PHYSICAL REVIEW D VOLUME 8, NUMBER 8 15 OCTOBER 1973

# Feynman Amplitudes as Power Series*

David S. Kershaw
*Center for Theoretical Physics, Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742*
(Received 7 June 1973)

It is shown that single-loop Feynman integrals have a simple, natural expression as generalized hypergeometric power series. It is shown how to continue these power series to regions of physical interest and how to use them to derive simple systems of differential equations satisfied by single-loop Feynman integrals. Applications to the evaluation of the box-graph contribution to the scattering of light by light are discussed.

## I. INTRODUCTION

If one examines any text on the theory of functions of several complex variables, one finds that a very basic tool in this theory is the representation of holomorphic functions by power series. This type of representation has received very little attention in the study of Feynman amplitudes, and so in this article we shall examine the representation of Feynman amplitudes by power series. We are motivated by a number of specific considerations. The recent work of Regge[1] has pointed out that Feynman amplitudes are generalized hypergeometric functions in the sense that their fundamental groups of analytic continuations are generalizations of the fundamental group for the ordinary hypergeometric function. This leads one to suspect that, as in the case for the ordinary hypergeometric function, the Feynman amplitudes might possess simple and elegant power-series expansions, power-series continuations, and differential equations. In the study of the ordinary hypergeometric function, the study of its fundamental group plays the role of the final abstraction or crowning glory of the theory, while most of the theory is developed using the power series representations and differential equation. Furthermore, the simplest way to numerically evaluate the hypergeometric function at any point is via the appropriate power series representation. Thus, by studying Feynman amplitudes as power series we hope to develop the idea of Feynman amplitudes as generalized hypergeometric functions in a powerful new direction and, at the same time, in a more practically useful direction. Our long-range goal is to develop a powerful and systematic method for the calculation and evaluation of higher-order Feynman integrals.

In Sec. II we study the single-loop diagrams, a class of diagrams whose fundamental group has been studied extensively by Regge.[2] In Sec. III we study a more complicated diagram.

## II. SINGLE-LOOP DIAGRAMS

Consider any single-loop Feynman graph with arbitrary internal masses. Let $P_1, \ldots, P_N$ be the $N$ external four momenta and $m_1, \ldots, m_N$ the adjacent internal masses. Then the amplitude for this graph may be written in the form[3]

$$F(P_1, \ldots, P_N) = \int d\mu\Big(\sum_{i<j} \alpha_i\alpha_j\, z_{ij} - \sum_i \alpha_i m_i{}^2 + i\epsilon\Big)^{2-N}, \quad (1)$$

where

$$\int d\mu = \prod_{i=1}^{N}\left[\int_0^1 d\alpha_i\right] \delta(1-\textstyle\sum\alpha_i) \ ,$$

and $z_{ij} = (P_i + P_{(i+1)} + \cdots + P_{(j-1)})^2$. We can obtain a power series for $F$ by expanding the integrand in Eq. (1) as a power series in $\{(z_{ij} - C_{ij})\}$ $i, j = 1, \ldots, N$ and $i<j$, and then performing the integration term by term. Here $\{C_{ij}\}$ is the point in the space of the $\frac{1}{2}[N(N-1)]$ variables $z_{ij}$, about which we wish to expand. Clearly this power series will converge so long as

$$\Big|\sum_{i<j} \alpha_i\alpha_j(z_{ij} - C_{ij})\Big| < \Big|\sum_i \alpha_i m_i{}^2 - \sum_{i<j}\alpha_i\alpha_j C_{ij}\Big|, \quad (2)$$

for all $\{\alpha_i\}$ in the region of integration. We assume that all masses are nonzero. By choosing $|C_{ij}|$ sufficiently small and by choosing $|z_{ij} - C_{ij}|$ sufficiently small, (2) can always be satisfied, so for sufficiently small $|C_{ij}|$, the power series obtained by this method will have a finite domain of convergence. Next we wish to choose $\{C_{ij}\}$ astutely so that the term-by-term integration will give simple, elegant coefficients for our power series. We note that in the theory of the ordinary hypergeometric function the simplest power series are obtained by expanding about points which are singular points of the function on some sheet. For example, the defining power series for ${}_2F_1(a, b, c; z)$,

