger(0)|0\rangle\,, \tag{40}$$

where $p_{i\mu}$ is the momentum of the intermediate state $|i\rangle$, $\bar{d}_0$ is the energy of the antideuteron, $j_d^\nu(0)$ is the deuteron source, and ϵ is an infinitesimal. We note that states contributing to the sum in Eq. (40) must be $J^P=0^-$ or $J^P=1^+$ states. We shall approximate the contributions of these states to the right-hand side of Eq. (40) by considering the contribution of the pion state ($J^P=0^-$) and a state $|\alpha\rangle$ ($J^P=1^+$), which we shall treat as a single-particle state.[18] It will be shown that the contribution of these two states to Eq. (40) can be described in terms of two form factors. Thus we need to determine $\langle n_1 n_2|j_d^{\nu\,\dagger}(0)|i\rangle$ and $\langle i|A_\mu^\dagger(0)|0\rangle$ for $|i\rangle$ equal to $|\pi\rangle$ and $|\alpha\rangle$. The contributions of the above-mentioned states to Eq. (40) will help to determine the form of $\langle nn|A_\mu^\dagger(0)|d\rangle$.

We first note that under a parity transformation, one finds

$$\begin{aligned}
\langle n_1,\vec{p}_1;n_2,\vec{p}_2|A_i^\dagger(0)|d,\vec{d}\rangle &= \langle n_1,\vec{p}_1;n_2,\vec{p}_2|\mathcal{P}^{-1}\mathcal{P}A_i^\dagger(0)\mathcal{P}^{-1}\mathcal{P}|d,\vec{d}\rangle \\
&= \langle n_1,-\vec{p}_1;n_2,-\vec{p}_2|A_i^\dagger(0)|d,-\vec{d}\rangle\,, \quad i=1,2,3 \\
\langle n_1,\vec{p}_1;n_2,\vec{p}_2|A_0^\dagger(0)|d,\vec{d}\rangle &= \langle n_1,\vec{p}_1;n_2,\vec{p}_2|\mathcal{P}^{-1}\mathcal{P}A_0^\dagger(0)\mathcal{P}^{-1}\mathcal{P}|d,\vec{d}\rangle \\
&= -\langle n_1,-\vec{p}_1;n_2,-\vec{p}_2|A_0^\dagger(0)|d,-\vec{d}\rangle\,.
\end{aligned} \tag{41}$$

Under a parity transformation, the left-hand side of Eq. (40) transforms like $\langle nn|A_0^\dagger(0)|d\rangle$ [$\langle nn|A_\mu^\dagger(0)|d\rangle$ transforms exactly like $\langle nn\bar{d}|A_\mu^\dagger(0)|0\rangle$] which places a constraint on $\langle i|A_\mu^\dagger(0)|0\rangle$ and $\langle n_1 n_2|j_d^{\nu\,\dagger}(0)|i\rangle$.

We shall make use of Eq. (41) in what follows.

The lowest-mass intermediate state which contributes to the right-hand side of Eq. (40) is the pion state $|i\rangle = |\pi\rangle$. The matrix element $\langle \pi^- | A_\mu^\dagger(0) | 0\rangle$ may be written as

$$\langle \pi^- | A_\mu^\dagger(0) | 0\rangle = i p_{\pi\mu} m_\pi \alpha_\pi / [(2\pi)^{3/2}(2\omega_\pi)^{1/2}], \tag{42}$$

where α_π $(=0.94 \pm 0.01)$ is the pion decay constant and m_π is the pion mass. The matrix element $\langle n_1 n_2 | j_d^{\nu\dagger}(0) | \pi\rangle$ may be written as

$$\langle n_1 n_2 | j_d^{\nu\dagger}(0) | \pi\rangle = i\bar{u}(p_1)\gamma_5 v(p_2) M^\nu \left(\frac{m}{E_1}\right)^{1/2}\left(\frac{m}{E_2}\right)^{1/2} \frac{1}{(2\omega_\pi)^{1/2}} \frac{1}{(2\pi)^{9/2}}, \tag{43}$$

where M^ν is to be determined. With these choices, the π^- contribution to Eq. (40) is

$$(2\pi)\xi_\nu(\bar{d})M^\nu \delta(q^2 - m_\pi^2) m_\pi \alpha_\pi \bar{u}\gamma_5 v q_\mu \left(\frac{m}{E_1}\right)^{1/2}\left(\frac{m}{E_2}\right)^{1/2}, \tag{44}$$

where $q_\mu = p_{1\mu} + p_{2\mu} + \bar{d}_\mu$.

