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Light-Cone Operator Expansions in Perturbation Theory. II*

John M. Cornwall† and George Tiktopoulos

Department of Physics, University of California, Los Angeles, California 90024

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A systematic investigation, in perturbation theory, is presented of the light-cone behavior of multiparticle matrix elements of time-ordered products of local fields: $\int d^4x\ e^{iqx}\ \langle\ \beta|T\psi_1(x)\psi_2(0)\ |\alpha\ \rangle$. In the limit $q_-\to\infty$, the contribution of any single Feynman graph is of the form $q_-^{\ \beta}(\ln q_-)^\gamma$. The main result here is a rule by means of which the integers β and γ can be read off from the topology of the graph. The implications of this investigation for local field theories are organized and discussed in operator language in a companion paper. A by-product of the special methods here developed to obtain asymptotic estimates in perturbation theory is a refinement of Weinberg's theorem for the *Euclidean* region: the determination of the logarithmic factors in the asymptotic form of Feynman amplitudes when a set of external momenta q_1, q_2, ... is allowed to approach infinity according to $q_i=\eta q'_i$, $\eta\to\infty$.

I. INTRODUCTION

This is the second of two papers[1] dealing with light-cone singularities of bilocal operator products as manifested in perturbation theory. The present paper is devoted exclusively to a systematic investigation of the asymptotic contribution of an arbitrary Feynman graph to Fourier-transformed multiparticle matrix elements of time-ordered products:

$$\int d^4x\, e^{iqx}\langle\beta\,|\,T\psi_1(x)\psi_2(0)\,|\,\alpha\rangle$$

in the generalized Bjorken-Johnson-Low (BJL) limit[2]: $q_-\to\infty$ with $q_+, \vec{q}_\perp$ fixed.

[For reasons of typographical clarity, the notation used in this paper differs from that in I. For any vector q^μ, the quantities $q_\pm$ are not defined as vector components, but rather as scalar products: $q_\pm=q^\pm=M_\pm\cdot q$, $M_\pm^\mu=2^{-1/2}(1,0,0,\mp 1)$. Thus q_- or q^- is numerically equal to the − component of the contravariant vector.]

The asymptotic behavior of any single Feynman graph turns out to be of the form $q_-^{\ \beta}(\ln q_-)^\gamma$. Our main result here is the formulation of a rule according to which the integers β and γ can be read off from the topology of any given graph. The application of this rule to classify asymptotically all Feynman graphs for various multiparticle matrix elements is then undertaken in detail in the context of a particular Lagrangian field theory with a scalar-spinor interaction of the type $\bar{\psi}\psi\phi$.

The asymptotic behavior in q space can be translated into light-cone behavior, using the results of Ref. 2 and the Appendix of I. The coefficient of $1/q_-$ is the Fourier transform of the singular part of the light-cone commutator.[2] However, there are other terms which do not lead to singularities in commutators, but only in T products or Wightman products. In the Appendix to I, it is shown that if the index β is less than two, the graph has no singularities on the light cone, but that there are singularities if $\beta \geq 2$ (e.g., $\ln x^2$ corresponds to $\beta = 2$, $\gamma = 0$). Such singularities occur in fermion bilocals in perturbation theory and violate Bjorken scaling.

The paper is organized as follows: In Sec. II we develop a topological formula for writing down an arbitrary Feynman integral in its parametric form. This is essentially a generalization to particles with spin of a previous formula for the purely scalar case given by Shimamoto.[3]

In Sec. III we discuss the asymptotic contribution of various regions in Feynman-parameter space for an arbitrary graph and locate the leading asymptotic behavior. Our discussion is summarized in a *general* topological rule. The application of this rule depends on the knowledge, within a *particular* Lagrangian theory, of certain characteristic integers, the *asymptotic indices* associated with a special class of graphs which we shall call *m graphs*. The concepts and methods introduced in this section for the light-cone limit actually have a wider applicability. We illustrate this with a brief discussion of asymptotic behavior in the *Euclidean* case, in which we obtain a refinement of Weinberg's theorem[4] by determining the logarithmic factors in the asymptotic limit.

In Sec. IV we focus on the $\bar{\psi}\psi\phi$ interaction. Estimating the asymptotic indices turns out to be, at least in this theory, particularly easy. In fact, they can be obtained from a study of a small number of *tree graphs*. As we are then able to survey the asymptotics of all Feynman graphs in this theory, there emerges a striking feature: The light-cone singularities of the operator product (i.e., the collection of all its multiparticle matrix elements) can be isolated in a small number of irreducible Bethe-Salpeter kernels (which are functions of a finite number of space-time coordinates). The detailed implications of this phenomenon for light-cone physics, if valid in all renormalizable field theories, are organized and discussed in operator language in the companion paper, where we formulate an expansion of bilocal operator products in terms of irreducible Bethe-Salpeter kernels.

The techniques developed in this paper enable one to extract the exact power of $\ln q_-$ in the $q_- \to \infty$ behavior of individual Feynman integrals and, moreover, to see that these logarithmic powers are associated with iterations of the irreducible kernels. However, we have not here addressed ourselves to the circumstances under which these logarithmic terms could build up to an essential modification of light-cone behavior. It is also appropriate to mention, at the outset, an important qualification of our present work on perturbation theory: We have only considered "skeleton" graphs, i.e., convergent Feynman integrals, with bare propagators and vertices. We, nevertheless, expect that the inclusion of the full renormalized propagators and vertex parts will not seriously affect our conclusions.

Sections II, III, and IV are rather long and involved. The reader who is not interested in these details is urged to turn to Sec. V, which is a brief summary of the main results.

II. PARAMETRIC FEYNMAN INTEGRALS

Our study of the asymptotic properties of Feynman integrals will be carried out in terms of their *parametric form*, namely, the form obtained by combining the denominators of all propagators through the Feynman identity and then explicitly performing the integration over internal momenta. Although the original defining form reflects the topology of the graph more directly, it does not seem to be appropriate for reliable asymptotic estimates. This is because the Feynman integrals in the original form are never absolutely convergent due to the indefiniteness of the Lorentz metric—even before the $i\epsilon \to 0$ limit is taken in the denominators. For this reason Weinberg's powerful theorem[4] on the asymptotic behavior of Feynman integrals is established only for Euclidean external momenta and cannot be directly applied where some or all of the external momenta are on the mass shell. In contrast, the parametric form is absolutely convergent (if $i\epsilon \neq 0$, as we will assume throughout our discussion); moreover, it is possible to give a relatively simple topological recipe for writing down the integrand as a function of the Feynman parameters and the external momenta. Such a prescription was given long ago by Shimamoto[3] and represented a refinement of previous work by Nambu,[5] Symanzik,[6] and Nakanishi.[7] Since Shimamoto's formula is, strictly speaking, applicable to graphs with only scalar particles, we shall need to generalize it in this section to the general case of particles with spin and in particular spin-$\frac{1}{2}$ particles.

It will be convenient to introduce certain preliminary topological concepts in relation to graphs. Let G be an arbitrary connected[8] Feynman graph with L independent loops and I internal lines with Feynman parameters $x_1, x_2, x_3, \ldots, x_I$.

Definition: A chord set h is a set of L internal lines $\{i_1, i_2, \ldots, i_L\}$ whose complement is loopless and connected (i.e., G-h is a tree). The product $x_{i_1}x_{i_2}\cdots x_{i_L}$ will be referred to as a chord-set product and will be denoted by $\chi(h)$.

Definition: A cut set c is a set of $L+1$ internal lines $\{i_1, i_2, \ldots, i_{L+1}\}$ whose complement consists of two vertex-disjoint, connected, loopless parts (i.e., G-c consists of two disjoint trees). The product $x_{i_1}x_{i_2}\cdots x_{i_{L+1}}$ will be referred to as a cut-set product and will be denoted by $\chi(c)$.

The sum of the external momenta attached to one of the two trees G-c will be referred to as the cut-set momentum P_c. In general, we shall further need to specify which one of the two trees P_c is associated with. Because of the over-all momentum conservation, this is only an over-all sign ambiguity and, for example, in Shimamoto's formula for the purely scalar case, only P_c^2 appears and no further specification is necessary.

Consider now a graph G with only scalar propagators as, for example, in a theory of scalar particles coupled by cubic and quartic interactions of the type ϕ^3 and ϕ^4. According to the Feynman rules (i) to each internal line carrying momentum Q_i there corresponds a propagator $(2\pi)^4 i\,(Q_j^2 - m_i^2 + i\epsilon)^{-1}$, where m_i is the mass of the associated particle, and (ii) to each vertex there corresponds a factor of $(2\pi)^4 i\,\delta^4(\sum Q)\times$(coupling constants), where $\sum Q$ is the sum of the momenta converging to that vertex.

