[4]P. A. M. Dirac, Can. J. Math. 2, 129 (1950).

[5]I. Gerstein, R. Jackiw, B. W. Lee, and S. Weinberg, Phys. Rev. D 3, 2486 (1971).

[6]J. L. Anderson, Phys. Rev. 94, 703 (1954).

[7]L. D. Faddeev and V. N. Popov, Kiev Report No. ITP-67-36 (unpublished).

[8]B. W. Lee and J. Zinn-Justin [Phys. Rev. D 5, 3121 (1972)] and A. Slarnov [Kiev Report No. ITP-71-83E (unpublished)] have derived Ward identities for the Yang-Mills fields. The identities for the currents are simpler and very easy to obtain. If the "classical" part of the action $I=\int d^4x\,(\mathcal{L}-f[A^a_\mu])$ is invariant under gauge transformations of the first kind, then from $\delta I=0$ we can define the Noether current $\partial_\mu j^{\mu a}=(\delta I/\delta A^b_\mu)\times \delta A^{ab}_\mu$. Here $\delta A^{ab}_\mu=\epsilon^{acb}A^c_\mu=\hat{A}^{ab}_\mu$. If we add, say, a fermion field coupling and the corresponding sources η, $\bar\eta$, we have

$$\partial_\mu j^{\mu a}=\frac{\delta I}{\delta A^b_\mu}\hat{A}^{ab}_\mu+\frac{\delta I}{\delta\psi}\,iT^a\psi-\frac{\delta I}{\delta\bar\psi}iT^a\bar\psi$$

$$=\frac{\delta I'}{\delta A^b_\mu}\hat{A}^{ab}_\mu+\frac{\delta I'}{\delta\psi}iT^a\psi-\frac{\delta I'}{\delta\bar\psi}\,iT^a\bar\psi+\frac{\delta\, i\ln\Delta}{\delta A^b_\mu}\hat{A}^{ab}_\mu\,,$$

where $I'=I-i\ln\Delta[A^a_\mu]$. But since $\Delta[A^a_\mu]$ is gauge-invariant we can always write the last term as a divergence: $\partial_\mu\mathcal{J}^{\mu a}$. A Ward identity for this "total current" is now easily derived by performing a gauge transformation in the fields in Eq. (28). Since the measure is invariant and a change of variables cannot change the value of the integral, we can set the variation with respect to $u^a(x)$ equal to zero and finally obtain

$$-\frac{1}{g}\,\partial_\mu\int[dA^a_\mu][d\psi][d\bar\psi]\frac{\delta S}{\delta A^a_\mu}\exp(iS)$$

$$+\int[dA^a_\mu][d\psi][d\bar\psi]\left(\frac{\delta I'}{\delta A^b_\mu}\hat{A}^{ab}_\mu+i\frac{\delta I'}{\delta\psi}T^a\psi-i\frac{\delta I'}{\delta\bar\psi}T^a\bar\psi\right)\exp(iS)$$

$$-\int[dA^a_\mu][d\psi][d\bar\psi]\,(J^{\mu b}\hat{A}^{ab}_\mu+i\bar\eta T^a\psi-i\eta T^a\bar\psi)\exp(iS)=0\,.$$

The first term is equal to zero. Substituting our expression for $\partial_\mu(j^{\mu a}+\mathcal{J}^{\mu a})$, we obtain the desired Ward identity:

$$\int[dA^a_\mu][d\psi][d\bar\psi]\,\partial_\mu(j^{\mu a}+\mathcal{J}^{\mu a})\exp(iS)$$

$$-\int[dA^a_\mu][d\psi][d\bar\psi]\,(J^{\mu b}\hat{A}^{ab}_\mu+i\bar\eta T^a\psi$$

$$-i\eta T^a\bar\psi)\exp(iS)=0.$$

[9]Similarly in the case of the gravitational field there are no ghost loops in the analogous "gauge" $g^{\mu 3}=\delta^{\mu 3}$.

