Quantization of the Yang-Mills Field in the Null-Plane Frame*

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The massless Yang-Mills field is quantized in the null-frame gauge $A_-^a = 0$, and Feynman rules are derived.

INTRODUCTION

Because of the current popularity enjoyed by Yang-Mills fields and light-cone techniques in particle physics, it might be of some interest to present here the quantization of such a field in the null-plane frame. The corresponding formalism for the case of the electromagnetic field has been given in Ref. 1. Here we shall confine ourselves to the self-interacting massless Yang-Mills field.[2] The coupling to other fields presents no difficulties.

We introduce the standard null coordinates

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^3), \quad \vec{x} = (x^1, x^2),$$

and the null metric

$$g_{+-} = g_{-+} = 1, \; g_{++} = g_{--} = 0,$$
$$g_{ij} = -\delta_{ij}, \quad i,j = 1,2.$$

The invariant inner product takes the form

$$A_\mu A^\mu = 2A_+A_- - A_iA_i \; .$$

I. CANONICAL QUANTIZATION

The Lagrangian for the massless Yang-Mills field is

$$\mathcal{L} = -\tfrac{1}{2}G^{\mu\nu a}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\hat{A}_\mu^{ab}A_\nu^b) + \tfrac{1}{4}G^{\mu\nu a}G_{\mu\nu}^a , \tag{1}$$

where $\hat{A}_\mu^{ab} \equiv \epsilon^{acb}A_\mu^c$. ϵ^{abc} are the structure constants of the internal-symmetry group. $\mathcal{L}$ is invariant under

$$A_\mu^a(x) \to A_\mu^{au}(x) = A_\mu^a(x) - \frac{1}{g}D_\mu^{ab}(x)u^b(x) + O(u^2), \tag{2}$$

where D_μ^{ab} is the covariant derivative operator $D_\mu^{ab}(x) = \delta_{ab}\,\partial_\mu + g\hat{A}_\mu^{ab}(x)$. We quantize in the null-plane gauge:

$$A_-^a = 0 , \tag{3}$$

where the field equations are

$$D_k^{ab}G^{+ka} = 0, \quad k = 1, 2, -, \tag{4}$$

$$D_\mu^{ab}G^{\mu ib} = 0, \quad i = 1, 2 . \tag{4a}$$

Equation (4a) is actually a constraint equation that can be solved for A_+^a:

$$A_+^a = -(\partial_-)^{-2}D_i^{ab}G^{+ib}, \quad i = 1, 2 . \tag{5}$$

To quantize the classical theory the commutators must be specified on the null plane. We cannot, however, adopt the usual canonical commutators postulated in the case of quantization on spacelike surfaces because canonically conjugate variables that are independent in one case may be related in the other. We may be able to guess the correct commutators, but we could also start with a formalism that can take into account constraint relations.[3,4] Such a formalism is presented in the Appendix and its somewhat tedious application to the Yang-Mills field gives [from Eq. (A8)]

$$[A_i^a(x), A_j^b(y)]_{x^+=y^+} = -\tfrac{1}{4}i\,\delta_{ij}\delta_{ab}\,\epsilon(x^- - y^-)\delta^2(\vec{x}-\vec{y}), \tag{6a}$$

$$[G^{+ia}(x), A_j^b(y)]_{x^+=y^+} = \tfrac{1}{2}i\,\delta_{ij}\delta_{ab}\,\delta(x^- - y^-)\delta^2(\vec{x}-\vec{y}), \tag{6b}$$

$$[G^{+ia}(x), G^{+jb}(y)]_{x^+=y^+} = \tfrac{1}{2}i\,\delta_{ij}\delta_{ab}\,\partial_-\delta(x^- - y^-)\delta^2(\vec{x}-\vec{y}). \tag{6c}$$

The corresponding Hamiltonian density is

$$\mathcal{H} = \tfrac{1}{4}G^{ija}G_{ij}^a - \tfrac{1}{2}(\partial_- A^{-a})(\partial_- A^{-a}) - (\partial_- A^{ia})(\partial_i A^{-a}) + g(\partial_- A_i^a)\hat{A}_+^{ab}A^{ib} . \tag{7}$$

Symmetrizations between products will not be indicated explicitly. A lengthy calculation shows that with the above commutation rules the correct Heisenberg equations of motion are satisfied.