To obtain an expression for M^ν we note that under a parity transformation one finds

$$\begin{aligned}
\langle n_1, \vec{p}_1; n_2, \vec{p}_2 | j_d^{i\dagger}(0) | \pi, \vec{p}_\pi\rangle &= \langle n_1, \vec{p}_1; n_2, \vec{p}_2 | \mathcal{P}^{-1}\mathcal{P} j_d^{i\dagger}(0)\mathcal{P}^{-1}\mathcal{P} | \pi, \vec{p}_\pi\rangle \\
&= -\langle n_1, -\vec{p}_1; n_2, -\vec{p}_2 | j_d^{i\dagger}(0) | \pi, -\vec{p}_\pi\rangle, \quad i = 1, 2, 3 \\
\langle n_1\vec{p}_1; n_2, \vec{p}_2 | j_d^{0\dagger}(0) | \pi\vec{p}_\pi\rangle &= \langle n_1, \vec{p}_1; n_2, \vec{p}_2 | \mathcal{P}^{-1}\mathcal{P} j_d^{0\dagger}\mathcal{P}^{-1}\mathcal{P}(0) | \pi, \vec{p}_\pi\rangle \\
&= \langle n_1, -\vec{p}_1; n_2, -\vec{p}_2 | j_d^{0\dagger}(0) | \pi, -\vec{p}_\pi\rangle .
\end{aligned} \tag{45}$$

From Eqs. (43) and (45) we see that

$$\begin{aligned}
M^j(\vec{p}_1, \vec{p}_2, \vec{p}_\pi) &= -M^j(-\vec{p}_1, -\vec{p}_2, -\vec{p}_\pi), \quad j = 1, 2, 3 \\
M^0(\vec{p}_1, \vec{p}_2, \vec{p}_\pi) &= M^0(-\vec{p}_1, -\vec{p}_2, -\vec{p}_\pi).
\end{aligned} \tag{46}$$

In addition to the constraint placed on M^ν by Eq. (46), the quantity M^ν must be chosen such that Eq. (41) is satisfied and such that $\gamma_5 C q_\mu M^\nu$ satisfies Eq. (5). [The latter requirement stems from a general one that the matrix elements of both the axial and vector currents must satisfy. See Eq. (4).] The simplest choice for M^ν which satisfies these conditions is $F_P Q^\nu$, where F_P is a form factor and $Q^\nu = p_1^\nu + p_2^\nu$ is a four-vector (defined earlier).

It can be shown by standard dispersion-theoretic arguments[18] that F_P has a pion pole. The form we have derived for the pion contributions to Eq. (40) [i.e., Eq. (44)] suggests that we choose

$$F_P(q^2, Q^2, P\cdot d)\bar{u}(p_1)\gamma_5 v(p_2) q_\mu \xi\cdot Q/M_d^2 \tag{47}$$

as one of the two terms of $\langle n_1 n_2 | A_\mu^\dagger(0) | d\rangle$; a factor of M_d^2 has been inserted in the above expression to make the form factor dimensionless.

Writing down $\langle n_1 n_2 | A_\mu^\dagger(0) | d\rangle$ in the impulse approximation we find

$$\begin{aligned}
\langle n_1 n_2 | A_\mu^\dagger(0) | d\rangle = \Big\langle \psi_{nn} \Big| \sum_{i=1}^{2} [\gamma_\mu^{(i)} F_A(q^2; n \to p) \\
+ q_\mu F_P(q^2; n \to p)] \\
\times \gamma_5^{(i)} \tau^{-(i)} e^{i\vec{q}\cdot\vec{r}(i)} \Big| \psi_d \Big\rangle .
\end{aligned} \tag{48}$$

It is seen that Eq. (49) contains a term proportional to q_μ. Furthermore $F_P(q^2; n \to p)$ is known to have a pion pole[18] so that the expression Eq. (47) corresponds to the term proportional to $F_P(q^2; n \to p)$ in the impulse approximation.