$$ {}_2F_1=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!}z^n , $$

is an expansion about $z=0$, which is a singular point of the continuation of $F$ obtained by circling the branch point at $z=1$. Generalizing this insight we shall let $C_{ij}=(m_i-m_j)^2$ since it is known that the single-loop amplitude has singularities on unphysical sheets at $z_{ij}=(m_i-m_j)^2$. For this choice of $C_{ij}$, the right-hand side of Eq. (2) becomes

$$ \left|\sum_i \alpha_i m_i^2-\sum_{i<j}\alpha_i\alpha_j(m_i-m_j)^2\right| $$

$$ =\left|\sum_i \alpha_i m_i^2+2\sum_{i<j}\alpha_i\alpha_j m_i m_j-\sum_i\alpha_i(1-\alpha_i)m_i^2\right| $$

$$ =\left|\sum_i(\alpha_i m_i)^2+2\sum_{i<j}(\alpha_i m_i)(\alpha_j m_j)\right| $$

$$ =\left|\sum_i \alpha_i m_i\right|^2>0 , $$

so the power series will have a finite radius of convergence. Let

$$ y_{ij}=z_{ij}-(m_i-m_j)^2. $$

$$ F(P_1,\ldots,P_N)=F(y_{ij}) $$

$$ =\int d\mu\left[\sum_{i<j}\alpha_i\alpha_j y_{ij}-\left(\sum\alpha_i m_i\right)^2\right]^{(2-N)} $$

$$ =(-1)^N\int d\mu\left(\sum\alpha_i m_i\right)^{(4-2N)}\times\left[1-\frac{\left(\sum_{i<j}\alpha_i\alpha_j y_{ij}\right)}{(\sum\alpha_i m_i)^2}\right]^{2-N} , $$

which, expanding by the binomial theorem, equals

$$ \frac{(-1)^N}{\Gamma(N-2)}\sum_{n=0}^{\infty}\frac{\Gamma(N-2+n)}{n!}\int d\mu\frac{\left(\sum_{i<j}\alpha_i\alpha_j y_{ij}\right)^n}{(\sum\alpha_i m_i)^{2(N-2+n)}} , $$

and, by the multinomial theorem, becomes

$$ \frac{(-1)^N}{\Gamma(N-2)}\sum_{\{n_{ij}\}=0}^{\infty}\Gamma(N-2+n)\prod_{i<j}\frac{y_{ij}^{n_{ij}}}{n_{ij}!}\times\int d\mu\frac{\prod_{i=1}^{N}\alpha_i^{n_i}}{(\sum\alpha_i m_i)^{2(N-2+n)}} , $$

where $n=\sum_{i<j}n_{ij}$, and $n_i=\sum_{j\,(j>i)}n_{ij}+\sum_{j\,(j<i)}n_{ji}$, and $\sum_{i=1}^{N}n_i=2n$. Now let us perform the integration. We have

$$ \int d\mu\frac{\prod_{i=1}^{N}\alpha_i^{n_i}}{(\sum\alpha_i m_i)^{2(N-2+n)}}=\frac{\Gamma(2(N-2))}{\Gamma(2(N+n-2))}\left[\prod_{i=1}^{N}\frac{\partial^{n_i}}{\partial m_i^{n_i}}\right]\int\frac{d\mu}{(\sum\alpha_i m_i)^{2(N-2)}} $$

$$ =\frac{(-1)^N\Gamma(N)}{\Gamma(2(N+n-2))}\left[\prod_{i=1}^{N}\frac{\partial^{n_i}}{\partial m_i^{n_i}}\right]\frac{\partial^{(N-4)}}{\partial x^{(N-4)}}\int\frac{d\mu}{(\sum\alpha_j m_j+x)^N}\bigg|_{x=0} $$

$$ =\frac{(-1)^N}{\Gamma(2(N+n-2))}\left[\prod_{i=1}^{N}\frac{\partial^{n_i}}{\partial m_i^{n_i}}\right]\frac{\partial^{(N-4)}}{\partial x^{(N-4)}}\left[\prod_{j=1}^{N}\frac{1}{(m_j+x)}\right]\bigg|_{x=0} $$