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Comments on Generalized Hamiltonian Dynamics*

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A reasonable generalization of Hamiltonian theory to $3m$-dimensional phase space suggests a geometrical structure giving the proper characteristic vector field. This structure, however, has only a single integral invariant, and implies no sensible generalization of either Poisson-bracket formalism, or Hamilton-Jacobi theory. Associated statistical mechanics and quantization are unlikely. The algebraic source of the difficulty is the lack of understanding of canonical expressions and classes of closed 3-forms.

I. INTRODUCTION

In an imaginative and suggestive paper, Nambu[1] has proposed generalization of classical Hamiltonian dynamics to $3m$-dimensional phase space. We have recently restudied Hamiltonian mechanics from the unifying geometric standpoint of modern differential geometry, in terms of which several quite deep insights are achieved.[2] We also had investigated possible generalization to $3m$-dimensional phase space, but for several reasons, connected with the insights into the geometrical structures involved, had been discouraged from pursuing the matter further. In the following we summarize the chief features of the geometrical approach, and then indicate the difficulties that must be met on generalization.

II. HAMILTONIAN STRUCTURES

We deal at first with structures related to a given vector field $\vec{V}$ in an even-dimensional manifold of n dimensions. When a holonomic dynamical system is said to have m degrees of freedom, its equations of motion are of second degree in m dependent variables and one independent variable, time. If the time variable enters autonomously, m auxiliary variables can be introduced so as to write the equations as a set of $n=2m$ first-order expressions that, geometrically, are the components of a vector field whose trajectories are the dynamical paths. If the time enters more generally or there is a nonholonomic constraint, both an auxiliary autonomous variable and a consistent equation of constraint can be introduced, so as again to achieve an autonomous set—vector components—in an even-dimensional space $[n = 2(m+1)]$.[2,3]

All classical integration and Hamiltonian techniques use scalars, vectors, and differential forms invariant with respect to the given field $\vec{V}$.[4] First consider invariant scalars, fields y, such that

$$\underset{\vec{V}}{\pounds} y = 0\,. \tag{1}$$

For scalars, vanishing of the Lie derivative is just the contraction $\vec{V}\lrcorner dy = 0$: These are co-moving coordinates, or classical first integrals (integral functions). Explicit solution for an independent set $\{y^i,\ i=1,\ldots,n-1\}$ would completely integrate the given problem expressed by $\vec{V}$.

A closed 2-form of maximum class invariant with respect to $\vec{V}$ can be constructed from a complete set of y^i and one more, independent, scalar field s chosen with constant normalization $\vec{V}\lrcorner ds = 1$. We denote the last one of the invariant scalars y^{n-1} as H and set

$$\phi \equiv dy^1\wedge dy^2 + dy^3\wedge dy^4 + \cdots + dy^{n-3}\wedge dy^{n-2} + dH\wedge ds\,, \tag{2}$$

so

$$\vec{V}\lrcorner\phi = -dH,\quad d\phi = 0,\quad \underset{\vec{V}}{\pounds}\phi = 0\,. \tag{3}$$

The first of these is Hamilton's equations of motion in differential geometric terminology. It determines the field $\vec{V}$ as the *characteristics* of the set of forms ϕ, dH (the Hamiltonian *structure*[5]).

Now the theorems of Pfaff and Darboux tell us that any closed, maximum-class 2-form can be expressed in canonical form as

$$\phi = dp_a\wedge dq^a \quad a = 1,\ldots,\tfrac{1}{2}n\,. \tag{4}$$

Since the p_a, q^a are a complete set of scalars, the scalar field H may be expressed functionally as $H(p_a, q^a)$. dH becomes $H_{,p_a}dp_a + H_{,q^a}dq^a$. The equations of motion take the customary canonical form when this is introduced into Eq. (4):

$$\begin{aligned} \dot{p}_a &\equiv \vec{V}\lrcorner dp_a = -H_{,q^a}\,,\\ \dot{q}^a &\equiv \vec{V}\lrcorner dq^a = H_{,p_a}\,. \end{aligned} \tag{5}$$

This construction makes clear that there is nothing mathematically very distinctive about the Hamiltonian H. *To any first integral* may be associated a "symplectic" 2-form ϕ, generating its associated sets of canonical scalar fields p_a, q^a, and so leading to all consequent formal results of classical Hamiltonian dynamics. The successes of statistical mechanics, built heuristically on this classical deterministic foundation, must surely, however, be due to the choice of a *particular* first integral for H, viz., the energy. Energy is known at a deeper level to be an invariant scalar by virtue of its connection with the symmetry group of the background space-time geometry—it must be this that provides the essentially physical justification for the success of coarse graining. Similarly, in quantization of a classical Hamiltonian system, deeper physical criteria must be invoked as to which first integral or integrals are to be preferred as Hamiltonian functions.