We obtain the parametric form by first combining the denominators[9] by means of the Feynman identity

$$\sum_{j=1}^{I} x_j\,(Q_j^2 - m_j^2 + i\epsilon)\,,$$

and then performing the momentum integrations. Shimamoto's formula for the resulting integral over $x_1, x_2, \ldots, x_I$ reads[10]

$$F(G) = (\text{coupling constant})\frac{(I-2L-1)!}{(4\pi)^L} \times \int_0^1 \frac{dx_1\cdots dx_I\,\delta(\sum x_j - 1)}{U^2\,|\,\phi/U - \sum x_j m_j^2 + i\epsilon\,|^{\,I-2L}}\,, \tag{1}$$

where

$$U = \sum_{\text{all } h}\chi(h)\,, \tag{2}$$

$$\phi = \sum_{\text{all } c} P_c^{\,2}\chi(c)\,. \tag{3}$$

For later use we note that ϕ/U is equal to the extremum of the sum $\sum x_j Q_j^2$ under variations of the components of all Q_j's subject to the four-momentum conservation constraints at the vertices. The extremizing values $\overline{Q}_j$ of these momenta satisfy Kirchhoff's law:

$$\sum x_j\,\overline{Q}{}_j^{\mu} = 0 \text{ along any loop}\,. \tag{4}$$

Another immediate consequence of the extremum property of which we shall later make use is

$$\frac{\partial}{\partial x_j}\left(\frac{\phi}{U}\right) = \overline{Q}_j^{\,2} \quad (\text{all } j)\,. \tag{5}$$

The formal correspondence to an electric circuit with the parameters $x_1, x_2, \ldots, x_I$ having the role of ohmic resistances has been frequently employed in the literature.[11]

We shall now proceed to a generalization of formula (1) for graphs having a number of internal particles with spin. The integrand of any Feynman integral with some polynomial in the momenta occuring in the numerator is a linear combination of terms of the form

$$\frac{Q_1^{\mu_1}Q_2^{\mu_2}\cdots Q_\sigma^{\mu_\sigma}}{(Q_1^2 - m_1^2 + i\epsilon)(Q_2^2 - m_2^2 + i\epsilon)\cdots(Q_I^2 - m_I^2 + i\epsilon)}\,. \tag{6}$$

To obtain the parametric form for this integral we make use of the scalar formula (1) by the device of writing

$$\frac{Q_j^{\mu}}{Q_j^2 - m_j^2 + i\epsilon} = \frac{1}{2}\int_{m_j^2}^{\infty} d\tau_j\,\frac{\partial}{\partial a_j^{\mu}}\,\frac{1}{(Q_j + a_j)^2 - \tau_j + i\epsilon}\bigg|_{a_i^{\mu}=0}$$

or symbolically,

$$\frac{Q_j^{\mu}}{Q_j^2 - m_j^2 + i\epsilon} = L_j^{\mu}\,\frac{1}{(Q_j + a_j)^2 - \tau_j + i\epsilon}\,,$$

which defines L_j^{μ} as an operator. We may then write the integrand (6) as

$$L_1^{\mu_1}L_2^{\mu_2}\cdots L_\sigma^{\mu_\sigma}\,\frac{1}{[(Q_1 + a_1)^2 - m_1^2 + i\epsilon][(Q_2 + a_2)^2 - m_2^2 + i\epsilon]\cdots}\,.$$

If we now, *before* the application of the operator $L_1^{\mu_1}L_2^{\mu_2}\cdots$, perform the momentum integrations by means of the Feynman identity, the resulting parametric form will obviously be that of a purely scalar graph with σ auxiliary external momenta $a_1, a_2, \ldots, a_\sigma$ (one for each fermion line) assigned as follows: If the ith fermion line joins the vertices v and v' and carries momentum Q_i^μ in the direction from v to v', then an auxiliary external momentum a_i^μ enters at v and exits at v'. The resulting parametric integral according to formula (1) reads

$$\frac{(I-2L-1)!}{(4\pi)^{2L}} L_1^{\mu_1} L_2^{\mu_2} \cdots L_\sigma^{\mu_\sigma} \int \frac{dx_1 dx_2 \cdots dx_I \, \delta(\sum x_j - 1)}{U^2 D^{I-2L}},$$

where

$$D \equiv \frac{\phi}{U} - \sum m_j^2 x_j + \frac{2}{U}\sum_{j=1}^{\sigma} a_j^\mu d_j^\mu + \frac{4}{U}\sum_{i<j}^{\sigma} (a_i a_j) r_{ij}$$

$$+ (\text{terms prop to } a_1^2, a_2^2, \ldots) + i\epsilon,$$

$$\phi \equiv \sum_{\text{all } c} P_c^2 \chi(c),$$

$$U \equiv \sum_{\text{all } h} \chi(h),$$

$$d_i^\mu \equiv \sum_{c \supset i} P_c^\mu \chi(c), \quad (7)$$

$$r_{ij} \equiv \sum_{c \supset i,j} \pm \chi(c). \quad (8)$$

Thus ϕ and U are the same expressions as in the purely scalar case. In addition, two new types of quantities appear, d_i^μ and r_{ij}, whose symbolic definitions as given above need further clarification. Let the ith fermion line join the vertices v_i and v_i' and be oriented, in the sense described above, from v_i to v_i'. Then the sum in the definition of d_i^μ runs over those cut sets c for which v_i and v_i' belong to *different* trees in G-c, and P_c^μ is the total external momentum (not including the auxiliary momenta) coming *into* the tree containing v_i. The sum in the definition of r_{ij} runs over those cut sets c for which either (i) v_i and v_j belong to one tree in G-c and v_i', v_j' to the other, or (ii) v_i and v_j' belong to one tree and v_i', v_j to the other. The + sign is appropriate in case (i) and the − sign in case (ii).

Note that the terms proportional to the squares a_1^2, a_2^2, ... can be dropped at this stage since all a_i's are to be set equal to zero after the differentiations implied by $L_1^{\mu_1} L_2^{\mu_2} \cdots L_\sigma^{\mu_\sigma}$.

The result of carrying out the operations $L_1^{\mu_1} L_2^{\mu_2} \cdots$ on D^{-I+2L} is

$$L_1^{\mu_1} L_2^{\mu_2} \cdots L_\sigma^{\mu_\sigma} D^{-I+2L} = \frac{d_1^{\mu_1}}{x_1 U}\frac{d_2^{\mu_2}}{x_2 U}\cdots\frac{d_\sigma^{\mu_\sigma}}{x_\sigma U} D_0^{-I+2L} + \left[\frac{r_{12} g^{\mu_1\mu_2}}{x_1 x_2 U}\frac{d_3^{\mu_3}}{x_3 U}\frac{d_4^{\mu_4}}{x_4 U}\cdots\frac{d_\sigma^{\mu_\sigma}}{x_\sigma U} + \text{permutations}\right]\frac{D_0^{-I+2L+1}}{I-2L-1}$$

$$+ \left[\frac{r_{12} g^{\mu_1\mu_2}}{x_1 x_2 U}\frac{r_{34} g^{\mu_3\mu_4}}{x_3 x_4 U}\frac{d_5^{\mu_5}}{x_5 U}\cdots\frac{d_\sigma^{\mu_\sigma}}{x_\sigma U} + \text{permutations}\right]\frac{D_0^{-I+2L+2}}{(I-2L-1)(I-2L-2)} + \cdots,$$

with

$$D_0 = \frac{\phi}{U} - \sum x_j m_j^2 + i\epsilon.$$

Here "+ permutations" mean that all *distinct* terms obtained from the given one by permutations of the indices 1, 2, ..., σ are to be added.

We may employ a shorthand notation to write this as

$$L_1^{\mu_1} L_2^{\mu_2} \cdots L_\sigma^{\mu_\sigma} D^{-I+2L} = \sum_\rho \frac{(I-2L-\rho-1)!}{(I-2L-1)!} D_0^{-I+2L+\rho}\left[\frac{d_1^{\mu_1 \cdot}}{x_1 U}\frac{d_2^{\mu_2 \cdot}}{x_2 U}\cdots\frac{d_\sigma^{\mu_\sigma}}{x_\sigma U} + \text{permutations}\right]_{\rho \text{ pairings}}$$

if we introduce the "pairing" dots as

$$d_i^{\mu \cdot} \cdots d_j^{\nu \cdot} = U g^{\mu\nu} r_{ij}. \quad (9)$$

Making use of double dots, triple dots, etc., to distinguish between different pairings, we see that the integrand (6) leads to the parametric integrand

$$\sum_\rho \frac{(I-2L-\rho)!}{(4\pi)^{2L}} \int \frac{dx_1 dx_2 \cdots dx_I \, \delta(\sum x_j - 1)}{U^2 [\phi/U - \sum x_j m_j^2 + i\epsilon]^{I-2L-\rho}} \left[\frac{d_1^{\mu_1 \cdot}}{x_1 U}\frac{d_2^{\mu_2 \cdot}}{x_2 U}\frac{d_3^{\mu_3 \cdot\cdot}}{x_3 U}\cdots\frac{d_\sigma^{\mu_\sigma}}{x_\sigma U} + \text{permutations}\right]_{\rho \text{ pairings}}.$$