$$ =\frac{(-1)^N}{\Gamma(2(N+n-2))}\left[\prod_{i=1}^{N}\frac{\partial^{n_i}}{\partial m_i^{n_i}}\right]\left(\sum_{i=1}^{N}\frac{\partial}{\partial m_i}\right)^{(N-4)}\left[\prod_{i=1}^{N}\frac{1}{m_i}\right] $$

$$ =\left(\sum_{i=1}^{N}\frac{\partial}{\partial m_i}\right)^{(N-4)}\frac{(-1)^N}{\Gamma(2(N+n-2))}\prod_{i=1}^{N}\left(\frac{n_i!}{m_i^{(n_i+1)}}\right). $$

Here we have assumed that $N\geqslant 4$. Thus,

$$ F(P_1,\ldots,P_N)=\frac{1}{\Gamma(N-2)}\left(\sum_{i=1}^{N}\frac{\partial}{\partial m_i}\right)^{(N-4)}\prod_{i=1}^{N}\left(\frac{1}{m_i}\right)\sum_{\{n_{ij}\}=0}\frac{\Gamma(N+n-2)}{\Gamma(2(N+n-2))}\left[\prod_{i=1}^{N}n_i!\right]\prod_{i<j}\frac{(y_{ij}/m_i m_j)^{n_{ij}}}{n_{ij}!} , $$

which, since $\Gamma(2z)=2^{(2z-1)}\Gamma(z)\Gamma(z+\frac{1}{2})/\pi^{1/2}$, becomes

$$ F(P_1,\ldots,P_N)=\frac{\pi^{1/2}2^{(5-2N)}}{\Gamma(N-2)}\left(\sum\frac{\partial}{\partial m_i}\right)^{(N-4)}\left(\prod m_i\right)^{-1}\sum_{\{n_{ij}\}}\frac{(\prod n_i!)}{\Gamma(N+n-\frac{3}{2})}\prod_{i<j}\frac{(u_{ij})^{n_{ij}}}{n_{ij}!} . \tag{3} $$

Here,

$$u_{ij}=\frac{y_{ij}}{4m_im_j}=\frac{z_{ij}-(m_i-m_j)^2}{4m_im_j}\ ,$$

and $u_{ij}=1$ gives the normal threshold in the variable $z_{ij}$. Thus we see that the single-loop Feynman amplitude is just the $(N-4)$th derivative of a single hypergeometric power series in $\frac{1}{2}[N(N-1)]$ variables.[4]

To illustrate the usefulness of the power-series method we will now confine ourselves to the case $N=4$, i.e., the box graph, and we shall consider various applications in detail. For the case $N-4$, Eq. (3) becomes

$$F(P_1,\ldots,P_4)=\frac{\pi^{1/2}}{8}\left(\Pi m_i\right)^{-1}\sum_{\{n_{ij}\}}\frac{(\Pi n_i\ !)}{\Gamma(n+\frac{5}{2})}\prod_{i<j}\frac{(u_{ij})^{n_{ij}}}{n_{ij}\ !}\ . \tag{4}$$

This formula is remarkable in itself, because the standard methods for evaluating Feynman integrals lead to $F(P_1,\ldots,P_4)=$the sum of 192 Spence functions of different rational algebraic functions of the invariants.[5] Note that Spence's function is a hypergeometric function of one variable, i.e.,

$$\mathrm{Sp}(z)=\sum_{n=1}^{\infty}\frac{z^2}{n^2}=\sum_{n=1}^{\infty}\frac{\Gamma(n)\Gamma(n)}{\Gamma(n+1)}\frac{z^n}{n\,!}.$$

We see that by expressing our amplitude in terms of generalized hypergeometric functions in several variables this sum of 192 ordinary hypergeometric functions in one variable collapses into just one simple hypergeometric function given in Eq. (4). Of course this is not quite the whole story, because the continuations of the ordinary hypergeometric function are known and thus the behavior of the sum of the 192 Spence functions is, in principle, known in every region, whereas all the continuations of the hypergeometric function in (4) are not known. In practice though the continuation of the sum of 192 Spence functions proves difficult because there are intricate cancellations between the singularities of the various terms while, as we shall see, the continuation of expression (4) is not too difficult and leads to equally simple and elegant power series in regions of physical interest. Finally, we shall see that in (4) and its continuations to other regions, the analytic behavior in a given region is simply and explicitly displayed, whereas in the sum of 192 Spence functions, individual terms have all sorts of spurious singularities which only cancel when one sums over all terms.