Further details such as the construction of the full energy-momentum tensor and the corresponding demonstration of the consistency of the commutation relations are straightforward and completely analogous to the electromagnetic case.[1]

To do scattering theory we will need to split H into a free part and an interaction part. Since A_+^a contains part of the interaction we set

$$A_+^a = \mathcal{A}_+^a + \phi^a , \tag{8}$$

where

$$\mathcal{A}_+^a = -(\partial_-)^{-2}\partial_i\,\partial_- A^{ai}\,, \tag{8a}$$

$$\phi^a = -(\partial_-)^{-2}(g\hat{A}_i^{ab}\,\partial_- A^{bi})\,. \tag{8b}$$

We may also define $\mathcal{A}^{ia} \equiv A^{ia}$, and $\mathcal{A}_-^a \equiv A_-^a = 0$ and $F^{\mu\nu a} = \partial^\mu \mathcal{A}^{\nu a} - \partial^\nu \mathcal{A}^{\mu a}$. Using these definitions and the constraint equation satisfied by A_+^a we find after some manipulation of the various terms of H that

$$H = H_0 + H_I\,, \tag{9}$$

where

$$H_0 = \int d^2x\,dx^-\,(\tfrac{1}{4}F^{ija}F_{ij}^a + \tfrac{1}{2}\mathcal{A}^{-a}\,\partial_-\partial_i\,\mathcal{A}^{ia}) \tag{9a}$$

and

$$\begin{aligned} H_I = & \int d^2x\,dx^-[\tfrac{1}{2}gF^{\mu\nu a}\hat{\mathcal{A}}_\mu^{ab}\mathcal{A}_\nu^b + \tfrac{1}{4}g^2(\hat{\mathcal{A}}^{\mu ab}\mathcal{A}^{\nu b})(\hat{\mathcal{A}}_\mu^{ac}\mathcal{A}_\nu^c)] \\ & - \int d^2x\,dx^- d\xi \tfrac{1}{4}g^2\,|\,x^- - \xi\,|\,(\hat{\mathcal{A}}_\mu^{ab}\partial_-\mathcal{A}^{b\mu})(x^+, \xi, \vec{x}) \\ & \times(\hat{\mathcal{A}}_\nu^{ac}\,\partial_-\,\mathcal{A}^{c\nu})(x^+, x^-, \vec{x})\,. \end{aligned} \tag{9b}$$

II. INTERACTION PICTURE AND FEYNMAN RULES

We may now pass to the interaction picture. We denote the interaction representation fields also by $\mathcal{A}_\mu^a$ with $\mathcal{A}_-^a = 0$. They satisfy free-field equations of motion:

$$\partial_\mu F^{\mu ia}(x) = 0\,, \tag{10}$$

with

$$\mathcal{A}_+^a = -(\partial_-)^{-2}\partial_i\,\partial_-\mathcal{A}^{ia} \tag{11}$$

and canonical commutation relations

$$[\mathcal{A}^{ia}(x), \mathcal{A}^{ib}(y)]_{x^+=y^+} = -\tfrac{1}{4}i\,\delta_{ab}\delta_{ij}\,\epsilon(x^- - y^-)\delta^2(\vec{x} - \vec{y})\,. \tag{12}$$

From the definition of $\mathcal{A}_+^a$ it then follows that

$$[\mathcal{A}^{-a}(x), \mathcal{A}^{ib}(y)]_{x^+=y^+} = -\tfrac{1}{4}i\,\delta_{ab}\,|\,x^- - y^-\,|\,\partial^i\delta^2(\vec{x} - \vec{y}). \tag{13}$$

