Focusing of Gravitational Radiation

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Under the assumptions that gravitational radiation is emitted by a pointlike source and focused by passage through the exterior or *interior* gravitational fields of a massive object, the intensity gain is shown to be of the order of $GM/\lambda c^2$ over a diffraction beam of small angular width.

Calculations[1,2,3] of the focusing of gravitational radiation by the gravitational field of a massive object ("lens") have usually been carried out in analogy with the corresponding geometrical-optics calculations of the focusing of light.[4] However, this analogy breaks down under certain circumstances: (i) Gravitational radiation can pass freely through the interior of the massive object (if not a black hole); under these conditions the deflection angle can *increase* with impact parameter and a true focal point is formed.[5] For light passing by the object, there is no true focus, only a caustic line. (ii) Some sources of gravitational radiation of size small compared with a wavelength (e.g., vibrating neutron stars) behave as *point* sources; the maximum focused intensity is then limited by diffraction effects. For the case of light the intensity is limited by the surface brightness of the source.[4]

In what follows we assume that the source is small compared with the wavelength ($\lambda \simeq 2\times 10^7$cm) and illuminates the lens fairly uniformly; the characteristic size of the lens is large compared with the wavelength (stationary-phase approximation applies); the gravitational field of the lens at the impact parameter of the ray is sufficiently weak so that it can be described by the linear approximation, and the lens is axially symmetric and the source is near the axis of symmetry.

To the extent that geometrical optics is valid, the the gain (defined as the factor by which the intensity of the radiation is increased by the presence of the lens) near a given ray is

$$(\text{gain}) \simeq w_1/w_2\,, \tag{1}$$

where w_1 is the angle, as seen from the source, between the axis and the impact point on the lens, and w_2 is the angle between the axis and the reception point. For rays that are received on the axis ($w_2 = 0$), the above formula fails and diffraction effects become important (the axis is a caustic line). The width of the diffraction beam over which the gain is large is approximately

$$w \simeq \lambda/\pi b\,, \tag{2}$$

where b is the impact parameter. If we assume that the deflection is produced by an exterior Schwarzschild field, then the deflection angle and impact parameter are related by

$$\theta = \frac{4GM}{bc^2}\,, \tag{3}$$

where M is the mass of the lens. In terms of the source-to-lens distance, $\theta = b/l_1$ and therefore

$$b = (4GMl_1/c^2)^{1/2}\,, \tag{4}$$

$$w \simeq \frac{\lambda}{\pi}\left(\frac{c^2}{4GMl_1}\right)^{1/2} \simeq 2.8\times 10^2\left(\frac{R_\odot M_\odot}{l_1 M}\right)^{1/2}. \tag{5}$$

For a *rough* estimate of the gain we can take $w_2 \simeq w$ and $w_1 \simeq b/l_1$ in (1) which results in

$$(\text{gain}) \simeq 4\pi GM/\lambda c^2\,. \tag{6}$$

A somewhat more precise calculation (stationary-phase approximation) increases this by a factor of 2π so that

$$(\text{gain}) \simeq 0.6 M/M_\odot\,. \tag{7}$$

A black hole of mass $10^6 M_\odot$ at the center of our galaxy[6] accompanied by a source at $l_1 \simeq 100 R_\odot$ could give a gain of $\sim 0.6\times 10^6$ over a beam of angular width 3×10^{-6}rad. The *a priori* probability for interception of this beam by the earth is very small and even if it is granted that the earth is placed in the beam at some time, the alignment would not persist.

In the exterior Schwarzschild field true focusing is not possible since the deflection angle *decreases* with impact parameter. A focal point can be obtained if the rays pass through a region where mass is present. For rays passing near the center of the mass distribution, the focusing condition is

$$1/l_1 + 1/l_2 = d\theta/db\,\big|_{b=0}\,, \tag{8}$$

where l_2 is the lens-to-receptor distance. The deflection angle for rays that pass through a mass is given by the formula

$$\theta(b) = \frac{4G}{bc^2} \times (\text{mass within impact parameter } b)\,. \tag{9}$$

The "focal distance" for a sphere of uniform density and radius R follows from (8) and (9)[5]:

$$\left.\frac{db}{d\theta}\right|_{b=0} = \left(\frac{Rc^2}{6GM}\right)R\,. \tag{10}$$

For a source placed at this focus, the width of the diffraction peak and the gain can be calculated by methods similar to the above, with the result

$$w \simeq 3.4\left(\frac{M_\odot}{M}\right)^{3/4}\left(\frac{GM}{Rc^2}\right)^{1/4}, \tag{11}$$

$$(\text{gain}) \simeq 0.98 M/M_\odot\,. \tag{12}$$

For example, if we arbitrarily set $GM/Rc^2 = 10^{-2}$, then $M = 10^9 M_\odot$ gives a gain of $\sim 10^9$ over a beam of width 2×10^{-7} rad.

These results on uniform mass-density spheres are interesting because the thin rotating disks of Bardeen and Wagoner[7] have a mass distribution which, in the nonrelativistic case, is exactly that obtained by projecting the mass of a uniform sphere on the plane of the disk; according to Eq. (9), the deflections and gains produced by sphere and disk are then the same.

To place an upper limit on the probability that the Weber pulses are due to focusing by a disk, we make the extreme assumption[8] that almost every galaxy contains a disk of mass $\sim 10^9 M_\odot$ in its core and that at the focal point of each disk pulses of gravitational radiation, each of energy $M_\odot$, are emitted by a point source. If Weber's apparatus can detect a pulse of energy $\sim M_\odot$ released at the galactic center, then it can detect a focused pulse of equal energy out to distances of 6×10^8 light years. Although there are $\sim 10^8$ galaxies in this volume, the probability for alignment is only $(2\times10^{-7})^2/4\pi$ and therefore the number of galaxies that might be expected to contribute to the focusing is $\sim 3\times10^{-7}$. This is a much larger number than that obtained by Lawrence[1] on the assumption of focusing by a black hole in the core of *our* galaxy, but the effect is still quite negligible.

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[1] J. K. Lawrence, Phys. Rev. D 3, 3239 (1971).
[2] J. K. Lawrence, Phys. Rev. D 7, 2275 (1973).
[3] G. A. Campbell and R. A. Matzner, J. Math. Phys. 14, 1 (1973).
[4] S. Liebes, Phys. Rev. 133, B835 (1964).
[5] This conclusion was also reached by J. K. Lawrence, Nuovo Cimento 6B, 225 (1971).
[6] D. Lynden-Bell, Nature (Lond.) 223, 690 (1969).
[7] J. M. Bardeen and R. V. Wagoner, Astrophys. J. 167, 359 (1971).
[8] The present focusing process requires that the lens have a quite homogeneous gravitational field. For a lens consisting of an ordinary galactic nucleus (many stars), the focusing is limited by the inhomogeneities in the lens rather than by diffraction effects, and our results do not apply.