As a first application of Eq. (4) we shall show how it may be used to write down differential equations for Feynman amplitudes. According to Bateman,[6] if we have a power series in $r$ variables,

$$f(\{z_i\})=\sum_{\{n_i\}=0}^{\infty}A(\{n_i\})\prod_{i=1}^{r}z_i^{\,n_i},$$

and if

$$\frac{A(n_1,\ldots,n_i+1,\ldots,n_r)}{A(n_1,\ldots,n_i,\ldots,n_r)}=\frac{P_i(\{n_i\})}{Q_i(\{n_i\})},$$

where $P$ and $Q$ are finite polynomials in the $\{n_i\}$, then $f$ satisfies the system of $r$ linear partial-differential equations:

$$\left[Q_i\left(\left\{z_i\frac{\partial}{\partial z_i}\right\}\right)z_i^{-1}-P_i\left(\left\{z_i\frac{\partial}{\partial z_i}\right\}\right)\right]f=0\ .$$

Applying this method to Eq. (4) we find

$$P_{ij}=(n_i+1)(n_j+1)$$

and

$$Q_{ij}=(n_{ij}+1)(n+\tfrac{5}{2})\ ,$$

and therefore the box-graph amplitude satisfies the system of 6 second-order linear differential equations,

$$\left[\left(u_{ij}\frac{\partial}{\partial u_{ij}}+1\right)\left(\sum_{k<l}u_{kl}\frac{\partial}{\partial u_{kl}}+\tfrac{5}{2}\right)(u_{ij})^{-1}-\left(\sum_{\substack{k\\(k\neq i)}}u_{(ik)}\frac{\partial}{\partial u_{(ik)}}+1\right)\times\left(\sum_{\substack{l\\(l\neq j)}}u_{(jl)}\frac{\partial}{\partial u_{(jl)}}+1\right)\right]F(\{u_{ij}\})=0\ ,$$

where

$$u_{(ik)}=\begin{cases}u_{ik} & \text{if } i<k,\\ u_{ki} & \text{if } i>k.\end{cases}$$

Simplifying, we find for the box graph,

$$\left[\sum_{(k<l)}u_{kl}\frac{\partial^2}{\partial u_{kl}\partial u_{ij}}-\sum_{\substack{k\\(k\neq i)}}\sum_{\substack{l\\(l\neq j)}}u_{(ik)}u_{(jl)}\frac{\partial^2}{\partial u_{(ik)}\partial u_{(jl)}}+(\tfrac{5}{2}-u_{ij})\frac{\partial}{\partial u_{ij}}-\sum_{\substack{k\\(k\neq i)}}u_{(ik)}\frac{\partial}{\partial u_{(ik)}}-\sum_{\substack{l\\(l\neq j)}}u_{(jl)}\frac{\partial}{\partial u_{(jl)}}-1\right]F(\{u_{ij}\})=0\ . \tag{5}$$

Thus, we have a simple way of finding differential equations for Feynman amplitudes.

Next, let us study the continuations of the power series to regions of physical interest. Let the external-leg masses be $\mu_1, \ldots, \mu_4$. Then

$$u_{12} = \frac{\mu_1^2-(m_1-m_2)^2}{4m_1m_2}, \quad u_{23} = \frac{\mu_2^2-(m_2-m_3)^2}{4m_2m_3},$$

$$u_{34} = \frac{\mu_3^2-(m_3-m_4)^2}{4m_3m_4}, \quad u_{14} = \frac{\mu_4^2-(m_1-m_4)^2}{4m_1m_4},$$

$$u_{13} = \frac{s-(m_1-m_3)^2}{4m_1m_3}, \quad u_{24} = \frac{t-(m_2-m_4)^2}{4m_2m_4}.$$