We next consider the contribution to Eq. (40) from a state[19] $|\alpha\rangle$ with $J^P = 1^+$. For this state we write [in analogy with Eqs. (42) and (43)]

$$\langle \alpha | A_\mu^\dagger(0) | 0\rangle = -i\sqrt{2}\xi_\mu(\alpha) a_\alpha / [2\omega_\alpha(2\pi)^3]^{1/2}, \tag{49a}$$

$$\begin{aligned}
\langle n_1 n_2 | j_d^{\nu\dagger}(0) | \alpha\rangle = i(m/E_1)^{1/2}(m/E_2)^{1/2}(2\pi)^{-9/2} \\
\times \xi_\rho(\alpha) N^{\rho\nu}\bar{u}(p_1)\gamma_5 v(p_2),
\end{aligned} \tag{49b}$$

where $\xi_\mu(\alpha)$ is the polarization vector for the state $|\alpha\rangle$, a_α is the decay constant of $|\alpha\rangle$, ω_α is the energy of $|\alpha\rangle$, and $N^{\rho\nu}$ is to be determined. Under a parity transformation we find

$$\begin{aligned}\langle n_1, \vec{p}_1; n_2, \vec{p}_2 | j_d^{i\dagger}(0) | \alpha, \vec{p}_\alpha\rangle &= \langle n_1, \vec{p}_1; n_2, \vec{p}_2 | \mathcal{P}^{-1}\mathcal{P} j_d^{i\dagger}(0)\mathcal{P}^{-1}\mathcal{P} | \alpha, \vec{p}_\alpha\rangle \\ &= \langle n_1, -\vec{p}_1; n_2, -\vec{p}_2 | j_d^{i\dagger}(0) | \alpha, -\vec{p}_\alpha\rangle\ , \ i=1,2,3 \\ \langle n_1, \vec{p}_1; n_2, \vec{p}_2 | j_d^{0\dagger}(0) | \alpha, \vec{p}_\alpha\rangle &= \langle n_1, \vec{p}_1; n_2, \vec{p}_2 | \mathcal{P}^{-1}\mathcal{P} j_d^{0\dagger}(0)\mathcal{P}^{-1}\mathcal{P} | \alpha, \vec{p}_\alpha\rangle \\ &= -\langle n_1, -\vec{p}_1; n_2, -\vec{p}_2 | j_d^{0\dagger}(0) | \alpha, -\vec{p}_\alpha\rangle\ .\end{aligned} \tag{50}$$

From the form of $\langle n_1 n_2 | j_d^{\nu\dagger}(0) | \alpha\rangle$ given by Eq. (49b) and the requirement that Eq. (50) be satisfied, one finds the following conditions on $N^{\rho\nu}$:

$$\xi_\rho(-\vec{p}_\alpha)N^{\rho j}(-\vec{p}_1, -\vec{p}_2, -\vec{p}_\alpha) = \xi_\rho(\vec{p}_\alpha)N^{\rho j}(\vec{p}_1, \vec{p}_2, \vec{p}_\alpha)\ , \tag{51}$$

$$\xi_\rho(-\vec{p}_\alpha)N^{\rho 0}(-\vec{p}_1, -\vec{p}_2, -\vec{p}_\alpha) = -\xi_\rho(\vec{p}_\alpha)N^{\rho 0}(\vec{p}_1, \vec{p}_2, \vec{p}_\alpha)\ .$$

If we choose $N^{\rho\sigma} = g^{\rho\sigma}$, Eq. (51) can be satisfied since

$$\begin{aligned}&\xi^j(-\vec{p}_\alpha) = \xi^j(\vec{p}_\alpha), \quad j=1,2,3 \\ &\xi^0(-\vec{p}_\alpha) = -\xi^0(\vec{p}_\alpha)\ .\end{aligned} \tag{52}$$

