Implications of the Decay $\eta \to \pi^+\pi^-\pi^0$ for Chiral-Symmetry Breaking*

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To account for the rate and slope parameter for $\eta \to \pi^+\pi^-\pi^0$ within the framework of effective Lagrangians, we show that the kinetic Lagrangian must break chiral SU(3) [but not SU(3) itself], and the mass term must contain the (3, 3*) + (3*, 3) and at least one other representation [for example, the (8, 8)]. Implications for π-π scattering are discussed.

The decay mode $\eta \to \pi^+\pi^-\pi^0$ has been a puzzle for a long time.[1] Its Dalitz plot has been explained in terms of a simple pole model,[2] but its decay rate is much harder to understand.[3] Analyses based upon current algebra and chiral dynamics[4,5] have not yielded the correct partial width even though most of them have included chiral-symmetry breaking of the (3, 3*) kind. Here we show that the difficulty can be removed only when more general forms of symmetry breaking are introduced into the pole model.

Within the framework of nonlinear effective Lagrangians, our result can be stated in two parts. Firstly, if all chiral-symmetry breaking occurs in the mass term of the Lagrangian (which means among other things that the η-π^0 vertex is a constant), then we cannot obtain both the correct slope and the correct decay rate no matter how many representations we introduce. Secondly, if we break chiral symmetry [but not SU(3)] in the kinetic part of the Lagrangian, we still need an admixture of the (3, 3*) and one other representation in order to accommodate the properties of η decay; this point is particularly important when we extract π-π scattering lengths from the Lagrangian. The conservation of SU(3) in the kinetic Lagrangian is necessary for the η-decay amplitude to remain invariant under redefinitions of the meson field.

To begin the discussion we assume that the η-π^0 vertex is both independent of energy and purely electromagnetic in origin. Its strength is then related to electromagnetic mass differences in the usual way:

$$\sqrt{3}\,\langle \pi^0 |\mathcal{L}_u|\eta^0\rangle = m^2(\pi^+) - m^2(\pi^0) + m^2(K^0) - m^2(K^+)\,. \tag{1}$$

Since the electromagnetic current is the U-spin scalar member of the $(8, 1) \oplus (1, 8)$ representation of SU(3)×SU(3), we take the Lagrangian $\mathcal{L}_u$ to belong to the (8, 8) representation, and adjust the admixtures of SU(3) octet and (27)-plet terms so as to fit the empirical masses. This part of the Lagrangian will contribute both a pole term, $\eta \to \pi^0$, and a direct, or "background" term $\eta \to 3\pi$. Up to fourth order in the meson field, it is given by[6]

$$\mathcal{L}_u = -\frac{\sqrt{2}}{\sqrt{3}}\left(\alpha - \frac{2\sqrt{2}}{5}\beta\right)\left\{(\sqrt{3}\,\Pi_3 + \Pi_8)\left[1 - \frac{X}{F_0^{\,2}}\,(2F_0F_x + \tfrac{3}{4})\right] + \left(\frac{\sqrt{3}\,\pi_3 + \pi_8}{6F_0^{\,2}}\right)Y\right\}$$
$$-\tfrac{2}{3}\beta\left\{(\sqrt{3}\,\pi_3 + \pi_8)^2\left[1 - \frac{2X}{F_0^{\,2}}(F_0F_x + \tfrac{1}{3})\right] + \frac{1}{4F_0^{\,2}}\,(\sqrt{3}\,\Pi_3 + \Pi_8)^2\right\} + \beta\left[\tfrac{1}{3}X - \frac{X^2}{3F_0^{\,2}}\,(2F_0F_x + \tfrac{7}{12})\right], \tag{2}$$

where F_0 is the pion decay constant ($\simeq m_\pi/\sqrt{2}$) and F_x is the coefficient of the quadratic term in the expansion of the chiral commutator[6]