If we assume stability of the external legs, then $\mu_1 < m_1 + m_2$, etc., and so $u_{12}$, $u_{23}$, $u_{34}$, and $u_{14} < 1$. If we wish to look at scattering in the $s$ channel, then $s > (\mu_1+\mu_2)^2$ and $s > (\mu_3+\mu_4)^2$, while $t < (\mu_2-\mu_3)^2$ and $t < (\mu_1-\mu_4)^2$. The stability condition ensures that the sum over $n_{12}$, $n_{23}$, $n_{34}$, and $n_{41}$ in Equation (4) will be convergent. If, in addition, $s < (m_1+m_3)^2 \to u_{13} < 1$ and $t$ is such that $|u_{24}| < 1$, then (4) will be a convergent power series in this region. If we wish to find the scattering amplitude in the vicinity of the normal threshold $s = (m_1+m_3)^2$ or $u_{13} = 1$, then we rewrite Eq. (4) in the form

$$F(P_1, \ldots, P_4) = \left(\prod \frac{1}{m_i}\right)\left(\frac{\pi^{1/2}}{8}\right) \sum_{\{n_{ij}\}}{}' n_2!\, n_4! \left[\prod_{i<j}{}' \frac{(u_{ij})^{n_{ij}}}{n_{ij}!}\right] \sum_{n_{13}=0}^{\infty} \frac{n_1!\, n_3!\, (u_{13})^{n_{13}}}{\Gamma(n+\frac{5}{2})n_{13}!},$$

where $\sum'$ and $\prod'$ mean do not sum over $n_{13}$ and do not include $(ij) = (13)$ in the product. From the theory of the ordinary hypergeometric function[7] it is known that

$$\sum_{n_{13}=0}^{\infty} \frac{n_1!\, n_3!\, u_{13}^{n_{13}}}{\Gamma(n+\frac{5}{2})n_{13}!} = {}_2F_1(n_1'+1, n_3'+1; n'+\tfrac{5}{2}; u_{13}) \frac{n_1'!\, n_3'!}{\Gamma(n'+\frac{5}{2})}$$

$$= {}_2F_1(n_1'+1, n_3'+1; -n_{24}+\tfrac{1}{2}; 1-u_{13}) \frac{n_1'!\, n_3'!\, \pi(-1)^{n_{24}}}{\Gamma(-n_{24}+\frac{1}{2})\Gamma(n'-n_1'+\frac{3}{2})\Gamma(n'-n_3'+\frac{3}{2})}$$

$$+ {}_2F_1(n'-n_1'+\tfrac{3}{2}, n'-n_3'+\tfrac{3}{2}; n_{24}+\tfrac{3}{2}; 1-u_{13}) \frac{(1-u_{13})^{n_{24}+1/2}}{\Gamma(n_{24}+\frac{3}{2})} \pi(-1)^{(n_{24}+1)},$$

where $n' = n - n_{13}$, $n_1' = n_1 - n_{13}$, and $n_3' = n_3 - n_{13}$. Therefore it follows that in the vicinity of the normal threshold,

$$F(P_1, \ldots, P_4) = (\pi^{3/2}/8\textstyle\prod m_i)\Bigg\{ \displaystyle\sum_{\{n_{ij}\}} \frac{(\prod n_i!)(-1)^{n_{24}}}{\Gamma(n_{13}-n_{24}+\frac{1}{2})\Gamma(n-n_1+\frac{3}{2})\Gamma(n-n_3+\frac{3}{2})} \left[\prod_{i<j}{}' \frac{(u_{ij})^{n_{ij}}}{(n_{ij})!}\right] \frac{(1-u_{13})^{n_{13}}}{n_{13}!}$$

$$+ \sum_{\{n_{ij}\}} \frac{n_2!\, n_4!\, \Gamma(n-n_1+n_{13}+\frac{3}{2})\Gamma(n-n_3+n_{13}+\frac{3}{2})}{\Gamma(n_{13}+n_{24}+\frac{3}{2})\Gamma(n-n_1+\frac{3}{2})\Gamma(n-n_3+\frac{3}{2})} (-1)^{(n_{24}+1)}$$

$$\times \prod_{i<j}{}' \frac{(u_{ij})^{n_{ij}}}{n_{ij}!} \frac{(1-u_{13})^{(n_{13}+n_{24}+1/2)}}{(n_{13})!} \Bigg\}.$$