If more than one first integral is for some reason to be distinguished in a generalized Hamiltonian structure, it is natural to introduce an invariant 3-form (or 4-form, etc.). The present suggestion is to consider the case $n = 3m$, and construct the following set of forms as a generalized Hamiltonian structure:

$$dH\wedge dG\,,$$

$$\psi \equiv dy^1\wedge dy^2\wedge dy^3 + dy^4\wedge dy^5\wedge dy^6 + \ldots + dy^{n-5}\wedge dy^{n-4}\wedge dy^{n-3} + dH\wedge dG\wedge ds\,. \tag{6}$$

Here again $y^1, y^2, \ldots, y^{n-3}, H, G$ are first integrals, the last two arbitrarily distinguished, and as before the set is completed with s, chosen so that $\vec{V}\lrcorner ds = 1$. If n is not divisible by three, this can as before be arranged with auxiliary variables and compatible constraints (e.g., $H=0$, $G=0$). Again,

$$\vec{V}\lrcorner\psi = dH\wedge dG,\quad d\psi = 0,\quad \underset{\vec{V}}{\pounds}\psi = 0\,. \tag{7}$$

Now unfortunately no adequate theory of a canonical expression for general closed 3-forms appears to be known. If we assume, however, that because of the construction (6) this ψ may be written in a complete set of adapted coordinates as

$$\psi = dp^A\wedge dq^A\wedge dr^A,\quad A = 1,\ldots,\tfrac{1}{3}n\,, \tag{8}$$

then substituting into the first of Eq. (7) gives

$$\begin{aligned} \dot{p}^A &\equiv \vec{V}\lrcorner dp^A = H_{,q^A}G_{,r^A} - H_{,r^A}G_{,q^A}\,,\\ \dot{q}^A &\equiv \vec{V}\lrcorner dq^A = H_{,r^A}G_{,p^A} - H_{,p^A}G_{,r^A}\,,\\ \dot{r}^A &\equiv \vec{V}\lrcorner dr^A = H_{,p^A}G_{,q^A} - H_{,q^A}G_{,p^A}\,, \end{aligned} \tag{9}$$

which is the posulated set of Nambu.[1] In the case $n = 3$ he finds a Liouville theorem. Also the variation of any scalar F along $\vec{V}$ is

$$\dot{F} \equiv \vec{V}\lrcorner dF = \sum_A \frac{\partial(F,H,G)}{\partial(p^A,q^A,r^A)}\,, \tag{10}$$

which looks like a sort of generalized Poisson bracket of three scalar fields. Nevertheless, we will see that neither a generalized theory of Poincaré's integral invariants, Poisson-bracket formalism, nor Hamilton-Jacobi theory go through satisfactorily for $n > 3$.

III. INTEGRAL INVARIANTS

If the boundary of a p-dimensional subspace is infinitesimally displaced by mapping points through $\epsilon\vec{V}$ along the congruence $\vec{V}$, (ϵ an infinitesimal constant), the numerical value of the integral of a p-form over the subspace is changed precisely by ϵ times the integral of the Lie derivative of the p-form. Thus we have immediately from the vanishing of the Lie derivative in Eq. (3) that the following integrals are invariant when their boundaries are mapped along $\vec{V}$:

$$\int_2 \phi,\ \int_4 \phi\wedge\phi,\ \ldots,\ \int_n \phi\wedge\phi\wedge\cdots\ \phi\,, \tag{11}$$

where the last integrand consists of $\frac{1}{2}n$ factors. Since ϕ is of maximum class, these all exist. These are Poincaré's absolute integral invariants in their most general form. Other integral invariants can easily be constructed from ϕ, H, and dH, and so-called *relative* integral invariants follow from the Stokes theorems. The last of Eq. (11) is taken over, in statistical mechanics, as Liouville's theorem of conservation of *a priori* probability.