Using Eqs. (10) – (13) it is straightforward to verify that the free part of H generates the correct equations of motion:

$$i[H_0, \mathcal{A}^{ia}] = \partial_+\mathcal{A}^{ia}\,. \tag{14}$$

From the free-field equations for $\mathcal{A}^{ia}$ and the definition (11), it follows that we can write free-field expansions for $\mathcal{A}^{\mu a}$ with the null-frame polarization vectors

$$\begin{aligned} \epsilon^\mu(p,1) &= \frac{1}{p_-}\,(0, p_-, 0, p^1), \\ \epsilon^\mu(p,2) &= \frac{1}{p_-}(0, 0, p_-, p^2)\,. \end{aligned} \tag{15}$$

The propagator is then found to be

$$\begin{aligned} \Delta^{\mu\nu ab}(x' - x) &= -i\langle 0\,|\,T_+(\mathcal{A}^{\mu a}(x')\mathcal{A}^{\nu b}(x))\,|\,0\rangle \\ &= -i[\langle 0\,|\,\mathcal{A}^{\mu a}(x')\mathcal{A}^{\nu b}(x)\,|\,0\rangle\,\theta(x^{+\prime} - x^+) + \langle 0\,|\,\mathcal{A}^{\nu b}(x)\mathcal{A}^{\mu a}(x')\,|\,0\rangle\,\theta(x^+ - x^{+\prime})] \\ &= \delta_{ab}\int \frac{d^4p}{(2\pi)^4}\,\frac{e^{-ip\cdot(x'-x)}}{p_\mu p^\mu + i\epsilon}\left[-g^{\mu\nu} + \frac{\delta_-^\mu p^\nu + \delta_-^\nu p^\mu}{p_-} - \frac{2p_-p_+ - \vec{p}^{\,2}}{p_-^{\,2}}\,\delta_-^\mu\delta_-^\nu\right]. \end{aligned} \tag{16}$$

Equation (9b) is to be used as the interaction Hamiltonian in the S matrix. The first two terms of this are the usual three- and four-point vertices of the Yang-Mills field. The third term represents an instantaneous "Coulomb"-type interaction.

It is also easy to see that the only vacuum expectation values additional to Eq. (16) we shall actually need in applying Wick's theorem to the expansion of $S = T_+[\exp(-i\int dx\,\mathcal{H}_I)]$ are

$$\langle 0\,|\,T_+(F^{+ia}(x)\mathcal{A}_j^b(y))\,|\,0\rangle = \partial_-^x\langle 0\,|\,T_+(\mathcal{A}^{ia}(x)\mathcal{A}_j^b(y))\,|\,0\rangle\,, \tag{17a}$$

$$\langle 0\,|\,T_+(F^{+ia}(x)F^{+jb}(y))\,|\,0\rangle = \partial_-^x\partial_-^y\langle 0\,|\,T_+(\mathcal{A}^{ia}(x)\mathcal{A}^{jb}(y))\,|\,0\rangle\,. \tag{17b}$$

We will now show that all graphs representing the Coulomb term in Eq. (9b) precisely cancel the contributions from the last term of the propagator Eq. (16) so that we are left with an effective interaction Hamiltonian density:

$$\overline{\mathcal{H}}_I = \tfrac{1}{2}gF^{\mu\nu a}\hat{\mathcal{A}}_\mu^{ab}\mathcal{A}_\nu^b + \tfrac{1}{4}g^2(\hat{\mathcal{A}}_{ab}^\mu\mathcal{A}_b^\nu)(\hat{\mathcal{A}}_\mu^{ac}\mathcal{A}_\nu^c)\,, \tag{18}$$

i.e., the two usual vertices, and a propagator

$$D^{\mu\nu ab}(x' - x) = \delta_{ab}\int\frac{d^4p}{(2\pi)^4}\,\frac{e^{-ip\cdot(x'-x)}}{p_\mu p^\mu + i\epsilon} \times\left[-g^{\mu\nu} + \frac{\delta_-^\mu p^\nu + \delta_-^\nu p^\mu}{p_-}\right]. \tag{19}$$