$$[K_a, \pi_b] = i\delta_{ab}(F_0 + XF_x + \cdots) + \cdots\,. \tag{3}$$

Also we have

$$X = \pi_a\pi_a;\quad Y = d_{abc}\,\pi_a\pi_b\pi_c\,,\ \ \Pi_a = d_{abc}\,\pi_b\pi_c\,, \tag{4}$$

and

$$2\sqrt{2}\,\alpha = \tfrac{2}{5}(m^2(\pi^0) - m^2(\pi^+)) + m^2(K^+) - m^2(K^0)\,,\quad 4\beta = m^2(\pi^0) - m^2(\pi^+)\,. \tag{5}$$

For the sake of generality, we take the isospin-invariant mass term of the Lagrangian to be a combination of the SU(3) singlet and octet components of the $(3,3^*)\oplus(3^*,3)$ representation, and of the $\{n,n^*\}\oplus\{n^*,n\}$ representation, where (n) denotes the SU(3) representation with characteristic numbers (μ_1,μ_2). The only constraint that we place upon the combination is that it yield the empirical masses of pseudoscalar mesons. Up to fourth order the mass Lagrangian is[6]

$$\begin{aligned}\mathcal{L}_B = &-\tfrac{1}{2}(A_3+A_n)X - \frac{\sqrt{3}}{2}(B_3+B_n+\tilde{B}_n)\Pi_8 + \frac{X^2}{F_0^{\,2}}\left[(A_3+A_n)F_0F_x + \tfrac{5}{24}A_3 + \tfrac{1}{40}(m_2+\tfrac{17}{3})A_n\right]\\ &+\sqrt{3}\,X\Pi_8\left[\tfrac{5}{24}B_3+\tfrac{1}{28}(m_2+\tfrac{19}{6})B_n+\tfrac{1}{21}\left(\frac{22m_2+57}{24}-\frac{2m_3^{\,2}}{m_2(2m_2+3)}\right)\tilde{B}_n+F_0F_x(B_3+B_n+\tilde{B}_n)\right]\\ &+\frac{\sqrt{3}\,\pi_8 Y}{3F_0^{\,2}}\left[\tfrac{1}{12}B_3+\tfrac{1}{28}(2m_2-3)B_n+\tfrac{1}{7}\tilde{B}_n\left(\frac{8m_3^{\,2}}{m_2(2m_2+3)}-\frac{2m_2+9}{12}\right)\right],\end{aligned} \tag{6}$$

where m_2 and m_3 are the Casimir eigenvalues of the representation (μ_1,μ_2).[6] The coefficients A_λ are associated with the SU(3) singlet parts of $(3,3^*)$ and (n,n^*), and the coefficients B_μ multiply the octet parts of these representations. [Notice that in general (n,n^*) contains two distinct octets.] In terms of these coefficients, the pseudoscalar masses are

$$\begin{aligned}&m_\pi^{\,2}+m_\eta^{\,2}=2(A_3+A_n),\\ &m_\pi^{\,2}-m_\eta^{\,2}=2(B_3+B_n+\tilde{B}_n),\\ &4m_K^{\,2}=m_\pi^{\,2}+3m_\eta^{\,2}.\end{aligned} \tag{7}$$

The chirally-invariant kinetic part of the Lagrangian is, up to fourth order in the meson field,[6]

$$\mathcal{L}_{\rm KE} = -\tfrac{1}{2}(\partial_\mu\pi_i)(\partial_\mu\pi_i)\left[1-\frac{2X}{F_0^{\,2}}(F_0F_x+\tfrac{1}{3})\right]+\frac{1}{2F_0^{\,2}}(\partial_\mu X)(\partial_\mu X)[F_0F_x+\tfrac{1}{6}]-\frac{1}{8F_0^{\,2}}\partial_\mu\Pi_k\partial_\mu\Pi_k\,, \tag{8}$$