In the proposed generalized Hamiltonian mechanics, it at first appears that we have, from (7), the analogous invariant integrals with respect to $\vec{V}$:

$$\int_3 \psi,\ \int_6 \psi\wedge\psi,\ \ldots,\ \int_n \psi\wedge\psi\wedge\cdots\ \psi, \tag{12}$$

where the last integrand consists of $\frac{1}{3}n$ factors. Since ψ is a 3-form, however, only the first of these is nonvanishing. Only in the case $n = 3$ does there seem to be a generalization of the Liouville theorem.

IV. CANONICAL TRANSFORMATION AND POISSON BRACKETS

Substitution of new canonical variables which preserve the canonical form (4) of ϕ is denoted canonical transformation (CT). The functional form of H is thereby changed. CT's form an infinite continuous group which, as is well known, can be found from generating potentials. The infinitesimal canonical transformations near the identity are generated by vector fields $\vec{U}$ satisfying

$$\underset{\vec{U}}{\pounds}\phi = 0\,. \tag{13}$$

An interesting Lie group contained in this is the symplectic group $\mathrm{Sp}(\frac{1}{2}n)$, restricted by the condition that in at least one canonical coordinate frame the components of $\vec{U}$ are linear in the coordinates

$$\underset{\vec{U}}{\pounds} p_a = A_a{}^b p_b + B_{ab} q^b, \quad a, b = 1, \ldots, \tfrac{1}{2}n$$

$$\underset{\vec{U}}{\pounds} q^a = C^{ab} p_b + D_b{}^a q^b\,. \tag{14}$$

Substituting into (13) imposes $\frac{1}{2}n(n-1)$ conditions on the n^2 constants, so one finds $\frac{1}{2}n(n+1)$ independent $\vec{U}$'s.

Equation (13) is the integrability condition for the association of a scalar field u to the vector $\vec{U}$, according to

$$\vec{U}\lrcorner\phi = du\,; \tag{15}$$

u is determined up to an additive constant. (15) may be read as the lowering of a contravariant index on the vector $\vec{U}$ to give the covariant gradient field du. The process is invertible: From a du one can find a unique $\vec{U}$, as is easily seen using the maximum-class expression for ϕ, Eq. (4).

Now any two such CT vector fields, $\vec{U}$ and $\vec{V}$, define a third, $\vec{W}$, by their Lie product (in this paragraph $\vec{V}$ is *any* CT vector)

$$[\vec{U}, \vec{V}] = \vec{W}\,. \tag{16}$$

$\vec{W}$ is again a CT vector, since

$$\underset{\vec{W}}{\pounds}\phi \equiv \underset{\vec{U}}{\pounds}\underset{\vec{V}}{\pounds}\phi - \underset{\vec{V}}{\pounds}\underset{\vec{U}}{\pounds}\phi = 0\,.$$

Also we note that the Jacobi identity is satisfied by such products. Taking the Lie derivative of (15) with respect to $\vec{V}$ gives

$$\underset{\vec{V}}{\pounds} du = \underset{\vec{V}}{\pounds}(\vec{U}\lrcorner\phi) = [\vec{V}, \vec{U}]\lrcorner\phi + \vec{U}\lrcorner\underset{\vec{V}}{\pounds}\phi$$

$$= -\vec{W}\lrcorner\phi = -dw\,. \tag{17}$$

But the operations d and $\pounds$ commute. The scalar field w associated to $\vec{W}$ is thus defined, up to a constant, in terms of u and v by

$$w = -\vec{V}\lrcorner du = \vec{U}\lrcorner dv\,. \tag{18}$$

Moreover, in the other order, v and u determine $-w$. The usual and convenient notation for the relation between these scalar fields u, v, w is to write, in analogy with (16)

$$[u, v] = w\,. \tag{19}$$

We then readily prove further relations such as

$$[uv, w] = u[v, w] + v[u, w]\,. \tag{20}$$

This is the symbolism of Poisson brackets, showing that in a space with a symplectic structure ϕ, there is a natural association of vectors—mapping operators—to scalar fields.

