Comments on Separable Potentials and Coulomb Interactions

M. Shah-Jahan*
Atomic Energy Center, P. O. Box-164, Dacca, Bangladesh
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The scattering of two charged particles via a separable potential has been discussed. It has been shown that equivalent local potentials for a large number of separable potentials in the presence of Coulomb interactions do not exist.

Separable potentials are extensively used in describing strong interactions of elementary particles and two- and three-body systems.[1-7] They are widely used because their use usually requires a small number of parameters to fit two-body scattering data and also because they can be handled in an extremely convenient manner. When the Coulomb interaction occurs along with the strong interaction, the solution of the two-body equation of quantum mechanics may not be as simple as the one in the absence of the Coulomb interaction. This problem has been treated in the past by various authors.[8,9] Recently Ali *et al.*[10] have considered this problem in the coordinate representation.

The purposes of this note are (i) to point out that even in the presence of the Coulomb interaction a neat exact expression relating the phase shift to the separable potential can easily be obtained in the momentum representation, and (ii) to show that an equivalent local potential (ELP) does not exist for a large number of (practically all interesting ones) separable potentials if the Coulomb interaction is present.

The radial Schrödinger equation for the strong interaction of two charged particles interacting via a separable potential is

$$\left(\frac{d^2}{dr^2}+k^2-\frac{l(l+1)}{r^2}-\frac{2nk}{r}\right)u_l(r)=\int_0^\infty \lambda q_l(r)q_l(r')u_l(r')dr', \quad (1)$$

where n and k are, respectively, the usual Coulomb parameter and the wave number.

The solution to this equation is

$$u_l(r)=F_l(kr)+a_l Z_l(r), \quad (2)$$

where

$$a_l=(\pi/2)^{1/2}\, q_{lC}(k)/(1-d_l), \quad (3)$$

$$q_{lC}(k)=(2/\pi)^{1/2}\int_0^\infty F_l(kr)q_l(r)dr,$$
$$d_l=\int_0^\infty Z_l(r)q_l(r)dr. \quad (4)$$

$Z_l(r)$ satisfies the equation

$$\left(\frac{d^2}{dr^2}+k^2-\frac{l(l+1)}{r^2}-\frac{2nk}{r}\right)Z_l(r)=\lambda q_l(r), \quad (5)$$

and is given by

$$Z_l(r)=\lambda\int_0^\infty G_l(r,r')q_l(r')dr', \quad (6)$$

with

$$G_l(r,r')=-\frac{1}{k}F_l(kr_<)H_l(kr_>), \quad (7)$$

$$H_l(kr)=G_l(kr)+iF_l(kr). \quad (8)$$

$F_l(kr)$ and $G_l(kr)$ are regular and irregular Coulomb functions, respectively. Now as $r\to\infty$,

$$F_l(kr)\sim\sin(kr-\tfrac{1}{2}l\pi-n\ln 2kr+\sigma_l), \quad (9)$$

$$G_l(kr)\sim\cos(kr-\tfrac{1}{2}l\pi-n\ln 2kr-\sigma_l), \quad (10)$$

where σ_l is the usual Coulomb phase shift.

We readily obtain

$$Z(r)\underset{r\to\infty}{\sim}-[\lambda(\pi/2)^{1/2}q_{lC}(k)]e^{i(kr-l\pi/2-n\ln 2kr+\sigma_l)}.$$

The S matrix is obtained from the asymptotic condition

$$u_l(r)\underset{r\to\infty}{\sim}-\frac{1}{2i}\left[e^{-i(kr-l\pi/2-n\ln 2kr+\sigma_l)}-S_l\, e^{i(kr-l\pi/2-n\ln 2kr+\sigma_l)}\right],$$
$$S_l=e^{2i\delta_l}, \quad (11)$$

whence we obtain

$$S_l=1-\lambda\frac{\pi}{k}iq_{lC}^2(k)/(1-d_l). \quad (12)$$

Now for elastic scattering,

$$e^{i\delta_l}\sin\delta_l=-\frac{\pi}{2k}\lambda q_{lC}^2(k)/(1-d_l). \quad (13)$$

Apart from a multiplicative factor, this is the two-body partial-wave on-shell T matrix. We now turn to the integral d_l:

$$d_l=\lambda\int_0^\infty\int_0^\infty G_l(r,r')q_l(r)q_l(r')drdr'. \quad (14)$$