and the total Lagrangian is

$$\mathcal{L}=\mathcal{L}_{\rm KE}+\mathcal{L}_B+\mathcal{L}_u\,. \tag{9}$$

Using $\mathcal{L}$ to evaluate the matrix elements for the processes

$$\eta\to\pi^0\to\pi^+\pi^-\pi^0\,, \tag{10a}$$

$$\eta\to\pi^+\pi^-\eta \;(\eta\to\pi^0)\,, \tag{10b}$$

$$\eta\to\pi^+\pi^-\pi^0\,, \tag{10c}$$

we find that the total amplitude for this η-decay mode is

$$A(\eta\to\pi^+\pi^-\pi^0)=g\left(1+h-\frac{2Q}{m_\eta}Y\right)\,, \tag{11}$$

where

$$\begin{aligned}g&=\frac{m_\eta^{\,2}}{3\sqrt{3}\,F_0^{\,2}(m_\eta^{\,2}-m_\pi^{\,2})}[m^2(\pi^0)-m^2(\pi^+)+m^2(K^+)-m^2(K^0)]\\ &\approx 0.12\end{aligned} \tag{12}$$

and

$$\begin{aligned}&hm_\eta^{\,2}=4\left[\frac{m_2}{3}-\frac{4m_3^{\,2}}{m_2(2m_2+3)}\right]\tilde{B}_n-m_\pi^{\,2}\,,\\ &Q=m_\eta-3m_\pi\,,\qquad Y=(3E_0-m_\eta)/Q\,.\end{aligned} \tag{13}$$

In Eq. (13), E_0 is the total energy of the neutral pion as measured in the η rest frame.

The first point to notice about the expression for $A(\eta\to\pi^+\pi^-\pi^0)$ is that it is independent of the parameter F_x of Eq. (2). Consequently the amplitude is independent of the way in which the meson field is defined.[6,7] The second point to notice is that the chiral-symmetry-breaking parameters (A_3, $\tilde{B}_n$, etc.) appear only in the constant term h of the amplitude [see Eqs. (11) and (13)]. Therefore we can either fit h to the observed slope of the Dalitz plot and then predict the decay rate, or we can fit h to the rate and then predict the slope. Only if we are very fortunate will these two procedures be consistent with one another.

To demonstrate our lack of fortune we note that the experimental value of the slope parameter

$$a=-2Q/m_\eta(1+h)\approx-0.5/(1+h) \tag{14}$$

is $-(0.540\pm0.007)$,[8] and it implies that h is very small ($h\approx-0.1$). The decay rate

$$\begin{aligned}\Gamma(\eta\to\pi^+\pi^-\pi^0)&=\frac{1}{384\sqrt{3}\,m_\eta}\left(\frac{gQ}{\pi}\right)^2\left[(1+h)^2+\frac{Q^2}{m_\eta^{\,2}}\right]\\ &\approx 67(1+h)^2\ \text{eV}\end{aligned} \tag{15}$$

for this value of h is 54 eV, well below the value of 600 eV given in the tables[9] and considerably smaller than a very recent measurement of 200 eV.[10] If we fit h to this latter rate, we find that $h\approx 0.7$ and therefore $\alpha\approx -0.30$; if we fit h to the larger rate ($h\approx 2$), we obtain an even smaller slope, $\alpha\approx -0.17$. Thus we cannot choose h, or equivalently the symmetry-breaking term $\mathfrak{L}_B$, in such a way as to fit both the rate and the Dalitz plot for η decay.