With this choice for $N^{\rho\sigma}$ the contribution to Eq. (40) from the $|\alpha\rangle$ state is

$$\delta(q^2 - m_\alpha^2)\bar{u}\gamma_5 v[m_\alpha^2\sqrt{2}a_\alpha \pi f_{\alpha nnd}\,\xi_\mu(d) - \xi\cdot Q q_\mu\sqrt{2}a_\alpha\pi f_{\alpha nnd}]\ , \tag{53}$$

where m_α is the $|\alpha\rangle$ mass and $f_{\alpha nnd}$ is the α-neutron-neutron-deuteron coupling constant. The term in Eq. (53) which is proportional to $(\xi\cdot Q)q_\mu$ can be included in the term Eq. (47). Because there is a term in the expression Eq. (53) proportional to $\xi_\mu(d)$, we include a term proportional to $\xi_\mu(d)$ in the matrix element $\langle nn | A_\mu^\dagger(0) | d\rangle$, i.e.,

$$F_A(q^2, Q^2, P\cdot d)\bar{u}(p_1)\gamma_5 v(p_2)\xi_\mu(d)\ . \tag{54}$$

This term satisfies Eqs. (5) and (41). Thus we write for the matrix element of the axial-vector current

$$\langle n_1 n_2 | A_\mu^\dagger(0) | d\rangle = \eta\bar{u}(p_1)\left(F_A\xi_\mu + F_P\frac{\xi\cdot Q q_\mu}{M_d^2}\right)\gamma_5 v(p_2)\ . \tag{55}$$

A comparison of Eq. (55) with Eq. (48), the impulse approximation form of $\langle n_1 n_2 | A_\mu^\dagger(0) | d\rangle$, shows that in both cases to the lowest order, the F_P term does not contribute. A comparison of the cases which yield lowest order contributions [for example, $\langle n_1, n_2;\ S=0,\ S_z=0 | A_\mu^\dagger(0) | d;\ S=1,\ S_z=1\rangle$] indicates that Eq. (55) and Eq. (48) agree to the lowest order as expected. The results are shown in Table III.

The form factors $F_A(q^2, Q^2, P\cdot d)$ and $F_P(q^2, Q^2, P\cdot d)$ can now be determined. We first note that there is no direct information about $F_A(0, Q^2, P\cdot d)$ since the deuteron does not undergo β decay. Thus we shall use impulse-approximation-based results to obtain F_A at all necessary values of q^2. From Table III we see that

$$\begin{aligned}-2R(\vec{q})F_A^I(q^2) &= NF_A(q^2, Q^2, P\cdot d)/\sqrt{2} \\ &= NF(Q^2, P\cdot d)f_A(q^2)/\sqrt{2}\ ,\end{aligned} \tag{56}$$

where the superscript I refers to the impulse approximation and where we have made use of Eq. (32) to factorize $F_A(q^2, Q^2, P\cdot d)$. A theorem[2] based on the impulse approximation relates $f_A(q^2)/f_A(0)$ to $F_M(q^2)/F_M(0)$, where F_M is the weak magnetism form factor. From Table III we have that

$$F_V(q^2; n\to p) + F_M(q^2; n\to p) \propto f_1(q^2)\ , \tag{57}$$

but this is the only relation available which connects F_M with f_1 and f_2. We, therefore, cannot determine which linear combination of f_1 and f_2 corresponds to F_M. Therefore, we make a weaker assumption,[20] that the q^2 dependence of $f_A(q^2)$, $f_A(q^2)/f_A(0)$ is given by

$$\frac{f_A(q^2)}{f_A(0)} = \frac{f_1(q^2) - f_2(q^2)}{f_1(0) - f_2(0)}\ . \tag{58}$$

From line 1 of Table III the following expression is obtained:

$$2R(\vec{q})\,[F_V(q^2; n\to p) + F_M(q^2; n\to p)] = \frac{Ni}{\sqrt{2}}[f_1(q^2) - f_2(q^2)]F(Q^2, P\cdot d)\ . \tag{59}$$

Setting $q^2=0$ and solving for $2R(\vec{q})/N$, we find

$$2R(\vec{q})/N = \frac{i\sqrt{2}F(Q^2, P\cdot d)}{\sqrt{2}[F_V(0; n\to p) + F_M(0; n\to p)]}\ . \tag{60}$$