This can be proved combinatorially by counting graphs. It is simpler however, to use one of the several functional techniques available. A convenient method[5] is to start with

$$S = T_+[\exp(-i\int dx\,\mathcal{H}_I)] \tag{20}$$

and replace $\mathcal{A}_+^a(x)$ in Eq. (20) with

$$\mathfrak{A}_+^a(x)+i\int d^4x'\,\Delta^{-\mu ab}(x-x')\frac{\delta}{\delta\mathfrak{A}^{\mu b}(x')}\,. \tag{21}$$

All the contractions between $\mathfrak{A}_+^a$ and $\mathfrak{A}_+^b$ are then automatically performed[6] and we have

$$S=:\exp\left\{\int d^4x(-i)\left[\tfrac{1}{2}gF^{ija}\hat{\mathfrak{A}}_i^{ab}\mathfrak{A}_j^b+\tfrac{1}{4}g^2(\hat{\mathfrak{A}}^{abi}\mathfrak{A}^{bj})(\hat{\mathfrak{A}}_i^{ac}\mathfrak{A}_j^c)\right.\right.$$
$$\left.\left.-\tfrac{1}{2}g^2(\hat{\mathfrak{A}}_i^{ab}F^{+ib})\partial_-^{-2}(\hat{\mathfrak{A}}_j^{ac}F^{+jc})+gF^{+ib}\hat{\mathfrak{A}}_i^{ab}\left(\mathfrak{A}_+^a+i\int d^4x'\,\Delta_{ac}^{-\mu}\frac{\delta}{\delta\mathfrak{A}^{\mu c}}\right)\right]\right\}:\,, \tag{22}$$

where : : denotes normal ordering, with respect to $\mathfrak{A}_+^a$ only. To save writing we shall use a symbolic matrix notation and denote the first two terms in the exponent by $\mathcal{H}'$ and $-igF^{+ib}\hat{\mathfrak{A}}_i^{ab}$ by G:

$$S=:\exp\left[\mathcal{H}'-\tfrac{1}{2}iG\partial_-^{-2}G+G\left(\mathfrak{A}_++i\Delta\frac{\delta}{\delta\mathfrak{A}_+}\right)\right]:\,. \tag{23}$$

Differentiating, with respect to $\mathfrak{A}_+$, we have $\delta S/\delta\mathfrak{A}_+=GS$, the solution of which is

$$S=C:\exp(G\mathfrak{A}_+):\,. \tag{24}$$

To find C we use Eqs. (23) and (24) to obtain

$$\left.\frac{\delta S}{\delta G}\right|_{\mathfrak{A}_+=0}=(-i\,\partial_-^{-2}G+i\Delta G)C=\frac{\delta C}{\delta G}\,,$$

and so

$$C=K\exp[-\tfrac{1}{2}iG\partial_-^{-2}G+\tfrac{1}{2}iG\Delta G]\,. \tag{25}$$

Choosing the arbitrary K so that $S|_{G=0}=T_+\exp\mathcal{H}'$, we finally obtain

$$S=:\exp(\mathcal{H}'-\tfrac{1}{2}iG\partial_-^{-2}G+\tfrac{1}{2}iG\Delta G+G\mathfrak{A}_+):\,. \tag{26}$$

From Eq. (16) we see that the last term in the expression $\frac{1}{2}G\Delta G$ precisely cancels the second term in the exponent in Eq. (26) so that we are left with

$$\bar{S}=:\exp(\mathcal{H}'+\tfrac{1}{2}iGDG+G\mathfrak{A}_+):\,. \tag{27}$$

According to Eqs. (22), (26), and (27) $\bar{S}$ is then the S matrix calculated with the Feynman rules [Eqs. (18) and (19)]. They hold to all orders, and since they were derived from the canonical formalism they yield a unitary S matrix. In particular, there are no "ghost loops" in our gauge. The simplicity of the Feynman rules in the null-plane gauge is, of course, obtained at the cost of lost manifest covariance. However, we expect the resulting S matrix to be covariant and gauge invariant. This can be verified by deriving the same rules from the path integral quantization method of Faddeev and Popov.[7] In this formalism, as is well known, the generating function $Z[J_\mu^a]$ of connected Green's functions is given by