The given field $\vec{V}$ is of course also a CT vector, satisfying Eq. (13), belonging by (3) to the scalar field $-H$. Hence for any other scalar field, u, we have from (18) and (19)

$$\dot{u} \equiv \vec{V} \lrcorner du = -w = [u, H] \ . \tag{21}$$

It is this particular case of Poisson bracket that might be generalized as Eq. (10).

Now the difficulty is that most of this just does not go through with the generalized Hamiltonian structure (6), in particular with the 3-form ψ. Generalized CT vector fields, say, $\vec{U}$ can be defined by requiring

$$\underset{\vec{U}}{\pounds}\psi = 0 \ , \tag{22}$$

and this is indeed the integrability condition for associating a 1-form u to $\vec{U}$ by writing

$$\vec{U} \lrcorner \psi = du \ . \tag{23}$$

However, the u is thereby algebraically quite specialized, and it is clear that the *converse association* cannot work for a general u. Moreover, this is for 1-forms—not scalar fields as before—and no expression generalizing Eq. (21) so as to yield Eq. (10) seems to exist. There appear to be no clues at all in the generalized formalism for associating operators to scalar fields.

V. HAMILTON-JACOBI THEORY

The set of forms dH, ϕ is called a Hamiltonian structure in even-dimensional n-space. Without belaboring details, we record that the Cartan characters[6] describing the Cauchy integrations to find regular integral manifolds of this set are $s_0 = 1$, $s_1 = 1, \ldots, s_{(n-2)/2} = 1$, and it follows that the *genus* g (dimensionality of the maximum-dimension regular integral manifold) is $g = \frac{1}{2}n$. The set of $\frac{1}{2}n$ variables $q_1, \ldots, q_{n/2}$ are found to be *in involution*, which means that they can be varied independently in these maximum regular integral manifolds—thus the remaining variables, $p_1, \ldots, p_{n/2}$, become dependent variables. This is the geometric basis for the existence of the Hamilton-Jacobi partial differential equation. It implies further the existence of an associated variational principle for the characteristics of the Hamiltonian structure (the trajectories of $\vec{V}$), since by a theorem of Cartan these must lie in the maximum regular integral manifolds.

Hamilton-Jacobi theory is closely connected with Schrödinger quantization, in which the maximum regular integral manifolds acquire amplitude and phase qualities. We must ask if this theory can be clearly generalized to $3m$-dimensional phase space. The form structure of Eq. (7) is one possibility (we have discovered no better), and it fails this test. For $n = 6$, we find $s_0 = 0$, $s_1 = 1$, $s_2 = 1$, $s_3 = 0$, and $g = 4$. For $n = 9$, $s_0 = 0$, $s_1 = 1$, $s_2 = 1$, $s_3 = 2$, $s_4 = 0$, and $g = 5$. For $n = 12$, $s_0 = 0$, $s_1 = 1$, $s_2 = 1$, $s_3 = 2$, $s_4 = 2$, $s_5 = 0$, and $g = 6$. This is not yet sufficient to establish the pattern for higher dimensionalities, but clearly no certain set of canonical variables *in involution* (such as the q_A) are at hand to be adopted as independent, in a generalized Hamilton-Jacobi theory.

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[1]Yoichiro Nambu, Phys. Rev. D 7, 2405 (1973).

[2]F. B. Estabrook and H. D. Wahlquist, report, 1973, (unpublished).

[3]J. L. Synge, in *Encyclopedia of Physics* (Springer, Berlin, 1960), Vol. III/1.

[4]B. K. Harrison and F. B. Estabrook, J. Math. Phys. 12, 653 (1971).

[5]C. Godbillon, *Géométrie Différentielle et Mécanique Analytique* (Hermann, Paris, 1969); R. Abraham, *Foundations of Mechanics* (Benjamin, New York, 1967).

[6]É. Cartan, *Les Systèmes Différentiels Extérieurs et Leurs Applications Géométriques* (Hermann, Paris, 1945).