Consider now a complete Feynman integral. For concreteness, assume that the only momentum-dependent factors in the numerator come from fermion propagators. The general form of the integral is then

$$F(G)=\int\frac{d^4k_1}{(2\pi)^4 i}\cdots\frac{d^4k_L}{(2\pi)^4 i}\,\frac{\not{Q}_1+m_1}{Q_1^2-m_1^2+i\epsilon}\,\Gamma_{12}\,\frac{\not{Q}_2+m_2}{Q_2^2-m_2^2+i\epsilon}\,\Gamma_{23}\cdots\frac{\not{Q}_\sigma+m_\sigma}{Q_\sigma^2-m_\sigma^2+i\epsilon}\,\frac{1}{Q_{\sigma+1}^2-m_{\sigma+1}^2+i\epsilon}\cdots,$$

where $k_1, k_2, \ldots, k_L$ is a set of independent loop moment and Γ_{12}, Γ_{23}, $\cdots$ are momentum-independent matrices. According to our previous analysis, the parametric form of $F(G)$ can be written as

$$F(G)=\sum_\rho\frac{(I-2L-\rho)!}{(4\pi)^{2L}}\int\frac{dx_1dx_2\cdots dx_I\,\delta(\sum x_j-1)}{U^2[\phi/U-\sum x_jm_j^2+i\epsilon]^{I-2L-\rho}}$$
$$\times\left\{\left(\frac{\not{d}_1}{x_1U}+m_1\right)^{\cdot}\Gamma_{12}\left(\frac{\not{d}_2}{x_2U}+m_2\right)^{\cdot}\Gamma_{23}+\cdots+\text{permutations}\right\}_{\rho\text{ pairings}}, \tag{10}$$

where we used the pairing dots for *fermion numerators*:

$$\cdots\left(\frac{\not{d}_j}{x_jU}+m_j\right)^{\cdot}\cdots\left(\frac{\not{d}_k}{x_kU}+m_k\right)^{\cdot}\cdots=\cdots\frac{r_{jk}}{x_jx_kU}\gamma_\mu\cdots\gamma^\mu\cdots. \tag{11}$$

The sum in the curly brackets in Eq. (10) is taken over all the terms obtained by forming exactly ρ pairs in all possible ways out of the set of fermion lines.

As an illustration we consider the 6th order graph of Fig. 1 for a $\bar\psi\psi\phi$ theory of fermions (solid lines) and scalars (dotted lines). We first write down the quantities U, ϕ, d_i, r_{ij} according to their topological definitions:

$$U=(x_1+x_3+x_5)(x_2+x_4+x_6+x_7)+x_6(x_1+x_2+x_3+x_4+x_5+x_7),$$
$$\phi=q_1^2[x_1x_5(x_2+x_4+x_6+x_7)+x_2x_5x_6]+q_2^2[x_3x_5(x_2+x_4+x_6+x_7)+x_4x_5x_6]$$
$$+p_1^2[x_2x_7(x_1+x_3+x_5+x_6)+x_1x_6x_7]+p_2^2[x_4x_7(x_1+x_3+x_5+x_6)+x_3x_6x_7]$$
$$+(p_1+p_2)^2[x_1x_3(x_2+x_4+x_6+x_7)+x_1x_4x_6+x_2x_3x_6+x_2x_4(x_1+x_3+x_5+x_6)]+(p_1+p_3)^2x_5x_6x_7,$$
$$\frac{d_1}{x_1}=x_5(x_2+x_4+x_6+x_7)q_1+[x_3(x_2+x_4+x_6+x_7)+x_4x_6](q_1+q_2)-x_6x_7p_1,$$
$$\frac{d_2}{x_2}=-x_7(x_1+x_3+x_5+x_6)p_1-[x_4(x_1+x_3+x_5+x_6)+x_3x_6](p_1+p_2)+x_5x_6q_1,$$
$$\frac{d_3}{x_3}=x_5(x_2+x_4+x_6+x_7)q_2+[x_1(x_2+x_4+x_6+x_7)+x_2x_6](q_1+q_2)-x_6x_7p_2,$$
$$\frac{d_4}{x_4}=-x_7(x_1+x_3+x_5+x_6)p_2-[x_2(x_1+x_3+x_5+x_6)+x_1x_6](p_1+p_2)+x_5x_6q_2,$$
$$\frac{r_{13}}{x_1x_3}=x_2+x_4+x_6+x_7,\quad\frac{r_{24}}{x_2x_4}=x_1+x_3+x_5+x_6,\quad\frac{r_{14}}{x_1x_4}=\frac{r_{23}}{x_2x_3}=\frac{r_{12}}{x_1x_2}=\frac{r_{34}}{x_3x_4}=x_6.$$

The parametric integral is

$$F(G)=\frac{1}{(4\pi)^4}\int\frac{dx_1\cdots dx_7}{U^2}\delta\left(\sum_{j=1}^7x_j-1\right)\left[\frac{3!}{D_0^3}\psi_{1234}+\frac{2!}{D_0^2}(\psi_{(12)34}+\psi_{(13)24}+\psi_{(14)32}+\psi_{(23)14}+\psi_{(24)13}+\psi_{12(34)})\right.$$
$$\left.+\frac{1}{D_0}(\psi_{(12)(34)}+\psi_{(13)(24)}+\psi_{(14)(23)})\right],$$

where $D_0=\phi/U-\sum x_jm_j^2+i\epsilon$ and the ψ's stand for the various ways of pairing fermion lines, in an obvious notation. For example, ψ_{1234} is the no-pairing term

$$\psi_{1234}=\left(\frac{\not{d}_1}{x_1U}+m_1\right)\left(\frac{\not{d}_2}{x_2U}+m_2\right)\left(\frac{\not{d}_3}{x_3U}+m_3\right)\left(\frac{\not{d}_4}{x_4U}+m_4\right).$$

A sample of some of the pairing terms is

$$\psi_{(12)34}=\frac{4r_{12}}{x_1x_2U}\left(\frac{\not{d}_3}{x_3U}+m_3\right)\left(\frac{\not{d}_4}{x_4U}+m_4\right),$$
$$\psi_{(13)24}=\frac{4r_{13}}{x_1x_3U}\gamma^\mu\left(\frac{\not{d}_2}{x_2U}+m_2\right)\gamma_\mu\left(\frac{\not{d}_4}{x_4U}+m_4\right),$$
$$\psi_{(14)(23)}=\frac{4r_{14}}{x_1x_4U}\frac{4r_{23}}{x_2x_3U}\gamma^\mu\gamma^\nu\gamma_\mu\gamma_\nu.$$

It is clear that an analogous notation for organizing various terms in the parametric form of a Feynman integral can be developed in the general case (e.g., derivative couplings, etc.) insofar as the integral can be written as a sum of terms of the form (6).

In general, writing out the parametric formula for the Feynman integral $F(G)$ will result in a considerable number of terms of the form[12]

$$\frac{r_{i_1 i_2} r_{i_3 i_4}\cdots r_{i_{2\rho-1} i_{2\rho}} d_{i_{2\rho+1}}\cdots d_{i_\tau}}{x_{i_1}x_{i_2}\cdots x_{i_\tau} D_0^{I-2L-\rho} U^{\tau-\rho+2}} . \tag{12}$$

For our subsequent discussion of asymptotic behavior it will be convenient to *further* decompose the integrand as follows. In the defining expression (7) for the "numerator momenta" d_i, we group cut-set products associated with the same cut-set momentum and write

$$d_i = \sum_\omega P_\omega \Big(\sum_{c_\omega \supset i} \chi(c_\omega) \Big)$$
$$\equiv \sum_\omega P_\omega d_i(\omega) ,$$

where P_ω is the sum of a subset ω of the external momenta and c_ω is a cut set with cut-set momentum P_ω. Accordingly, each of the terms (12) is decomposed into a sum of terms like

$$\frac{r_{i_1 i_2}\cdots r_{i_{2\rho-1} i_{2\rho}} d_{i_{2\rho+1}}(\omega_{i_{2\rho+1}})\cdots d_{i_\tau}(\omega_{i_\tau})}{x_{i_1}x_{i_2}\cdots x_{i_\tau} D_0^{I-2L-\rho} U^{\tau-\rho+2}} ,$$

each of which is specified by (i) the set of numerator momenta d which are paired and the particular pairing pattern, (ii) the set of unpaired d's and the corresponding P_ω's.

We shall call the specifications (i) and (ii) a numerator pattern P and we shall denote by $F(G,P)$ the corresponding contribution to $F(G)$:

$$F(G) = \sum_P F(G,P) .$$

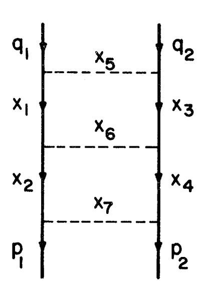

FIG. 1. A 6th-order graph in $\bar\psi\psi\phi$ theory used in the text to illustrate the topological construction of the parametric integral with spin.