The Green's function $G_l(r,r')$ has the following integral representation:

$$G_l(r,r') \underset{\epsilon\to 0}{=} \frac{2}{\pi}\int_0^\infty \frac{F_l(k'r')F_l(k'r)dk'}{k^2-k'^2+i\epsilon}\,. \qquad (15)$$

Using this representation for $G_l(r,r')$ we can write d_l as

$$d_l = \lambda\int_0^\infty \frac{q_{lC}{}^2(k')dk'}{k^2-k'^2+i\epsilon}\,. \qquad (16)$$

So we can write

$$e^{i\delta_l}\sin\delta_l = -\frac{\pi}{2k}\,\frac{\lambda q_{lC}{}^2(k)}{1-\lambda\int_0^\infty q_{lC}{}^2(k')dk'/(k^2-k'^2+i\epsilon)}\,. \qquad (17)$$

This result has the usual form of the two-body on-shell T matrix for separable potentials except for the fact that $q_{lC}(k)$ is the Coulomb transform of the form factor $q_l(r)$ instead of the usual Bessel transform. The same result would have been easily obtained by Harrington[8] if he had intended to write a formal solution to this equation for the Coulomb corrected two-body T matrix. Following Harrington[8] one can easily obtain the off-shell partial-wave T matrix for this problem. The off-shell T matrix turns out to be

$$T_l(k,k',s) = \frac{\lambda q_{lC}(k)q_{lC}(k')}{1-\lambda\int_0^\infty dq\, q_{lC}{}^2(q)/(s/2\mu - q^2+i\epsilon)}, \qquad (18)$$

where s is the energy.

We now consider the construction of ELP. Following Husain and Ali[11] and Husain,[12] we write down the ELP, $U_l(r)$ for a single-term separable potential when the Coulomb interaction is present:

$$U_l(r) = -\frac{q_l'(r)}{q_l(r)}\frac{g_l'(r)}{g_l(r)} + 3\left(\frac{g_l'(r)}{g_l(r)}\right)^2 + \frac{3}{2}\frac{I}{g_l{}^2}, \qquad (19)$$

where

$$I = \lambda B q_l(r)[b_l F'(kr) + a_l G_l'(kr)],$$
$$b_l = \frac{c_l}{1-d_l}, \quad c_l = -\int_0^\infty q_l(r)G_l(kr)dr, \qquad (20)$$

$$d_l = \lambda\int_0^\infty\int_0^\infty G_l(r,r')q_l(r)q_l(r')drdr', \qquad (21)$$

$$B = -\frac{1}{k+\lambda b_l q_{lC}(k)(\pi/2)^{1/2}},$$

$$g_l{}^2(r) = -B\left[k+\lambda b_l\int_0^r F_l(kr')q(r')dr' + \lambda a_l\int_r^\infty G_l(kr')q_l(r')dr'\right]. \qquad (22)$$

Here we shall keep in mind that the expression for the ELP has been written following the notations of Husain and Ali[11] and Husain[12] who use the Green's function

$$G_l(r,r') = -\frac{1}{k}F_l(kr_<)G_l(kr_>)\,. \qquad (23)$$

This Green's function satisfies boundary conditions for scattering processes, but cannot be readily extended to the bound-state case.

The derivations of Husain[12] and Husain and Ali[11] rest upon the fact that all the integrals involved in $U_l(r)$ are finite. In fact, c_l does not exist for a large number of choices of $q_l(r)$. The divergence originates from the singularities of $G_l(kr)$ for small kr. Only those $q_l(r)$ which can overcome the logarithmic singularities of $G_l(kr)$ near the origin yield a finite c_l. However, in most cases of interest, this is not so. We have tested the convergence of c_l for various useful choices of $q_l(r)$ available in the literature[5-7] and have always found that c_l diverges. Because of this divergence, the methods developed in Refs. 11 and 12 to obtain an ELP cannot be extended to the Coulomb case. The above mentioned formalism works in many cases in the absence of the Coulomb interaction simply because $G_l(kr)$, in the absence of the Coulomb interaction, does not have this logarithmic singularity near the origin but a simple type of singularity $\sim r^{-l}$, and quite often the structure of $q_l(r)$ is sufficient to overcome this. This, then, makes c_l finite.