Substituting this expression in Eq. (56) and solving for f_A, we find

$$f_A(q^2) = F_A^I(q^2)\sqrt{2}/[F_V(0, n\to p) + F_M(0, n\to p)]\ , \tag{61}$$

where[21] $F_V(0, n\to p) = 1.00$ and $F_M(0, n\to p) = 3.70$. Writing

$$f_A(q^2) = \frac{F_A^I(0)[F_A^I(q^2)/F_A^I(0)]\sqrt{2}}{[F_V(0, n\to p) + F_M(0, n\to p)]}\ , \tag{62}$$

we have that

$$f_A(q^2)/f_A(0)=F_A^I(q^2)/F_A^I(0)\,. \qquad (63)$$

Using Eq. (58) and the fact that [from Eqs. (30), (32), and (39)]

$$f_a(q^2)-f_b(q^2)=\frac{f_1(q^2)-f_2(q^2)}{f_1(0)-f_2(0)}\,, \qquad (64)$$

we find

$$f_A(q^2)=\frac{F_A^I(0)[f_a(q^2)-f_b(q^2)]\sqrt{2}}{F_V(0,n\to p)+F_M(0,n\to p)}\,, \qquad (65)$$

but $F_A^I(0)=1.23\pm0.01$, so $f_A(q^2)$ is completely determined.

We next consider $F_P(q^2, Q^2, P\cdot d)$. Taking the matrix element of the axial-current divergence, we obtain

$$\langle n_1n_2|\,\partial^\mu A_\mu^\dagger(x)\,|d\rangle|_0=\eta\bar{u}(p_1)\left(F_A\xi\cdot Q+\frac{F_P}{M_d^2}q^2\xi\cdot Q\right)\times\gamma_5 v(p_2)\,. \qquad (66)$$

The hypothesis of partially conserved axial-vector current (PCAC), particularly an argument due to Nambu,[22] is used to obtain

$$F_P=-M_d^2F_A/(q^2-m_\pi^2) \qquad (67)$$

from Eq. (66). Thus the matrix element of the axial current is

$$\langle nn|\,A_\mu^\dagger(0)\,|d\rangle=\eta F_A(q^2,Q^2,P\cdot d)\bar{u}(p_1)\times\left[\xi_\mu-\frac{q_\mu\xi\cdot Q}{(q^2-m_\pi^2)}\right]\gamma_5 v(p_2)\,. \qquad (68)$$

V. THE DOUBLET AND QUARTET MUON-CAPTURE RATES IN DEUTERIUM

We now have determined M, the transition matrix element [Eq. (2)]. Since the muon and deuteron may be assumed to interact at rest in muon capture, the doublet and quartet spin states of the muon-deuteron system may be projected out by inserting into Eq. (2) the following projection operators:

$$P_d=\tfrac{1}{3}(\delta^{\alpha\beta})-\tfrac{1}{2}\sigma^{ij}S_{ij}^{\alpha\beta})\,,\qquad P_q=\tfrac{2}{3}(\delta^{\alpha\beta}+\tfrac{1}{4}\sigma^{ij}S_{ij}^{\alpha\beta})\,, \qquad (69)$$

with $\sigma^{\mu\nu}=\frac{1}{2}i[\gamma^\mu,\gamma^\nu]$ and $S_{ij}^{\alpha\beta}=g_i^\alpha g_j^\beta-g_i^\beta g_j^\alpha$, and where d stands for doublet and q for quartet. We obtain the following results for the doublet and quartet transition matrix elements squared:

$$|M|_d^2=\frac{2}{3m_\mu m_\nu m^2}\left\{F_A^2(p_1\cdot p_2+m^2)\left[9m_\mu E_\nu+\frac{6m_\mu^2\vec{Q}^2}{(q^2-m_\pi^2)}+\frac{\vec{Q}^2E_\nu m_\mu^3}{(q^2-m_\pi^2)^2}\right]+2\vec{Q}^2m_\mu E_\nu[F_1-F_2]^2\right\}, \qquad (70a)$$

$$|M|_q^2=\frac{4}{3m_\mu m_\nu m^2}\left\{F_A^2(p_1\cdot p_2+m^2)\frac{\vec{Q}^2E_\nu m_\mu^3}{(q^2-m_\pi^2)^2}+\tfrac{1}{2}(\vec{Q}^2m_\mu E_\nu)[F_1-F_2]^2\right\}, \qquad (70b)$$

where E_ν is the neutrino energy.