III. ASYMPTOTIC BEHAVIOR OF FEYNMAN INTEGRALS

We are interested in the light-cone behavior of time-ordered products of local field operators. Consider, for example, the case of two Dirac fields (one of which may be an adjoint field):

$$\int d^4x\, e^{iqx} \langle B\,|\, T(\psi_{1\alpha}(x)\psi_{2\beta}(0))\,|\,A\rangle = C_{\alpha\beta}(q; P_1, P_2, \ldots) .$$

In perturbation theory, contributions to the connected part of $C_{\alpha\beta}$ correspond to connected graphs G. We explore the light cone in the limit $q_- \to \infty$, with q_+, $\vec{q}_\perp$; $p_1, p_2, \ldots, p_n$ fixed. If we display explicitly the external propagators associated with the fields ψ_1 and ψ_2,

$$F(G) = \left(\frac{\not q + m_1}{q^2 - m_1^2} \right)_{\alpha\gamma} \left(\frac{\not q + \not p + m_2}{(q+p)^2 - m_2^2} \right)_{\beta\delta}$$
$$\times I_{\gamma\delta}(q; p_1, p_2, \ldots, p_n) ,$$

the quantity $I_{\gamma\delta}$ will be associated with the "internal" lines of G. For ease of reference, it will be convenient to combine the two propagators for ψ_1 and ψ_2 with the ones in $I_{\gamma\delta}$ by the Feynman identity, although this is not necessary. In general the contribution of a particular numerator pattern P of a graph G with L independent loops and I propagators is of the form

$$F(G,P) = \int \frac{dx_1\cdots dx_I \delta(\sum x_j - 1)}{U^2[\phi/U - \sum x_j m_j^2 + i\epsilon]^{I-2L-\rho(P)}}$$
$$\times M_{G,P}(x_1, \ldots, x_I) H_{G,P}(q; p_1, p_2, \ldots) ,$$

where $\rho(P)$ is the number of pairings and the dependence on the external momenta of the numerator is factored in the quantity $H_{G,P}$, which is a polynomial in the components of $q, p_1, p_2, \ldots, p_n$.

Since we are only interested in the $q_- \to \infty$ behavior of $F(G,P)$, we may simplify this expression for $F(G,P)$ as follows:

(1) We replace $H_{G,P}$ by its large-q_- behavior: $H_{G,P} \to q_-^{N(G,P)} \hat{H}_{G,P}(q_+, \vec{q}_\perp; p_1, p_2, \ldots, p_n)$.

(2) Since $\max m_j^2 \geq \sum x_j m_j^2 \geq \min m_j^2$, we replace the sum $\sum x_j m_j^2$ in the denominator by a constant m^2 (this amounts to assuming that all the masses are equal to m).

(3) The quantity ϕ in the denominator is a linear combination of the Lorentz scalar products

$$q^2, qp_1, qp_2, \ldots, p_1^2, p_1p_2, \ldots ,$$

with coefficients depending on the parameters $x_1, x_2, \ldots, x_I$. Asymptotically we have

$$q^2, qp_1, \ldots \to q_-q_+, q_-p_1, \ldots \sim q_- ; \quad p_ip_j \text{ fixed.}$$

We may separate the q_--dependent terms by writing

$$\phi = fq_- + (\text{terms proportional to fixed momenta}),$$

$$f = \sum_c \chi(c) P_c^+ .$$

We now drop the terms which are proportional to fixed momenta and simply write $\phi = fq_-$, as this will not affect the leading asymptotic behavior of $F(G,P)$.

Note that $q_- f/c - m^2$ is the exact denominator if (1) all internal masses are equal to m and (2) $p_1, p_2, \ldots, p_n$ only have a + component and also $\vec{q}_\perp = 0$.

Thus we may write

$$F(G,P) \sim \int dx_1 \cdots dx_I\, \delta\left(\sum x_j - 1\right) \times \frac{M_{G,P}(x) q_-^{N(G,P)} \hat{H}_{G,P}}{U^2 [f/U q_- - m^2 + i\epsilon]^{I-2L-\rho(P)}}. \quad (13)$$

The reader should be warned at this point that, although in so simplifying the integrand the *leading* term in $F(G,P)$ is not altered, it is conceivable that in the sum $F(G) = \sum_P F(G,P)$, the leading terms could cancel, in which case our subsequent result for the asymptotic behavior of $F(G,P)$ would be an *overestimate*. It would then be necessary to consider the next leading terms in the $F(G,P)$'s. Such cancellations, as well as cancellations between *different* graphs, cannot be dismissed as totally unlikely.[13]

Let us now consider the contribution from various regions in parameter space in the limit $q_- \to \infty$. It is clear that the contribution of any region where $f/U \neq 0$ behaves like $q_-^{\beta(G,P)}$, where

$$\beta(G,P) = N(G,P) - I + 2L + \rho(P).$$

The integer $\beta(G,P)$ will be called the *asymptotic index* of G associated with the numerator pattern P.

Next, let $f/U = 0$ at $x_i = \bar{x}_i$, $(i = 1, 2, \ldots, I)$. Assume first that

$$\sum_i \bar{x}_i \left[\frac{\partial}{\partial x_i}\left(\frac{f}{U}\right)\right]^2_{\bar{x}} \neq 0,$$

and consider the straight line

$$x_i(\xi) = \bar{x}_i + \eta_i \xi \text{ with } \eta_i = \bar{x}_i \frac{\partial}{\partial x_i}\left(\frac{f}{U}\right)_{\bar{x}}. \quad (14)$$

[Note that points on this line satisfy $\sum x_i = 1$ because

$$\sum \eta_i = \sum \bar{x}_i \frac{\partial}{\partial x_i}\left(\frac{f}{U}\right)_{\bar{x}} = \left(\frac{f}{U}\right)_{\bar{x}},$$

where the last equality follows from the fact that f/U is a homogeneous function of the x_i's of degree one.] We look now at the values of f/U on the line in the neighborhood of $\xi = 0$ by expanding

$$\frac{f}{U} = \xi \sum \eta_i \frac{\partial}{\partial x_i}\left(\frac{f}{U}\right)_{\bar{x}} + O(\xi^2).$$

Since the coefficient of ξ is not zero by assumption, f/U *changes sign* at $\xi = 0$. From Eq. (14) it further follows that if, for some i, $\bar{x}_i = 0$ then also $x_i(\xi) = 0$ so that a finite segment of the line containing $\xi = 0$ lies in the integration domain $\{x_i \geq 0; \sum x_i = 1\}$. If we now take ξ as one of the integration variables (by setting, e.g., $x_i = \bar{x}_i + \eta_i \xi$ for some i for which $\eta_i \neq 0$) our integral takes the form

$$\int dx \cdots \int_{\xi_1}^{\xi_2} \frac{d\xi Q(\xi, x)}{[q_- f/U - m^2 + i\epsilon]^{I-2L-\rho(P)}} q_-^{N(G,P)}, \quad (15)$$

where $\xi_1 < 0 < \xi_2$, and f/U, in its dependence on ξ, is analytic with a nonvanishing derivative at $\xi = 0$. Under these circumstances the path of the ξ integration may be distorted in a small semicircle about $\xi = 0$ in the lower or upper half plane [depending on the sign of $q_- \partial(f/U)/\partial\xi$] and it follows that the integral behaves like $q_-^{\beta(G,P)}$ just as from regions where $f/U \neq 0$.

At this point a clarification is necessary. We have taken for granted that f/U is analytic at $x_i = \bar{x}_i$. Since f and U are both polynomials, this is true except when $U = 0$. From its definition, U vanishes if and only if the Feynman parameters of at least one loop vanish. In that case f *also* vanishes and f/U is indeterminate. This formal difficulty can be overcome by introducing a scaling parameter λ. Let, for example, $x_1, x_2, \ldots, x_m$ be the parameters of the loop and set $x_i = \lambda x'_i$, $i = 1, 2, \ldots, m$ with $\sum x'_i = 1$. For small λ we have $f \sim \lambda$ and $U \sim \lambda$ so that, in terms of the new variables λ and x'_i, the ratio f/U is analytic at $\lambda = 0$ (and the x'_i's cannot vanish simultaneously). Our previous argument then applies. (If the parameters of more than one loop vanish, we simply introduce one scaling parameter for each loop.)