Note how the square-root branch point is explicitly displayed in the second term. Similarly, if we wish to look at the behavior of the scattering amplitude in the vicintiy of $s \to \infty$, we simply use the formula for continuation of ${}_2F_1$ to the neighborhood of $u_{13} = \infty$ and we obtain a power series which converges rapidly as $s \to \infty$, and explicitly displays the asymptotic behavior.[8]

As a simple application of these methods, let us consider the box-graph contribution to the scattering of light by light. According to Karplus and Neuman,[9] the scattering amplitude may be written as a sum of amplitudes of the form

$$A(P_1, \ldots, P_4) = \prod_{i=1}^{4} \int_0^1 d\alpha_i (\alpha_i)^{\sigma_i} \frac{\delta(1-\sum\alpha_i)}{(\alpha_1\alpha_3 s + \alpha_2\alpha_4 t - m^2 + i\epsilon)^2},$$

where the $\sigma_i$ are non-negative integers and $\sum_{i=1}^4 \sigma_i = 4$, $s = (P_1+P_2)^2$, $t = (P_2+P_3)^2$, $P_i^2 = 0$. Applying our method to this expression we find,

$$A(P_1, \ldots, P_4) = m^{-4} \sum_{p,q=0}^{\infty} \frac{(p+q+1)!\,(p+\sigma_1)!\,(q+\sigma_2)!\,(p+\sigma_3)!\,(q+\sigma_4)!}{(2p+2q+7)!\, p!q!} \left(\frac{s}{m^2}\right)^p \left(\frac{t}{m^2}\right)^q. \tag{6}$$

Using

$$\Gamma(2p+2q+8)=2^{2(p+q)+7}\pi^{-1/2}\Gamma(p+q+4)\Gamma(p+q+\tfrac{9}{2})\ ,$$

we find

$$A(P_1,\ldots,P_4)=\left(\frac{\pi^{1/2}}{2^7m^4}\right)\sum_{p,q=0}^{\infty}\frac{(p+q+1)!}{(p+q+3)!}\ \frac{(p+\sigma_1)!\,(q+\sigma_2)!\,(p+\sigma_3)!\,(q+\sigma_4)!}{\Gamma(p+q+\frac{9}{2})p!q!}\,x^p y^q\ ,$$

where $x=s/4m^2$ and $y=t/4m^2$. Thus we see that the amplitude for the scattering of light by light is just a simple generalization of the Appell function $F_3$.[10]

The continuations of Eq. (6) to other regions may be easily obtained by the method described above. In the low energy limit, $s,\,t\ll 4m^2$ and the $p=q=0$ term of Eq. (6) gives the low energy approximation of Karplus and Neuman while the higher-order terms give all corrections to that approximation.[11] In the case of forward scattering, $t=0$, and we have

$$\begin{aligned}A(P_1,\ldots,P_4)&=m^{-4}\,\sigma_2!\,\sigma_4!\\&\quad\times\sum_{p=0}^{\infty}\frac{(p+1)!(p+\sigma_1)!(p+\sigma_3)!}{(2p+7)!\,p!}\left(\frac{s}{m^2}\right)^p\\&=\frac{m^{-4}}{7!}\left[\prod\sigma_1!\right]\\&\quad\times{}_3F_2(\sigma_1+1,\,\sigma_3+1,\,2\,;\,4,\tfrac{9}{2};\,(s/4m^2)),\end{aligned}$$

where ${}_3F_2$ is the generalized hypergeometric series in one variable whose continuations may be looked up in the Bateman manuscript.[12] The clarity and suitability for numerical computation of Eq. (6) may best be appreciated by comparing it with the equivalent expression by Karplus and Neuman for $A(P_1,\ldots,P_4)$.[13]