There is still one remaining problem. We know the form factors only in spacelike region. The phase-space integration necessary to obtain the capture rates Γ_d and Γ_q involves values of q^2 in the timelike region as well (in particular $m_\mu^2\geq q^2$). Thus some assumption must be made about the behavior of $f_a(q^2)-f_b(q^2)$ in the timelike region. The simplest assumption that can be made is to assume that in this timelike region the form factors may be analytically continued from the spacelike region so that we continue to have [see Eq. (39)]

$$|f_a(q^2)-f_b(q^2)|=1/(1-q^2/M^2)^2\,,\qquad M=224\pm25\text{ MeV}\,,\quad q^2\geq0\,. \qquad (71)$$

Using Eq. (71) to represent $f_a(q^2)-f_b(q^2)$ in the timelike region and using a computer to perform numerically the phase-space integrations, we obtain the following values for the doublet and quartet muon-capture rates, Γ_d and Γ_q, respectively:

$$\Gamma_d=374\pm18\text{ sec}^{-1}\,, \qquad (72a)$$

$$\Gamma_q=6.07\pm0.04\text{ sec}^{-1}\,. \qquad (72b)$$

We note that Γ_d and Γ_q are very sensitive[23] to the behavior of $F_A(q^2,Q^2,P\cdot d)$ near $q^2=-m_\mu^2$. For this value of q^2, we have $q_0=0$. From Eqs. (32), (35), (68), (70a), and (70b), as well as the discussion preceding Eq. (38), we see that F_A, and hence $|M|_d^2$ and $|M|_q^2$, go to infinity at this value. Using electrodisintegration data near threshold[16] to obtain a good fit for $F^2(\theta)$ [see Eqs.(35)–(37) *et seq.*] near $q^2=-m_\mu^2$, we replaced the factor $1/k=1/q_0$ in Eq. (35) by

$$\frac{1}{q_0}\to\frac{q_0+2.76\text{ MeV}}{(q_0+4.98\text{ MeV})^2} \qquad (73a)$$

to yield the values of Γ_d, Γ_q in Eqs. (72a), (72b). If another less good but possible fit is used, for example,

$$\frac{1}{q_0}\to\frac{1}{(q_0^2+20.25\text{ MeV}^2)^{1/2}}\,, \qquad (73b)$$

a value of $\Gamma_d = 490\ \mathrm{sec}^{-1}$ results. Thus, the results are very sensitive to the behavior of the form factors (particularly F_A) at $q^2 = -m_\mu^2$.

VI. CONCLUSION

The doublet muon-capture rate in deuterium has been determined experimentally by two groups.[24] Their results are given below:

$$\Gamma_d^{\mathrm{exp}} = 365 \pm 91\ \mathrm{sec}^{-1}; \tag{74a}$$

$$\Gamma_d^{\mathrm{exp}} = 451 \pm 70\ \mathrm{sec}^{-1}, \tag{74b}$$

where the superscript exp stands for experimental. Thus our result, Eq. (72a), is seen to be consistent with experiment.

Moreover, if we set the form factors equal to zero for $q^2 \geq 0$, we find a value for Γ_d of

$$\Gamma_d = 272\ \mathrm{sec}^{-1},\ q^2 \leq 0 \tag{75}$$

i.e., the timelike region contributes to Γ_d the amount 102 sec^{-1} or 27%. This is not a negligible amount. Although the accuracy of the experimental data is not sufficient to show that the form factors can be analytically continued from the spacelike to the timelike region, such a continuation is seen from Eq. (72a) and Eqs. (74a) and (74b) to be consistent with the experimental data.

In conclusion we note that if more accurate and complete electrodisintegration data were available in the accessible region ($0 \geq q^2 \geq -m_\mu^2$), particularly if $d\sigma/d\Omega_e\, d\Omega_p$ (where the subscript p stands for proton) were measured at a number of proton angles, it would be possible to obtain the form factors F_1 and F_2 free from any errors inherent in using impulse-approximation based results. If, in addition, more accurate measurements of the muon-capture rates were available, it would be possible to obtain much more accurate information about the form factors in the timelike region. We are currently continuing this work by calculating the cross section for the process $\nu_\mu + d \to p + p + \mu^-$, using the form factors we have obtained from studying muon-capture in deuterium.

ACKNOWLEDGMENTS

The author would like to thank Professor C. W. Kim for reading this manuscript, and to thank Professor C. W. Kim, Professor Gordon Feldman, and Professor Wayne Repko for useful conversations. He would also like to thank Professor Behram Kurşunoğlu for the kind hospitality extended to him at the Center for Theoretical Studies, Coral Gables.