It remains to discuss what happens when

$$\sum x_i \left[\frac{\partial}{\partial x_i}\left(\frac{f}{U}\right)\right]^2 = 0$$

for $x_i = \bar{x}_i$. This means that, for every i, *either* $\bar{x}_i = 0$ or

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial U}\right)\bigg|_{\bar{x}} = 0.$$

Recall, however, that [see Eq. (5)]

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial U}\right)_{\bar{x}} = \bar{Q}^2 = (\bar{Q}_i)_+ (\bar{Q}_i)_-,$$

where $\{\bar{Q}_i\}$ extremizes $\sum \bar{x}_i Q_i^2$ [with $\vec{q}_\perp = (\vec{p}_i)_\perp = (p_i)_- = 0$]. In terms of the electric circuit analogy this means that the lines of our graph are divided into *three disjoint sets*: (1) lines of zero resistance (i.e., shortcircuited), (2) lines carry-

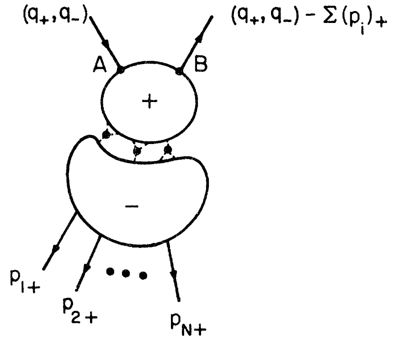

FIG. 2. The + and − components of the extremizing momenta flow in disjoint parts of the graph.

ing only + current, and (3) lines carrying only − current. Since Kirchhoff's laws hold separately for the + and − currents, it is easy to see that the potential between the points A and B (where q_- and q_+ flow in and out of the circuit) must be the same (see Fig. 2). Thus the resistance between A and B must be zero, namely, the *parameters of a set of lines forming a continuous path from A to B must vanish.*

Before going into a systematic discussion of the asymptotic contributions of such vanishing sets of parameters, it will perhaps be helpful for the reader to visualize the relevant mechanism in a general way. Let $x_1, x_2, \ldots, x_n$ be the parameters of a set of lines forming exactly l-independent loops and containing a continuous path from A to B. Introduce a scaling parameter by $x_i = \lambda x_i'$, $i = 1, 2, \ldots, n$ with $\sum x_i' = 1$ as above. From the topological definition of f and U it follows that $f \sim \lambda^{l+1}$ and $U \sim \lambda^l$ for small λ. Accordingly, we have a contribution to $F(G,P)$ from the neighborhood of $\lambda = 0$ of the form

$$q_-^{N(G,P)} \int_0^\epsilon \frac{d\lambda\, \lambda^{n-1+\tau}}{(\lambda q_- - m^2 + i\epsilon)^{I-2L-\rho(P)}},$$

where λ^τ is the small-λ behavior of $M_{G,P}/U^2$ [see Eq. (13)]. Clearly if $n+\tau > I - 2L - \rho(P)$ this integral still behaves like $q_-^{(G,P)}$. However, in contrast with the integral of Eq. (15), if $n+\tau \leq I - 2L - \rho(P)$ the behavior is $q_-^{N(G,P)-n-\tau}$ (times $\ln q_-$ if the equality sign holds). This is because the lower limit of the λ integration is zero, i.e., the vanishing of the coefficient of q_- in the denominator cannot be avoided by a distortion of contours into the complex plane.

These *end-point* contributions are familiar in the literature from the study of the high-energy behavior of scattering amplitudes in perturbation theory.[14] In the case of scattering amplitudes where all external momenta are on the mass shell and, in particular, for fixed momentum transfer, in addition to end-point contributions, there appears another mechanism, the so-called pinch-contributions, associated with a certain class of nonplanar graphs.[15] This is essentially because, in the high-energy fixed-momentum-transfer limit of scattering amplitudes, the + and − currents can flow in separate pieces of the graph without the vanishing of all cut-set products in the coefficient of the large variable in the denominator of the integral. The so-called Mandelstam graphs, which are associated with moving cuts in the complex angular momentum plane and the Gribov-Pomeranchuk singularities, correspond, in electric circuit language, to a kind of double Wheatstone bridge (see Fig. 3). *It is an important simplification that, as we have shown above, there are no pinch-contributions in the light-cone limit* $q_- \to \infty$. This phenomenon can be traced to the simple fact that the + and − components of q are always present at the external vertices A and B (the pinch contributions would reappear for $q_+ = 0$ but this would not be the light-cone limit for local operators).

We proceed to discuss the end-point contributions in a systematic way. Consider an end-point set of lines E with parameters $x_1, x_2, \ldots, x_n$. Suppose a subset σ of E with parameters $x_1, x_2, \ldots, x_m$ is *disconnected* from A and B (i.e., there is no continuous path of lines of E from A to any of the lines of σ). Introducing a scaling parameter, as above, for the parameters of σ we have, for small λ $f \sim \lambda^l$ and $U \sim \lambda^l$, where l is the number of independent loops of σ. Therefore f/U does not vanish for small λ. Clearly then, in terms of the new variables $\lambda, x_1', x_2', \ldots, x_m', x_{m+1}, \ldots, x_n$, it is just the vanishing of $x_{m+1}, x_{m+2}, \ldots, x_n$ that makes f/U vanish. We may thus restrict ourselves from now on to *connected* end-point sets of lines. Moreover, it is easy to see that these sets need not contain "redundant" lines, i.e., lines whose removal from the set does not destroy any of its loops or the continuous path from A to B. We therefore introduce the following definition.

Definition: A *minimal end-point subgraph* or briefly an *m subgraph* is a set of lines of G with the properties:

(1) It is connected.

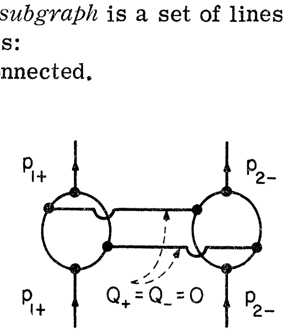

FIG. 3. Electrical-circuit analog of the Mandelstam graphs.

(2) It contains a set of lines forming a continuous path from A to B.

(3) The removal of any of its lines destroys *either* one of its loops or the path from A to B. We shall include G itself in the set of its m subgraphs.

Now let $(1, 2, 3, \ldots, I)$ be the lines of a Feynman graph G with L-independent loops. By definition $F(G, P)$ is given by a parametric integral associated with the numerator pattern P. Also let $(1, 2, \ldots, n)$ be the lines of a m subgraph g with l independent loops. In order to isolate and study the asymptotic contribution of the manifold $x_1 = x_2 = \ldots = x_n = 0$ (corresponding to the vanishing of the parameters of g) we introduce the quantity $F_{\epsilon,g}(G, P)$ as the value of the parametric integral for $F(G, P)$ when the domain of the integration is restricted by

$$x_1/\lambda, x_2/\lambda, \ldots, x_n/\lambda, x_{n+1}, x_{n+2}, \ldots, x_1 > \epsilon,$$

where $\lambda = \sum_1^n x_i$ and $1/I > \epsilon > 0$.

Next we consider the Feynman integral $F(g)$ associated with g itself with external momenta as shown in Fig. 4. Besides the external momenta q and $-q$ at the vertices A and B (just as in G), we assign a set of external momenta $p_1', p_2', \ldots$ at those vertices which g has in common with its complement in G. [When we later refer to the $q_- \to \infty$ behavior of $F(g)$ it will be meant under fixed but general values of $p_1', p_2', \ldots$.] $F(g)$ is itself a sum over parametric integrals of various numerator patterns P' of g:

$$F(g) = \sum_{P'} F(g, P').$$

We shall focus on those P' which are *compatible* with P. By this we shall mean the following relation between P' and P to be denoted by $P' \subset P$:

(1) If two numerator momenta are paired in g under P', their counterparts in G are also paired under P.

(2) If $d_i'(\omega')$ is an unpaired numerator momentum in P', its counterpart $d_i(\omega)$ in D is also unpaired and ω' contains q if and only if ω contains q.

Our first step will be to obtain the asymptotic behavior of $F_{\epsilon,g}(G, P)$. According to our analysis in Sec. II, $F(G, P)$ is of the form

$$F(G,P) \sim \int \frac{d\alpha_1 d\alpha_2 \cdots d\gamma_1 d\gamma_2 \cdots \delta(\sum \alpha + \sum \gamma - 1)}{U^2[f/Uq_- - m^2 + i\epsilon]^{I-2L-\rho(P)}} \frac{A_1}{\alpha_1 U} \frac{A_2}{\alpha_2 U} \cdots \frac{C_1}{\gamma_1 U} \frac{C_2}{\gamma_2 U} \cdots \frac{A_i^{\cdot} A_j^{\cdot}}{\alpha_i \alpha_j U} \cdots \frac{A_i^{\cdot} C_j^{\cdot}}{\alpha_i \gamma_j U} \cdots \frac{C_i^{\cdot} C_j^{\cdot}}{\gamma_i \gamma_j U} \cdots$$

$$= \int \frac{d\alpha \cdots d\gamma \cdots \delta(\sum \alpha + \sum \gamma - 1)}{U^2[f/Uq_- - m^2 + i\epsilon]^{I-2L-\rho(P)}} M_{G,P}(\alpha, \gamma) \hat{H}_{P,G} q_-^{N(G,P)},$$

where, for ease of reference, we denoted by $\alpha_1, \alpha_2, \ldots, \alpha_n$ the parameters of g and by $\gamma_1, \gamma_2, \ldots$ those of G-g. Also we denoted by A_i and C_i the numerator momenta "d_i" for g and G-g, respectively. (We have omitted the vector indices on A_i and C_j for simplicity.) If we introduce a scaling parameter for the lines of g:

$$\alpha_1 = \lambda \alpha_i'; \quad \sum \alpha_i' = 1, \qquad d\alpha_1 d\alpha_2 \cdots d\alpha_n = \lambda^{n-1} d\lambda \, d\alpha_1' d\alpha_2' \cdots d\alpha_n' \delta\left(\sum \alpha_i' - 1\right),$$

we have, by definition,

$$F_{\epsilon,g}(G,P) \sim \int_{\alpha', \beta > \epsilon} \frac{\lambda^{n-1} d\lambda \, d\alpha_1' \cdots d\gamma_1 \cdots \delta(\sum \alpha' - 1) \delta(\sum \gamma + \lambda - 1)}{U^2[f/Uq_- - m^2 + i\epsilon]^{I-2L-\rho(P)}} M_{G,P}(\lambda \alpha', \gamma) \hat{H}_{P,G} q_-^{N(G,P)}.$$

It is obvious that the $q_- \to \infty$ behavior of $F_{\epsilon,g}(G, P)$ depends on the small-λ behavior of f, U, and $M_{G,P}$. We shall examine each of these quantities separately.