## III. MORE COMPLICATED DIAGRAMS

Let us consider the lowest-order self-energy graph in $\phi^4$ theory. There are three internal lines with equal masses, and after renormalization, one finds that the corresponding Feynman amplitude is given by

$$F(P^2)=a+bx+\frac{1}{3!}\sum_{n=2}^{\infty}(n-2)!\,C_n\left(\frac{x^n}{n!}\right)\ ,$$

where $a$ and $b$ depend on the mass and wave-function renormalization, $x=(P^2-m^2)/m^2$, and

$$C_n=\int d\mu\,\prod_{i=1}^{3}\left(\frac{\alpha_i}{1-\alpha_i}\right)^{(n-1)}.$$

We have defined $x$ so that $x=0$ gives the unphysical-sheet Landau singularity. The coefficients $C_n$ may be evaluated in the following manner: First, by repeatedly multiplying and dividing by $\sum_{i=1}^{3}(1-\alpha_i)=2$, we can reduce $C_n$ to a sum of integrals of the form

$$\int d\mu\,\frac{(\alpha_1\alpha_2\alpha_3)^{(n-1)}}{(1-\alpha_1)^{r_1}(1-\alpha_2)^{r_2}}\,.$$

Then, by using relations of the type

$$\begin{aligned}\int d\mu\,&\frac{\prod\alpha_i^{q_i}}{(1-\alpha_1)^{r_1}(1-\alpha_2)^{r_2}}\\&=\int\prod d\alpha_i\,\delta(1-\textstyle\sum\alpha_i)\,\frac{\prod\alpha_i^{q_i}}{(1-\alpha_1)^{r_1}}\,\frac{\partial}{\partial\alpha_2}\,\frac{(1-\alpha_2)^{(1-r_2)}}{(r_2-1)}\\&=\frac{1}{(r_2-1)}\int d\mu\,\frac{\alpha_1^{q_1}\alpha_2^{(q_2-1)}\alpha_3^{(q_3-1)}(q_3\alpha_2-q_2\alpha_3)}{(1-\alpha_1)^{r_1}(1-\alpha_2)^{(r_2-1)}}\,,\end{aligned}$$

we can further reduce $C_n$ to a sum of integrals of the form

$$\int d\mu\,\frac{\prod\alpha_i^{n_i}}{(1-\alpha_1)(1-\alpha_2)}\,.$$

Finally, writing $\alpha_1=1-(1-\alpha_1)$, $\alpha_2=1-(1-\alpha_2)$, $\alpha_3=(1-\alpha_1)+(1-\alpha_2)-1$, and expanding, we obtain $C_n$ as a sum of integrals of the form

$$\int d\mu\,\prod\alpha_i^{p_i}=\frac{p_i\,!}{(\sum p_i+2)!}\ ,$$

and

$$\int d\mu\left(\frac{1}{1-\alpha_1}\right)=1\ ,$$

$$\int d\mu\,\frac{1}{(1-\alpha_1)(1-\alpha_2)}=\tfrac{1}{6}\pi^2.$$

Thus we see that $C_n=p_n+\pi^2 q_n$, where $p_n$ and $q_n$ are both rational numbers, but $p_n$ and $q_n$ are no longer simple products of $\Gamma$ functions as was true in the single-loop case. To investigate the convergence of the power series we note that $\prod\alpha_i/(1-\alpha_i)$ has a maximum at $\alpha_1=\alpha_2=\alpha_3=\frac{1}{3}$, the center of the triangle of integration, and goes to zero on the boundary of the triangle. Therefore for large $n$, let $\epsilon_i=\alpha_i-\frac{1}{3}$, and we find that

$$\begin{aligned}C_n&=\int d\mu\prod\left(\frac{\alpha_i}{1-\alpha_i}\right)^{(n-1)}\\&=(\tfrac{1}{8})^{(n-1)}\int_{-1/3}^{+2/3}\prod d\epsilon_i\,\delta(\textstyle\sum\epsilon_i)\displaystyle\prod\left(\frac{1+3\epsilon_i}{1-\frac{3}{2}\epsilon_i}\right)^{(n-1)}\\&\approx(\tfrac{1}{8})^{(n-1)}\int_{-\infty}^{+\infty}d\epsilon_1 d\epsilon_2\,e^{-(n-1)(27/4)(\epsilon_1^{\,2}+\epsilon_2^{\,2}+\epsilon_1\epsilon_2)}\\&=\frac{\pi}{27[3(n-1)]^{1/2}}(\tfrac{1}{8})^{n-2}\ .\end{aligned}$$