We have shown that a simple expression for the Coulomb-modified two-body T matrix can be given in the momentum representation. This can be very easily obtained from Harrington's original formulation.[8] The approximations used in Refs. 8 and 9 are not essential for the numerical evaluation of the phase shifts. Thus, these works do not involve any essential approximation. Their approximations are simply to facilitate the numerical evaluation of physically interesting quantities.

We also have shown that the Coulomb-modified ELP of a large number of separable potentials does not exist. However, this fact is not very worrisome since the ELP is of academic interest and in no way governs the dynamics of the system.

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*Present address: Department of Physics, Indiana University, Bloomington, Indiana 47401.
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Errata

Hadronic Final States of Deep-Inelastic Processes in Simple Quark-Parton Models, John Kogut, D. K. Sinclair, and Leonard Susskind [Phys. Rev. D 7, 3637 (1973)].

1. Page 3655: At the end of Case 2 the following sentence should be added: The final mass-shell condition $(k_n+p)^2=m^2$ requires that $\alpha=\pm 1$.

2. Page 3656: In Case 2 the three parenthetic remarks "(or $Q-k_{j+}\leq\sim Q^{-\alpha}$)," "(or $Q-k_{i+}\leq\sim Q^{-\alpha}$)," and "[or $k_{i+}=Q+O(Q^{-\alpha})$]" should be deleted.

3. Page 3650: Directly before Acknowledgments, the following sentence should be inserted: We have recently had brought to our attention related work in the context of the nonperturbative parton model.[39,40]

4. Page 3657: The following two references should be added to the end of the list:

[39]P. V. Landshoff and J. C. Polkinghorne, Nucl. Phys. B33, 221 (1971).
[40]C. F. A. Pantin, Nucl. Phys. B46, 205 (1972).

Nonlinear Bootstrap in Dual Models, J. Maharana and R. Ramachandran [Phys. Rev. D 7, 3670 (1973)]. There is a need for a clarifying note and an erratum in the above paper. For the process $\pi^-K^+\to\pi^-K^+$ it is possible to consider the corresponding inclusive reaction $\pi^-+K^+\to K+$anything, rather than the reaction $\pi^-+K^+\to\pi^0+$anything described in Sec. II D. This then requires the use of strangeness-conservation sum rules, and the bootstrap equation (2.12), now applicable, takes the form

$$\sigma_t^{\text{Res}}(\pi^-K^+)=\sum_{K=K^+,K^0}\int d^3p_K\frac{d\sigma^{\text{Res}}}{d^3p_K}(\pi^-K^+\to K+x)-\sum_{\bar{K}=K^-,\bar{K}^0}\int d^3p_{\bar{K}}\frac{d\sigma^{\text{Res}}}{d^3p_{\bar{K}}}(\pi^-K^+\to\bar{K}+x)\,.$$

The main advantage in using this sum rule rather than the one used in text is that it justifies the neglect of the contribution arising from Fig. 2(d) to the right-hand side of Eq. (2.12). Apart from a different set of Chan-Paton factors, this introduces no change in the body of the paper. As a result there is a factor $\frac{1}{2}$ which multiplies the right-hand side of Eq. (3.9), and a factor 2 multiplies the right-hand side of Eq. (3.10).

This changes the result $G_{K^*K^*\pi}{}^2/4\pi$ from the value of 1.99 to

$$\frac{G_{K^*K^*\pi}{}^2}{4\pi}=3.98\,,$$

which is in fact in better agreement with the experimental value of 3.25 ± 1.25.

Similarly, for $\pi\eta\to\pi\eta$, described in Sec. II E, the charge-conservation sum rule may be used. The resulting change in the Chan-Paton factor is $\frac{1}{6}$, thus yielding a value for $G_{A_2\rho\pi}{}^2/4\pi$ given by

$$\frac{G_{A_2\rho\pi}{}^2}{4\pi}=610\text{ GeV}^{-4}$$

(instead of 101.68 GeV^{-4}), which is to be compared with the experimental number of 313.4 GeV^{-4}.

The present address of J. Maharana is as follows: Department of Physics, University of Tokyo, Tokyo, Japan.