In order to rectify this situation we must introduce another parameter into the amplitude, and, to avoid merely redefining h in Eq. (11), we must ensure that the additional parameter contributes to the energy-dependent term. The only way of doing this is to break the chiral symmetry in the kinetic part of the Lagrangian. Now, detailed calculations[11] show that if this symmetry-breaking term also violates SU(3), then the resultant $\eta\to 3\pi$ amplitude will depend on the parameter F_x of Eq. (2), and hence upon the definition of the meson field. On the other hand, if the extra term is SU(3)-invariant, the amplitude will be independent of F_x. Therefore, to preserve invariance under redefinitions of the meson field, we take the chiral-symmetry-breaking part of the kinetic Lagrangian to be an SU(3) singlet; and we construct it from the $(3, 3^*)\oplus(3^*, 3)$ combination of $\partial_\mu z_a$ and $\partial_\mu \bar{z}_b$. To fourth order it is given by[6]

$$\mathfrak{L}'_{\mathrm{KE}} = -\frac{\alpha_3}{2}\partial_\mu\pi_i\partial_\mu\pi_i\left[1-\frac{2X}{F_0^{\,2}}(F_0F_x+\tfrac{1}{3})\right] + \frac{\alpha_3}{2F_0^{\,2}}\left[(F_0F_x)\partial_\mu X\partial_\mu X+\tfrac{1}{4}\partial_\mu\Pi_k\partial_\mu\Pi_k\right], \quad (16)$$

where α_3 is an arbitrary constant.

The presence of these additional terms forces us to "renormalize" the fields π_a so that the quadratic terms in $(\mathfrak{L}_{\mathrm{KE}}+\mathfrak{L}'_{\mathrm{KE}})$ take the standard form $(-\frac{1}{2}\partial_\mu\pi_i^R\partial_\mu\pi_i^R)$. Accordingly we define[12]

$$\pi_i^R=(Z)^{1/2}\pi_i\,,\quad Z=1+\alpha_3\,, \quad (17)$$

and we regard π_i^R rather than π_i as corresponding to physical mesons. We can still use the Lagrangians of Eqs. (2) and (6) as the mass terms, but because of the renormalization in Eq. (17), their coefficients are related to physical masses in a slightly different way. Thus for $\mathfrak{L}_u$ we now have

$$\begin{aligned}&2\sqrt{2}\,\alpha-\tfrac{8}{5}\beta=Z[m^2(K^+)-m^2(K^0)]\,,\\&4\beta=Z[m^2(\pi^0)-m^2(\pi^+)]\,,\end{aligned} \quad (18)$$

instead of Eq. (5), and for $\mathfrak{L}_B$ we have

$$\begin{aligned}&2(A_3+A_n)=Z(m_\pi^{\,2}+m_\eta^{\,2})\,,\\&2(B_3+B_n+\tilde{B}_n)=Z(m_\pi^{\,2}-m_\eta^{\,2})\,,\end{aligned} \quad (19)$$

rather than Eq. (7).

Using the modified Lagrangian we can now compute the amplitude for $\eta\to\pi^+\pi^-\pi^0$ corresponding to the three processes in Eq. (10). We obtain

$$A'(\eta\to\pi^+\pi^-\pi^0)=\frac{g}{Z}\left\{1+h'-\frac{2(2-Z)}{Z}\frac{Q}{m_\eta}\,y\right\}, \quad (20)$$

where g is defined in Eq. (12) and

$$m_\eta^{\,2}h'=\frac{4\tilde{B}_n}{Z}\left[\frac{m_2}{3}-\frac{4m_3^{\,2}}{m_2(2m_2+3)}\right]-m_\pi^{\,2}. \quad (21)$$

The empirical value of the slope parameter, namely, $\alpha\approx -0.5$, implies that

$$1+h'\approx\frac{2}{Z}-1\,, \quad (22)$$

and the decay rate is now given by [see Eqs. (15) and (20)]

$$\Gamma(\eta\to\pi^+\pi^-\pi^0)\approx\frac{70(1+h')^2}{Z^2}\ \mathrm{eV}\,. \quad (23)$$