Therefore, the power series converges if $x<8$, and diverges if $x>8$. This is as expected since the physical-sheet Landau singularity lies at $P^2=(3m)^2$, or $x=8$. We also note that the power series

$$f(x)=\sum_{n=2}^{\infty}(n-2)!\,C_n\frac{x^n}{n!}$$

may be continued by the method of Mellin and Barnes. That is,

$$f(x)=\frac{1}{2\pi i}\int d\mu\int_{-i\infty+3/2}^{+i\infty+3/2} ds\,\Gamma(s-1)\Gamma(-s)(-x)\times\left(\prod_{i=1}^{3}\frac{\alpha_i}{1-\alpha_i}\right)^{s-1},$$

where $|\arg(-x)|<\pi$, as is easily seen by closing the contour with an infinite semicircle in the region $\mathrm{Re}\,s>\frac{3}{2}$.

## IV. CONCLUSION

It has been shown that all single-loop Feynman diagrams can be represented by a single generalized hypergeometric power series. The continuations of these power series to various physically-interesting regions has been given. A simple method for finding the system of linear differential equations satisfied by each graph is presented.[14] A simple application to the scattering of light by light is given.

## ACKNOWLEDGMENTS

I wish to thank P. K. Mitter and O. W. Greenberg for many helpful discussions.

---

*Work supported in part by the National Science Foundation, Center for Theoretical Physics under Grant No. NSF GU 2061.

[1] T. Regge, in *Battelle Rencontres*, 1967 Lectures in Mathematics and Physics (Benjamin, Reading, Mass., 1968), pp. 433–458. See particularly p. 442.

[2] T. Regge, Commun. Math. Phys. 18, 1 (1970).

[3] R. J. Eden *et al.*, *The Analytic S-Matrix* (Cambridge Univ. Press, Cambridge, England, 1966), p. 33.

[4] See *Higher Transcendental Functions* [(Bateman Manuscript Project), edited by A. Erdelyi *et al.* (McGraw-Hill, New York, 1953), Vol. I, pp. 222–224], for the definition of hypergeometric power series in several variables.

[5] A. C. T. Wu, K. Dan Vidensk. Selsk. Mat.-Fys. Medd. 33, 72 (1961), Appendix A.

[6] *Higher Transcendental Functions* (Bateman Manuscript Project) (Ref. 4), Vol. I, p. 232, Sec. 5.9.

[7] *Higher Transcendental Functions* (Bateman Manuscript Project) (Ref. 4), Vol. I, p. 108, Sec. 2.10, Eq. (1).

[8] *Higher Transcendental Functions* (Bateman Manuscript Project) (Ref. 4), Vol. I, p. 109, Sec. 2.10, Eqs. (7) and (8).

[9] R. Karplus and M. Neuman, Phys. Rev. 80, 380 (1950). See Eq. (46).

[10] *Higher Transcendental Functions* (Bateman Manuscript Project) (Ref. 4), Vol. I, p. 222, Sec. 5.7.

[11] R. Karplus and M. Neuman, Phys. Rev. 83, 776 (1951). See especially Sec. IV.

[12] *Higher Transcendental Functions* (Bateman Manuscript Project) (Ref. 4), Vol. I, Secs. 4.1 thru 4.7.

[13] R. Karplus and M. Neuman, Phys. Rev. 83, 776 (1951). See pp. 782–784.

[14] In a recent paper, G. Barnucchi and G. Ponzano [J. Math. Phys. 14, 396 (1973)] have derived a set of differential equations for single-loop Feynman diagrams by working directly with the integral representations. Their differential equations are much more complicated than ours. For example, for the box graph, they define a set of 15 distinct (having distinct Speer $\lambda$ parameters) generalized box-graph amplitudes and then find a system of 150 differential equations for these 15 functions of 10 variables (they regard the internal masses as variables), whereas with our method we obtain a system of 6 equations for one function (the physical box-graph amplitude) of 6 variables.