$$e^{Z[\vec{J}_\mu]}=\int[dA_\mu^a]\exp(iS)$$
$$=\int[dA_\mu^a]\Delta[A_\mu^a]\times\exp\left\{i\int d^4x(\mathcal{L}-f[A_\mu^a]-J_\mu^aA^a)\right\}. \tag{28}$$

$f[A_\mu^a]$ is a gauge term that breaks the invariance of $\mathcal{L}$, whereas the gauge-invariant Jacobian $\Delta[A_\mu^a]$ satisfies the condition

$$\Delta[A_\mu^a]\int[du]\exp\left(i\int f[A_\mu^{au}]\right)=\text{constant}. \tag{29}$$

In a perturbation expansion $\Delta[A_\mu^a]$ gives rise to the well-known ghost loops of non-Abelian field theories. J_μ^a is an external boson source. The resulting S matrix is independent of the form of the gauge term.

The path integral method is very well suited for studying the consequences of gauge invariance, in particular global Ward-Takahashi identities.[8]

We now write Eq. (28) for the case of quantization in the null-plane frame. Our gauge condition $A_-^a=0$ can be incorporated in a term of the form

$$\int[dC^a]\exp\left(i\int d^4x\,C^aA_-^a\right)=\delta(A_-^a)\,. \tag{30}$$

Then from Eq. (29) and the fact that we need the value of $\Delta[A_\mu^a]$ only in the neighborhood of $A_-^a=0$, we get

$$\Delta[A_\mu^a]\int\delta(A_-^{au})[du]=\Delta[A_\mu^a]\int\delta\left(-\frac{1}{g}\partial_-u^a\right)[du]$$
$$=\text{constant}. \tag{31}$$

Since the integral does not depend on A_μ^a we can set $\Delta[A_\mu^a]=1$.[9] The generating functional Eq. (28) then becomes

$$e^{Z[\vec{J}_\mu]}=\int[dA_\mu^a]\delta(A_-^a)\times\exp\left[i\int d^4x(-\tfrac{1}{4}G^{\mu\nu a}G_{\mu\nu}^a-J_\mu^aA^{\mu a})\right] \tag{32}$$

from which one immediately obtains the Feynman rules [Eqs. (18) and (19)].

Finally we might mention that, following Kogut and Soper, one could use the old-fashioned perturbation series as an alternative way of calculating the S matrix in the null-plane frame. H_I of Eq. (9b), with the Heisenberg operators evaluated at $x^+=0$, is to be taken as the potential. The proof of the equivalence of the resulting series with the Feynman rules is exactly analogous to that given in Ref. 1, and need not be repeated here.

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APPENDIX

The constraints encountered in the case where the momenta are not independent functions of the velocities are of two kinds. First there are the constraints associated with the invariance of the Lagrangian with a certain group of transformation. The requirement that they are satisfied at all times may lead to further constraints. All these relations are, in Dirac's terminology, first-class constraints. They result in a number of arbitrary functions in the theory and are usually dealt with by imposing a set of "gauge conditions" on the dynamical variables.

The second kind of constraints arise, for example, in cases where one treats (time) derivatives of fields as independent variables or where time derivatives enter the Lagrangian linearly for some reason such as the nature of the metric—this is actually the case in the null-plane frame. It is with these "second class" constraints that we are essentially concerned below. They may result in a modification of the elementary canonical commutation rule.[3,4]