---

*Research supported in part by the National Science Foundation and in part by the Air Force Office of Scientific Research Grant No. AFOSR 71-1993.

†Present address.

[1]For example, see C. W. Kim and H. Primakoff, Phys. Rev. 140, B566 (1965); J. Frazier and C. W. Kim, Phys. Rev. 177, 2568 (1969); C. W. Kim and S. L. Mintz, Phys. Lett. 31B, 503 (1970).

[2]C. W. Kim and H. Primakoff, Phys. Rev. 140, B566 (1965).

[3]There have been a number of treatments of muon capture in deuterium using various forms of the impulse approximation. See, for example, I-T. Wang, Phys. Rev. 139, B1539 (1965); E. Cremmer, Nucl. Phys. B2, 409 (1967); P. Pascual, R. Tarrach, and F. Vidal, Nuovo Cimento 12B, 241 (1973).

[4]H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento 1, 1425 (1955).

[5]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965), p. 115.

[6]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (Ref. 5), p. 116.

[7]R. P. Feynman and M. Gell-Mann, Phys. Rev. 109, 193 (1958).

[8]In the case of matrix elements of the form $\langle f | J_\mu^{(0)} | i \rangle$ where i and f are single-particle states and J_μ is either a vector or an axial vector, a transformation to a frame where the three-momenta $\vec{p}_i$ and $\vec{p}_f$ are collinear can be made. In this frame an angular-momentum argument can be used to show that only three form factors in general are necessary to describe the matrix element. For the case of two hadrons in the final state, these arguments cannot be used.

[9]See, for example, M. Chemtob and M. Rho, Nucl. Phys. A163, 1 (1971).

[10]In the impulse approximation both F_V and F_M properly normalized are of the same order of magnitude. We note $F_V(0,\ n \leftrightarrow p) = 1.00$ and $F_M(0,\ n \leftrightarrow p) = 3.70$.

[11]C. W. Kim and H. Primakoff, Phys. Rev. 139, B1447 (1965).

[12]See, for example, I-T. Wang, Phys. Rev. 139, B1539 (1965). The author would like to thank I-T. Wang for making details of his wave functions available via private communication.

[13]J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (Wiley, New York, 1952), pp. 608–613.

[14]E. A. Whalin, B. D. Schriever, and A. D. Hanson, Phys. Rev. 101, 377 (1956); D. R. Dixon and K. C. Bandtel, *ibid.* 104, 1730 (1956); J. A. Galey, *ibid.* 117, 763 (1960).

[15]The phenomenological form given by Eq. (37) is given in E. A. Whalin, B. D. Schriever, and A. D. Hanson, Phys. Rev. 101, 377 (1965), see p. 381. The forms given in Eq. (38) are the author's.

[16]G. A. Peterson and W. C. Barber, Phys. Rev. 128, 812

(1962); L. Katz, G. Ricca, T. E. Drake, and H. S. Kaplan, Phys. Lett. 28B, 114 (1968).

[17]J. Goldemberg and C. Schaerf, Phys. Lett. 20, 193 (1966).

[18]See, for example, C. W. Kim and M. Ram, Phys. Rev. 162, 1584 (1967).

[19]The state $|\alpha\rangle$ need not correspond to a real particle. It is possible to approximate the $J^P = 1^+$ contribution to Eq. (41) by a pole of appropriate mass. See Ref. 18.

[20]We note that in the impulse approximation only the nucleon form factors are used. For the nucleons

$$\frac{F_M(q^2, n \leftrightarrow p)}{F_M(0; n \leftrightarrow p)} \approx \frac{F_A(q^2, n \leftrightarrow p)}{F_A(0; n \leftrightarrow p)}$$

$$\approx (1 - q^2/M_A{}^2)^{-2},$$

with $M_A{}^2 \cong 1.1$ BeV^2. If, assuming a dipole fit, we write $[F_V(q^2; n \leftrightarrow p) + F_M(q^2; n \leftrightarrow p)]/[F_V(0; n \leftrightarrow p) + F_M(0; n \leftrightarrow p)] \approx (1 - q^2/M^2)^{-2}$, we find that $M^2 \cong 0.71$ BeV^2, so that

$$\frac{F_M(q^2; n \leftrightarrow p)}{F_M(0; n \leftrightarrow p)} \approx \frac{[F_V(q^2; n \leftrightarrow p) + F_M(q^2; n \leftrightarrow p)]}{[F_V(0, n \leftrightarrow p) + F_M(0; n \leftrightarrow p)]}.$$

Thus Eq. (58) may be a reasonably good assumption.