For $Z=0.8$ and $h'\approx 0.5$ the rate is 250 eV, and for $Z=\frac{2}{3}$, $h'\approx 1$, the rate is 630 eV. These values of Z and h' imply that the first term in Eq. (21) does not vanish, and hence the mass Lagrangian contains a contribution from some representation other than the $(3, 3^*)$. The Casimir coefficient of $\tilde{B}_n$ vanishes when (n) is a triangular representation of SU(3), and so this additional representation cannot belong to the series $(6, 6^*)$, $(10, 10^*)$, ...; however, it could be the $(8, 8)$.[13]

Besides $\eta\to 3\pi$, our Lagrangian also gives rise to meson-meson scattering, and so we can examine its predictions for S-wave scattering lengths. For isoscalar and isotensor π-π scattering we find that

$$\begin{aligned}&a_0=(21Z+5C)/48\pi Z^2m_\pi\,,\\&a_2=(-6Z+2C)/48\pi Z^2m_\pi\,,\\&C\,m_\pi^{\,2}=2(A_8+2\tilde{B}_8)\,,\end{aligned} \quad (24)$$

where A_8 and $\tilde{B}_8$ are the coefficients in $\mathfrak{L}_B$ when the extra symmetry-breaking term belongs to the $(8, 8)$ [see Eqs. (6) and (19)]. These expressions contain a parameter, namely, A_8, that does not appear in the η-decay amplitude and so the scattering lengths are not completely determined by the values of Z and h'. If, however, we set $a_2=0$, we find that

$$a_0=\frac{0.26}{Z}m_\pi^{\,-1}=\begin{cases}0.32\,m_\pi^{\,-1} & (Z=0.8)\\ 0.39\,m_\pi^{\,-1} & (Z=\frac{2}{3})\,.\end{cases} \quad (25)$$

Both values are consistent with present experimental data,[14] but the smaller value, which corresponds to $\Gamma(\eta\to\pi^+\pi^-\pi^0)\approx 250$ eV, is more satisfactory than the larger one, for which $\Gamma(\eta \to \pi^+\pi^-\pi^0) \approx 630$ eV.

It is important to note that if we were to set the (3, 3*) mass term to zero (i.e., $A_3 = B_3 = 0$), then the parameters A_8 and $\tilde{B}_8$ would be completely determined from the mass relations of Eq. (19) [$B_8 = 0$ for the (8, 8)]. The values of h', a_0, a_2 would then be fixed, and moreover they would all be highly unsatisfactory ($h' \approx -4$, $a_0 \approx -1.5 m_\pi^{-1}$, and $a_2 \approx -0.7 m_\pi^{-1}$). We may therefore summarize what we have learned from the $\pi^+\pi^-\pi^0$ mode of η-decay by saying that the kinetic Lagrangian must break SU(3)×SU(3) [but not SU(3)], and that the mass Lagrangian must contain the $(3, 3^*) \oplus (3^*, 3)$ representation and one other representation such as the (8, 8). To find out more about this other representation we shall need accurate data on π-π, K-π, and K-K scattering.

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Pion-Nucleon Charge-Exchange Scattering in a Regge-Regge-Cut Model

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We have obtained a fit to most of the available data on pion-nucleon charge-exchange scattering using a model with Regge-Regge cuts. This shows quantitatively the importance of the double-Regge cuts, as opposed to the Regge-Pomeron cuts. We make comments on a recent work by Worden claiming that, in certain idealized limits, the Regge-Regge cuts in this process cancel. It is suggested that because of the strong assumptions in that work, many of which are probably not valid, the conclusions are not necessary. Comparisons are also made with several other works on this process.

I. INTRODUCTION

For a long time pion-nucleon charge-exchange scattering has been a favorite reaction for testing models of Regge poles and cuts. Recent polarization data[1] have, however, shown strong disagreement with the original predictions of two of the currently popular Regge-cut models, namely, the weak-cut[2] and the strong-cut models.[3] Both of these models in their unmodified form basically include the contributions of the Regge cut due to the simultaneous exchange of the ρ and the Pomeron trajectories and predict large negative polarization in the vicinity of $|t| \approx 0.4$ (GeV/c)^2.