It will be convenient then to write all Lagrangians in a form linear in time derivatives:

$$L=\tfrac{1}{4}i\,(q\Lambda\,\dot{q}-\dot{q}\Lambda\,q)-H(q,t)\,, \qquad \text{(A1)}$$

where Λ is an imaginary and antisymmetric matrix and q is the column matrix of the dynamical variables. If Λ is nonsingular, the equations of motion that follow from (A1) can be solved for $\dot{q}$ and we can set

$$[q,\dot{q}]=\hbar\Lambda^{-1}\,. \qquad \text{(A2)}$$

In most cases of interest, however, Λ is singular. In this case we introduce (Ref. 3) a projection operator P on the zero space of Λ:

$$P\Lambda=\Lambda P=0,\quad P^2=P \qquad \text{(A3)}$$

and

$$\overline{P}=I-P=\overline{P}^2\,. \qquad \text{(A3a)}$$

The Lagrangian equations decompose into two sets:

$$i\,(\overline{P}\Lambda\,\overline{P})(\overline{P}\dot{q})-\overline{P}H_q=0\,, \qquad \text{(A4)}$$

$$PH_q=0\,. \qquad \text{(A5)}$$

H_q is the matrix $\partial H/\partial q$. We also use H_{qq} for $\partial^2 H/\partial q\,\partial q$, etc. Equation (A5) is actually a constraint equation that should be solved for Pq in terms of $\overline{P}q$. The condition for this is that $PH_{qq}P$ be nonsingular in the subspace on which P projects. If this condition holds it is easy to show that

$$[\overline{P}q,q\overline{P}]=\hbar(\overline{P}\Lambda\,\overline{P})^{-1} \qquad \text{(A6a)}$$

or

$$[\Lambda q,q\Lambda]=\hbar\Lambda \qquad \text{(A6b)}$$

are the appropriate commutators.

If the condition does not hold we introduce a new projection Q onto the zero space of $PH_{qq}P$ and the corresponding $\overline{Q}=I-Q$. Differentiating Eq. (A5) with respect to time and using Eq. (A4) one can derive another (secondary) constraint equation:

$$-i\,QPH_{qq}\overline{P}(\overline{P}\Lambda\,\overline{P})^{-1}\overline{P}H_q+QPH_{qt}=0\,, \qquad \text{(A7)}$$

which is to be solved for QPq. If we assume for convenience that $\overline{P}H_q$ is linear in QPq the relevant condition is that the matrix $T=QPH_{qq}\overline{P}(\overline{P}\Lambda\,\overline{P})^{-1}\times\overline{P}H_{qq}PQ$ be nonsingular in the subspace on which QP projects. If this is the case, one can check that the appropriate ansatz for the commutators is

$$\frac{1}{\hbar}[\Lambda q,q\Lambda]=\Lambda-\overline{P}H_{qq}PQT^{-1}QPH_{qq}\overline{P}\,. \qquad \text{(A8)}$$

If the secondary constraints are not solvable then differentiation, with respect to time, leads to tertiary constraints and so on. However (A8) is sufficient for our purposes.

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[8]B. W. Lee and J. Zinn-Justin [Phys. Rev. D 5, 3121 (1972)] and A. Slarnov [Kiev Report No. ITP-71-83E (unpublished)] have derived Ward identities for the Yang-Mills fields. The identities for the currents are simpler and very easy to obtain. If the "classical" part of the action $I=\int d^4x\,(\mathcal{L}-f[A^a_\mu])$ is invariant under gauge transformations of the first kind, then from $\delta I=0$ we can define the Noether current $\partial_\mu j^{\mu a}=(\delta I/\delta A^b_\mu)\times\delta A^{ab}_\mu$. Here $\delta A^{ab}_\mu=\epsilon^{acb}A^c_\mu=\hat{A}^{ab}_\mu$. If we add, say, a fermion field coupling and the corresponding sources η, $\bar{\eta}$, we have

$$\partial_\mu j^{\mu a}=\frac{\delta I}{\delta A^b_\mu}\hat{A}^{ab}_\mu+\frac{\delta I}{\delta\psi}\,iT^a\psi-\frac{\delta I}{\delta\bar{\psi}}iT^a\bar{\psi}$$