[21]J. Frazier and C. W. Kim, Phys. Rev. 177, 2560 (1969).

[22]Y. Nambu, Phys. Rev. Lett. 4, 380 (1960).

[23]We also note that the results for Γ_d and Γ_q depend on M, the mass used in F_A [see Eq. (39)]. Our value of $M = 224 \pm 25$ MeV is in reasonable agreement with an effective value of $M = 293$ MeV obtained from an impulse-approximation calculation, i.e.,

$$1/(1 + m_\mu{}^2/M^2)^2 = \frac{\left\langle \psi_f \left| \sum_{i=1}^{2} F_A(-m_\mu{}^2, n \leftrightarrow p) e^{i\vec{q}\cdot\vec{r}^{(i)}} \tau^{-(i)} \right| \psi_i \right\rangle \Big|_{(\vec{q}^2 = -m_\mu{}^2)}}{\left\langle \psi_f \sum_{i=2}^{2} F_A(0, n \leftrightarrow p) e^{i\vec{q}\cdot\vec{r}^{(i)}} \tau^{-(i)} \middle| \psi_i \right\rangle \Big|_{\vec{q}^2 = 0}}$$

[24]I-T. Wang *et al.* [Phys. Rev. 139, B1528 (1965)] give result (74a); A. Placci, E. Zavattini, A. Bertin, and A. Vitale [Phys. Rev. Lett. 25, 475 (1970)] give result (74b).

PHYSICAL REVIEW D VOLUME 8, NUMBER 9 1 NOVEMBER 1973

Prediction of the Width and Slope of the Decay $X^0 \to \eta\pi\pi$ by Finite Dispersion Relations*

R. H. Graham and Toaning Ng

Department of Physics, University of Toronto, Toronto, Canada

(Received 9 March 1973; revised manuscript received 23 June 1973)

A model of the $X^0 \to \eta\pi\pi$ decay amplitude is constructed using finite dispersion relations. The predicted decay width and energy dependence of the Dalitz plot are consistent with the latest data.

I. INTRODUCTION

Our understanding of scattering amplitudes has been greatly extended over the past few years through the application of Cauchy's theorem over (either effectively or explicitly) finite contours. In particular, finite-energy sum rules (FESR) have led to interesting relations between the high-energy (Regge) and low-energy (resonance) forms of scattering amplitudes.[1,2] In addition, the utility of finite dispersion relations (FDR), as a means of exploiting a knowledge of the Regge and resonance parts of an amplitude to determine its low-energy behavior, has come to be recognized.[3-6]

One of the most promising applications of FDR is to three-body decays, where the decay amplitude is related by crossing to the corresponding two-body scattering amplitude. This approach was used by Aviv and Nussinov[3] to describe the decay $\omega \to 2\pi\gamma$, with encouraging results. Later applications of FDR to $\eta \to \pi^0\gamma\gamma$ (Ref. 4) and $\eta \to \pi\pi\gamma$ (Ref. 5) have yielded results in good agreement with experiment.

In view of the above successes, we were led to apply FDR to the decay $X^0(957) \to \eta\pi\pi$, where the X^0 is assumed to have $J^P = 0^-$. Whereas previous attempts[7-15] to describe $X^0 \to \eta\pi\pi$ could predict only the width *or* the slope of the decay distribution, we have attempted to predict *both* the width *and* the slope. Our results are consistent with the latest data.

In Sec. II we give the details of our model for the $X^0 \to \eta\pi\pi$ decay amplitude. Our results are presented in Sec. III, where they are compared with the experimental data. Section IV contains a discussion of our predictions in which comparison is made with other theoretical work on $X^0 \to \eta\pi\pi$.

II. DETERMINATION OF THE AMPLITUDE

We begin by considering the two-body scattering process

$$X(p) + \pi(-q_1) \to \eta(k) + \pi(q_2) \tag{2.1}$$

(see Fig. 1). The respective momenta of the parti-