(1) Every chord set of G must contain at least l lines belonging to g because its removal destroys all the loops of g. It follows that

$$U = \lambda^l \, \tilde{U}(\lambda, \alpha', \gamma),$$

where $\tilde{U}$ is a polynomial in λ, α', and γ. Note that $\tilde{U}(0, \alpha', \gamma)$ does not vanish identically because there is always at least one chord set which contains exactly l lines belonging to g.

(2) Every cut set of G in f contains at least $l+1$

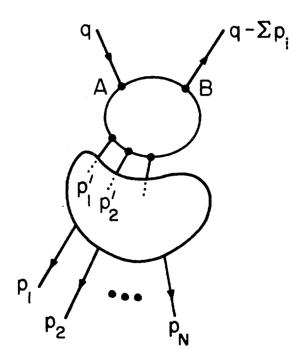

FIG. 4. External momenta assigned to an m subgraph g.

lines belonging to g because its removal destroys all the loops in g *and* disconnects A from B. It follows that

$$f=\lambda^{l+1}\tilde{f}(\lambda,\alpha',\gamma),$$

where again $\tilde{f}$ is a polynomial which does not vanish identically for $\lambda=0$.

(3) The small-λ behavior of M will be obtained by looking at the quantities $A_i/\alpha_i U$, $C_i/\gamma_i U$, etc., individually.

Each cut-set product in A_i, $A_i^{\cdot}A_j^{\cdot}$, or $A_i^{\cdot}C_i^{\cdot}$ contains, as factors, $l+1$ α parameters associated with one of several numerator patterns P' compatible with P. It follows that

$$\frac{A_i}{\alpha_i U}\sim\lambda^0 P_{\omega_i},\quad \frac{A_i^{\cdot}A_j^{\cdot}}{\alpha_i\alpha_j U}\sim\lambda^{-1},\quad \frac{A_i^{\cdot}C_j^{\cdot}}{\alpha_i\gamma_j U}\sim\lambda^0 .$$

The pairing C_iC_j contains cut-set products with l α's (but *none* with less than l α's) so that

$$\frac{C_i^{\cdot}C_j^{\cdot}}{\gamma_i\gamma_j U}\sim\lambda^0 .$$

Finally, if and only if the cut-set momentum in a given C_i contains q, each of the associated cut sets contains $l+1$ lines forming cut sets of g (of compatible patterns, i.e., containing q). It follows that

$$\frac{C_i}{\gamma_i U}\sim\lambda P_{\omega_i}\ \text{ if } \omega_i \text{ contains } q;$$

$$\frac{C_i}{\gamma_i U}\sim\lambda^0 P_{\omega_i}\ \text{ if } \omega_i \text{ does not contain } q.$$

This means that a power of λ multiplies every extra power of q_- that the integrand of $F(G,P)$ picks up in excess of the $q_-^{N(g,P')}$ of a compatible pattern P' of g. Thus the small-λ behavior of $M_{G,P}$ can be expressed by writing

$$M_{G,P}(\lambda',\gamma)q_-^{N(G,P)}=\sum_{P'\subset P}\lambda^{-\rho(P')}(\lambda q_-)^{N(G,P)-N(g,P')}q_-^{N(g,P')}\psi_{P'}(\lambda,\alpha',\gamma),$$

where the quantities $\psi_{P'}$ are not identically zero for $\lambda=0$.

Collecting powers of λ and recalling that $\beta(g,P')=N(g,P')-n+2l+\rho(P')$, we have

$$F_{\epsilon,g}(G,P)\sim\sum_{P'\subset P}\int_{\alpha',\gamma>\epsilon}d\alpha_1'\cdots d\gamma_1\cdots\int_0^{\epsilon}d\lambda\,\frac{\lambda^{-\beta(g,P')}(\lambda q_-)^{N(G,P)}\psi_{P'}(\lambda,\alpha',\gamma)}{\tilde{U}^2[\lambda f/\tilde{U}q_- - m^2+i\epsilon]^{N(G,P)-\beta(G,P)}} .$$

Because of the restriction α', $\gamma>\epsilon$, no end-point set of parameters for $\tilde{f}/\tilde{U}$ can vanish[16] so that the leading $q_-\to\infty$ behavior comes from the neighborhood of $\lambda=0$. The problem is thus reduced to a study of elementary integrals of the form

$$\int_0^{\epsilon}d\lambda\,\frac{\lambda^{-\beta(g,P')}(\lambda q_-)^{N(G,P)}}{[\lambda q_-+1]^{N(G,P)-\beta(G,P)}}\quad (q_-\to\infty).$$

The result is as follows: Let $\beta=\max_{P'\subset P}\beta(g,P')$; then

$$\begin{aligned}F_{\epsilon,g}(G,P)&\sim q_-^{\beta(G,P)}\ \text{ if } \beta<\beta(G,P)\\ &\sim q_-^{\beta}\ln q_-\ \text{ if } \beta=\beta(G,P)\\ &\sim q_-^{\beta}\ \text{ if } \beta>\beta(G,P).\end{aligned}$$

Observe that the asymptotic behavior of $F_{\epsilon,g}(G,P)$ depends only on the asymptotic indices of g and G and not on $N(G,P)$ or any other detail of the parametric forms. We shall, in fact, proceed to show that the knowledge of the asymptotic indices associated with all m subgraphs of G determines the asymptotic behavior of $F(G)$.

We begin by considering a numerator pattern P of G such that $F(G,P)$ behaves asymptotically like $F(G)$, i.e., $F(G)/F(P,G)\to$ finite $\neq 0$.[17] Because of the constraint $\sum_1^I x_i=1$ the leading behavior should be unaffected[18] if the range of a certain parameter, x_I say, is restricted away from zero: $x_I>\epsilon$. Let g_1 be the largest m subgraph contained in the set of lines $\{1,2,\ldots,I-1\}$; g_1 is well defined as the union of all m subgraphs contained in this set of lines. Without loss of generality we may take g_1 to consist of the lines $\{1,2,\ldots,n_1\}$. We introduce a scaling parameter λ_1 for g_1:

$$x_1=\lambda_1 x_1^{(1)},\quad i=1,2,\ldots,n_1;\quad \sum x_i^{(1)}=1,$$

$$dx_1dx_2\cdots dx_{n_1}=\lambda_1^{n_1-1}d\lambda_1dx_1^{(1)}\cdots dx_{n_1}^{(1)}\delta\left(\sum x_i^{(1)}-1\right).$$

Again because of the restriction $\sum x_i^{(1)}=1$ we may restrict the range of integration of a particular $x_i^{(1)}$, e.g., $x_{n_1}^{(1)}>\epsilon$ without affecting the leading behavior.

Next, let $g_2=\{1,2,\ldots,n_2\}$ be the largest m subgraph contained in the set $\{1,2,\ldots,n_1-1\}$ and introduce a second scaling parameter λ_2:

$$x_i^{(1)}=\lambda_2 x_i^{(2)},\quad i=1,2,\ldots,n_2;\quad \sum x_i^{(2)}=1,$$

$$dx_1^{(1)}dx_2^{(1)}\cdots dx_{n_2}^{(1)}=\lambda_2^{n_2-1}d\lambda_2dx_1^{(2)}dx_2^{(2)}\cdots dx_{n_2}^{(2)}\times\delta\left(\sum x_i^{(2)}-1\right).$$

Continuing this way we construct a sequence of "nested" m subgraphs:

$$G > g_1 > g_2 > \cdots > g_k ,$$

which terminates when we finally arrive at an m subgraph having no proper m subgraph, i.e., g_k is a tree graph.