$$=\frac{\delta I'}{\delta A^b_\mu}\hat{A}^{ab}_\mu+\frac{\delta I'}{\delta\psi}iT^a\psi-\frac{\delta I'}{\delta\bar{\psi}}\,iT^a\bar{\psi}+\frac{\delta i\ln\Delta}{\delta A^b_\mu}\hat{A}^{ab}_\mu\,,$$

where $I'=I-i\ln\Delta[A^a_\mu]$. But since $\Delta[A^a_\mu]$ is gauge-invariant we can always write the last term as a divergence: $\partial_\mu \mathcal{J}^{\mu a}$. A Ward identity for this "total current" is now easily derived by performing a gauge transformation in the fields in Eq. (28). Since the measure is invariant and a change of variables cannot change the value of the integral, we can set the variation with respect to $u^a(x)$ equal to zero and finally obtain

$$-\frac{1}{g}\,\partial_\mu\int[dA^a_\mu][d\psi][d\bar{\psi}]\frac{\delta S}{\delta A^a_\mu}\exp(iS)$$

$$+\int[dA^a_\mu][d\psi][d\bar{\psi}]\left(\frac{\delta I'}{\delta A^b_\mu}\hat{A}^{ab}_\mu+i\frac{\delta I'}{\delta\psi}T^a\psi-i\frac{\delta I'}{\delta\bar{\psi}}T^a\bar{\psi}\right)\exp(iS)$$

$$-\int[dA^a_\mu][d\psi][d\bar{\psi}]\,(J^{\mu b}\hat{A}^{ab}_\mu+i\bar{\eta}T^a\psi-i\eta T^a\bar{\psi})\exp(iS)=0\,.$$

The first term is equal to zero. Substituting our expression for $\partial_\mu(j^{\mu a}+\mathcal{J}^{\mu a})$, we obtain the desired Ward identity:

$$\int[dA^a_\mu][d\psi][d\bar{\psi}]\,\partial_\mu(j^{\mu a}+\mathcal{J}^{\mu a})\exp(iS)$$

$$-\int[dA^a_\mu][d\psi][d\bar{\psi}]\,(J^{\mu b}\hat{A}^{ab}_\mu+i\bar{\eta}T^a\psi$$

$$-i\eta T^a\bar{\psi})\exp(iS)=0.$$

[9]Similarly in the case of the gravitational field there are no ghost loops in the analogous "gauge" $g^{\mu 3}=\delta^{\mu 3}$.

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Comments on Generalized Hamiltonian Dynamics*

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A reasonable generalization of Hamiltonian theory to $3m$-dimensional phase space suggests a geometrical structure giving the proper characteristic vector field. This structure, however, has only a single integral invariant, and implies no sensible generalization of either Poisson-bracket formalism, or Hamilton-Jacobi theory. Associated statistical mechanics and quantization are unlikely. The algebraic source of the difficulty is the lack of understanding of canonical expressions and classes of closed 3-forms.

I. INTRODUCTION

In an imaginative and suggestive paper, Nambu[1] has proposed generalization of classical Hamiltonian dynamics to $3m$-dimensional phase space. We have recently restudied Hamiltonian mechanics from the unifying geometric standpoint of modern differential geometry, in terms of which several quite deep insights are achieved.[2] We also had investigated possible generalization to $3m$-dimensional phase space, but for several reasons, connected with the insights into the geometrical structures involved, had been discouraged from pursuing the matter further. In the following we summarize the chief features of the geometrical approach, and then indicate the difficulties that must be met on generalization.

II. HAMILTONIAN STRUCTURES

We deal at first with structures related to a given vector field $\vec{V}$ in an even-dimensional manifold of n dimensions. When a holonomic dynamical system is said to have m degrees of freedom, its equations of motion are of second degree in m dependent variables and one independent variable, time. If the time variable enters autonomously, m auxiliary variables can be introduced so as to write the equations as a set of $n=2m$ first-order expressions that, geometrically, are the components of a vector field whose trajectories are the