If l_i is the number of independent loops in g_i, the small-λ_1, $\lambda_2, \ldots, \lambda_k$ behavior of f and U can be factored out as follows:

$$f = \lambda_1^{l_1+1} \lambda_2^{l_2+1} \cdots \lambda_k^{l_k+1} \tilde{f} ,$$

$$U = \lambda_1^{l_1} \lambda_2^{l_2} \cdots \lambda_k^{l_k} \tilde{U} .$$

Our restrictions on the integration domain mean, effectively, that $\tilde{f}/\tilde{U}$ can vanish at no end-point manifolds.

According to the analysis given in connection with $F_{\epsilon,g}(G,P)$ the small-λ_1, $\lambda_2, \ldots$ behavior of the expression M/U^2 in the integrand of $F(G,P)$ can be displayed as a sum over *all* sequences of compatible patterns:

$$\frac{M}{U^2} = \sum_{P \supset P_1 \supset P_2 \cdots} \lambda_1^{N(G,P) - \beta(g_1,P_1)} \times \lambda_2^{N(G,P) - \beta(g_2,P_2)} \cdots \times \psi_{P_1,P_2,\ldots} / \tilde{U}^2 .$$

$F(G,P)$ is thus essentially reduced to a set of elementary integrals of the form

$$J(P_1, P_2, \ldots, P_k) = \int_0^\epsilon d\lambda_1 \int_0^\epsilon d\lambda_2 \cdots \int_0^\epsilon d\lambda_k \frac{\lambda_1^{-\beta(g_1,P_1)} \lambda_2^{-\beta(g_2,P_2)} \cdots \lambda_k^{-\beta(g_k,P_k)} (\lambda_1\lambda_1 \cdots q_-)^{N(P,G)}}{[\lambda_1\lambda_2 \cdots \lambda_k q_- + 1]^{N(G,P) - \beta(G,P)}} ,$$

whose asymptotic behavior is easily evaluated. Let $\beta = \max_{(g_i, P_i)} \beta(g_i, P_i)$ and let γ be the number of $\beta(g_i, P_i)$'s equal to β. Then

$$J \sim q_-^{\beta(G,P)} \quad \text{if } \beta < \beta(G,P),$$

$$J \sim q_-^{\beta} (\ln q_-)^{\gamma} \quad \text{if } \beta = \beta(G,P),$$

$$J \sim q_-^{\beta} (\ln q_-)^{\gamma - 1} \quad \text{if } \beta > \beta(G,P).$$

Note that (1) the behavior again depends on asymptotic indices only and not explicitly on $N(G,P)$; (2) in a given sequence P_1, $P_2, \ldots$ the m subgraphs with $\beta(g_i, P_i) < \max \beta(g_i, P_i)$ are irrelevant to the leading behavior; (3) obviously the leading power of q_- will come from those pattern sequences with the largest possible β.

From our construction then, we can deduce a general rule for the $q_- \to \infty$ behavior of $F(G)$.

Rule: The asymptotic behavior of $F(G)$ as q_- approaches ∞ is given by $q_-^{\beta} (\ln q_-)^{\gamma}$, where $\beta = \max \beta(g, P')$, the maximum being taken over all m subgraphs g of G (including G itself) and associated numerator patterns P'. To determine γ consider the *longest* sequences $g_1 g_2 \cdots g_k$ of nested m subgraphs and corresponding numerator patterns $P_1 P_2 \cdots P_k$ such that $\beta(g_1, P_1) = \beta(g_2, P_2) = \cdots = \beta(g_k, P_k) = \beta$. Then $\gamma = k - 1$. (g_1 may coincide with G.)

In this paper we are concerned with the $q_- \to \infty$ limit. However, it should be pointed out that the concepts and methods we have introduced in this section have a wider applicability. As an illustration we shall devote the rest of this section to a brief discussion of asymptotic limits for large Euclidean momenta, a case of great importance for the renormalization program. With our method we not only recover Weinberg's theorem, but are also able to refine it by determining the logarithmic factors in the asymptotic limit.

Consider a "Euclidean" Feynman integral $F(G)$ (as obtained by rotating the integration paths of all energy variables to the imaginary axis) with external momenta q_1, $q_2, \ldots, q_n$, p_1, $p_2, \ldots, p_m$. Of interest is the asymptotic behavior of $F(G)$ as the subset $\{q_1, q_2, \ldots, q_n\}$ is allowed to approach ∞ according to

$$q_i = \eta \bar{q}_i , \quad \eta \to \infty .$$

In the expression (4) for the quantity ϕ in the parametric integral, we note that if a cut-set momentum P_c contains at least one of the large momenta q, then $P_c^2 \sim \eta^2$, otherwise P_c^2 is fixed. Accordingly, we define a quantity f by writing

$$\frac{\phi}{U} = \frac{f}{U} \eta^2 + (\text{terms proportional to fixed momenta } p_i).$$

Contributions to $F(G,P)$ from regions in parametric space where $f/U \neq 0$ behave asymptotically like $\eta^{\beta(G,P)}$, i.e., like the integrand of the numerator pattern in question. We may define the exponent $\beta(G,P)$ as the asymptotic index. Actually, because of the Euclidean metric, we may just focus on the patterns P with the largest $\beta(G,P)$. The number $\beta(G) \equiv \max_{(P)} \beta(G,P)$ is easily seen to be what Weinberg calls the "dimensionality" of G and can be obtained *without* reference to the parametric form by simply power counting: Each propagator contributes according to its large momentum behavior, e.g., -2 for scalar propagator, -1 for spin-$\frac{1}{2}$ propagator, etc., and $+4$ for each independent loop integration. (Any momentum factors arising from derivative couplings are assumed to be appropriately assigned to propagators.)

Next we note that, because of the Euclidean nature of the metric, $P_c^2 > 0$ for all cut sets c so that f/U vanishes only if all cut-set products in f vanish. Thus there are no complications from "pinch" contributions just as in the $q_- \to \infty$ limit. We may therefore focus on m subgraphs. In this case an *m subgraph* is a set of lines which (i) are connected, (ii) contain a continuous path between any two of the n vertices at which the large momenta $q_1, q_2, \ldots, q_n$ are applied, and (iii) are strongly connected (i.e., contain no line[19] which does not belong to any of its loops). Note that our m subgraphs are identical with the subgraphs of Weinberg's theorem.

The resulting asymptotic behavior is $F(G) \sim \eta^{\beta}(\ln\eta)^{\gamma}$, where $\beta = \max\beta(g)$, the maximum being taken over all m subgraphs of G (including G itself). To determine γ, consider the *longest* sequences of nested m subgraphs $g_1 \supset g_2 \supset g_3 \supset \cdots \supset g_k$ with $\beta(g_i) = \beta$. Then $\gamma = k-1$ (g_1 may coincide with G).

IV. ASYMPTOTIC INDICES IN $\bar{\psi}\psi\phi$ THEORY

In Sec. III we reduced the problem of determining the asymptotic behavior of an arbitrary Feynman integral $F(G)$ to the knowledge of the asymptotic indices $\beta(g, P)$ of its m subgraphs for various numerator patterns P. Clearly these asymptotic indices depend on the particular local field theory we are considering. In this section we shall discuss in some detail the simplest type of scalar-spinor interaction, namely, $\bar{\psi}\psi\phi$. It turns out that, in this case, it is fairly easy to calculate the asymptotic indices on a systematic basis.

Minimal Trees. We begin with the tree graphs.[20] By definition, trees have no loops and therefore there is only one continuous path joining any two vertices. In particular, there is a unique continuous path of (internal) lines joining the vertices A and B (where the large momentum q enters and exits). Since q can only be routed along this path, the rest of the graph is irrelevant insofar as we are interested in the large q_- behavior. Another way of expressing this is to say that the lines along the q_- route from the one and only m subgraph of the tree (which may, of course, coincide with the entire tree). We shall therefore focus on *minimal or m trees* defined as those trees which coincide with their m subgraph.

Consider now a general m-tree contribution to matrix elements of $\psi(x)\Gamma_i\psi(0)$ or $\bar{\psi}(x)\Gamma_i\psi(0)$, where $\{\Gamma_i\}$ is a complete set of 4×4 matrices. The corresponding Fourier transforms have the form

$$\mathrm{Tr}[\Gamma_i S(\pm q) I(q, p_1, p_2, \ldots) S(\pm q + p)],$$

where the fermion propagators $S(q)$ correspond to the fields $\psi(x)$ and $\psi(0)$, and I is associated with the internal lines of the graph.

It will turn out that the strongest asymptotic behavior is associated with $\Gamma_i = \gamma_-$ or $\gamma_-\gamma_\perp$ or $\gamma_-\gamma_5$. This can be anticipated in view of the identities

$$\not{q} = \gamma_+ q_- + \gamma_- q_+ + \vec{\gamma}_\perp \cdot \vec{q}_\perp,$$

$$\gamma_+^2 = \gamma_-^2 = 0,$$

$$\{\gamma_\pm, \gamma_\perp\} = \{\gamma_\pm, \gamma_5\} = 0,$$

from which it follows[21] that in the product $\not{q}\Gamma_i\not{q}$, the $\gamma_+ q_-$ terms from the $\not{q}$'s can both contribute if and only if $\Gamma_i = \gamma_-$. In Fig. 5 we have drawn a general m tree whose contribution is

$$\gamma_- S(q) S(q_1) S(q_2) \cdots u(p_n) \frac{1}{q_n^2 - m^2} \times u(p_{n+1}) S(q_{n+1}) \cdots S(q_{N-1}) S(q),$$

where

$$q_i = q + (\text{fixed external momenta}).$$

It is clear by inspection that the numerator pattern with the most powers of q_- corresponds to the term $\gamma_- \not{q}\not{q}_1\not{q}_2 \cdots$. Moreover, because $\gamma_+^2 = \gamma_-^2 = 0$, only alternate factors can contribute a power of q_-:

$$\gamma_-(\gamma_+ q_-)(\gamma_- q_+)(\gamma_+ q_-) \cdots .$$

This means that in each chain of fermion propagators, only about half the numerators grow like q_-, whereas all denominators grow like $q^2 \to q_-$. Consequently the longer the chain, the weaker the asymptotic behavior (i.e., the smaller the assymptotic index). In fact, the strongest asymptotic contribution to $\psi\gamma_-\psi$ at the tree level is associated with the two second-order graphs of Fig. 6 whose behavior is

$$\frac{\not{q}}{q^2}\frac{1}{q^2}\frac{\not{q}}{q^2} \to \frac{1}{q_-}.$$

The strongest asymptotic contribution to $\bar{\psi}\gamma_-\psi$

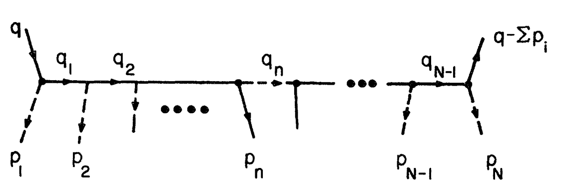

FIG. 5. Assignment of momenta to a general m tree.

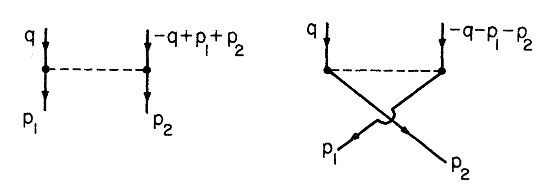

FIG. 6. Leading m trees for $\psi\gamma_-\psi$.

is given by the four simple graphs of Fig. 7 whose behavior is also $1/q_-$. Thus we see that, at the m-tree level, the leading behavior $1/q_-$ (i.e., an asymptotic index of -1) is only present in the two-particle matrix elements $\langle\beta\,|\,\psi\gamma_-\psi\,|\,\alpha\rangle$ and $\langle\beta\,|\bar{\psi}\gamma_-\psi\,|\,\alpha\rangle$, i.e., for $N\equiv N_\alpha+N_\beta=2$, where N_α, N_β are the number of particles in the states α and β.

All other m trees behave like $1/q_-^2$ or weaker. In each case the leading behavior can be readily obtained by appropriately assigning a power of q_- to alternate fermion numerators along the chain and a power of $1/q_-$ to every (fermion or scalar) denominator. The asymptotic indices for various other numerator patterns can be similarly estimated. Identical results hold for $\Gamma=\gamma_-\gamma_\perp$ or $\gamma_-\gamma_5$.

A similar reasoning for the case $\Gamma=\gamma_\perp$ or γ_5 shows that the only $1/q_-$ contribution comes from the one-particle graph in Fig. 8(a), whereas the leading behavior of the two- and three-particle graphs in 8(b), 8(c), 8(d), and 8(e) (as well as those obtained by crossing) is $1/q_-^2$.

All other graphs behave like $1/q_-^3$ or weaker. Here again the asymptotic index decreases as N increases.

Lastly, we record the following leading behaviors for the "least singular" case $\Gamma=\gamma_+$ or $\gamma_+\gamma_\perp$ or $\gamma_+\gamma_5$:

$N=1 \quad 1/q_-^2,$

$N=2 \quad 1/q_-^3,$

$N=3 \quad 1/q_-^4.$

Analogous results are obtained for matrix elements of the products $\psi\phi$ and $\phi\phi$: asymptotic indices decrease with increasing N.

m graphs with loops. We now consider m graphs with loops, i.e., connected graphs with loops which can occur as subgraphs of a general Feynman graph. Take the case $\psi\gamma_-\psi$ or $\psi\gamma_-\psi$, $N=2$ (the nontree graphs with $N=1$ are not skeleton graphs). We focus, first, on graphs with no internal fermion loops. In Fig. 9 we have drawn

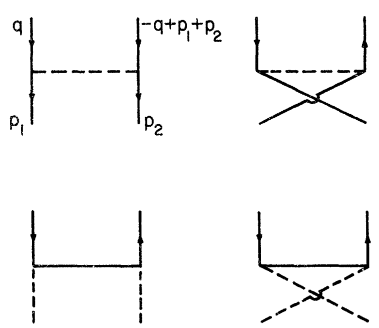

FIG. 7. Leading m trees for $\bar{\psi}\gamma_-\psi$.

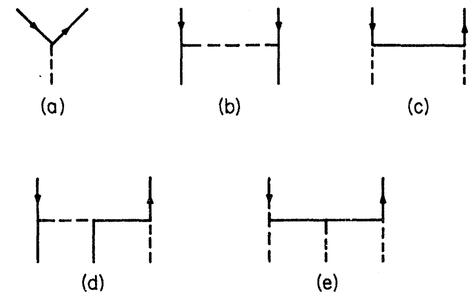

FIG. 8. Leading m trees for $\bar{\psi}\gamma_\perp\psi$ or $\bar{\psi}\gamma_5\psi$.

general graphs of this type with L loops and $3L+1$ internal lines. There are $2L$ internal fermion propagators for $\psi\psi$ and $2L+1$ fermion propagator for $\bar{\psi}\psi$. It is readily seen that in numerator patterns with no pairings the largest number of powers of q_- one can have is L (including the two external propagators of the fermion fields) obtained when successive d's in the fermion chains carry the large moment q_-. Since the no-pairing $(\rho(P)=0)$ denominator in the parametric integrand grows like $q_-^{\,I-2L-\rho(P)}=q_-^{\,L+1}$, we have

$$\beta(G,P)\leq -1 \text{ for } \rho(P)=0\,,$$

where the equality is actually attained for some P. This estimate persists for graphs with internal fermion loops: An internal fermion loop with $2v$ vertices on it yields up to v powers of q_- (for certain patterns) in the numerator but $I-2L$ also increases by v. It is also easy to see that the result is the same for *any number of pairings*. One can argue inductively: In a no-pairing numerator with $\beta(P,G)=-1$, all d's contribute (alternatively) by their γ_+q_- or γ_-q_+ term. But in pairing two d's, a γ_+ is always contracted to a γ_- so that we lose one power of q_- in the numerator which counterbalances the increase of $\rho(P)$ by one. Thus the behavior of the integrand is again $1/q_-$.

Without going into the details we state that the situation with the other choices of Γ and number of particles N is the same, namely: "The maxi-

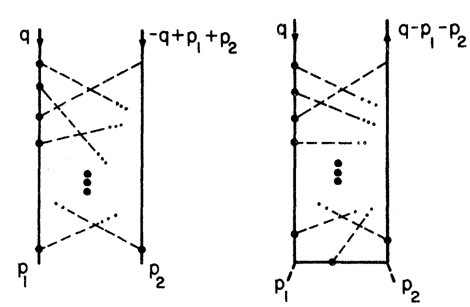

FIG. 9. Two-particle graphs with no internal fermion loops.

mum asymptotic index of nontree m graphs for the N-particle matrix element of the time-ordered product of two fields is equal to the maximum asymptotic index of the corresponding m trees."

It is now fairly easy to apply the rule obtained in Sec. III for the asymptotic behavior of a general graph. Consider, for example, the "ladder" graph for $\bar{\psi}\gamma_-\psi$ shown in Fig. 10(a). Since the asymptotic index for $N=2$ is -1, the graph itself and the L m subgraphs obtained by cutting the fermion chains between any two successive rungs form the "dominant" sequence $g_1 \supset g_2 \supset g_3 \supset \cdots \supset g_{L+1}$. Thus the asymptotic behavior[22] is $(\ln q_-)^L/q$. Similarly the "crossed" ladder graph of Fig. 10(b) behaves like $(\ln q_-)^{n-1}/q_-$, where n is the number of "crosses." Note, in fact, that the only contributions to an N-particle matrix element of $\bar{\psi}\gamma_-\psi$ (with $N>2$) of order $1/q_-$ (apart from powers of $\ln q_-$) come from graphs with "two particle intermediate states," which can be classified as shown in Fig. 11. The "blobs" K and I represent the sum of all "two-particle irreducible" graphs with $N=2$ and $N>2$, respectively. The full N-particle matrix element can thus be expanded in terms of iterations of an irreducible Bethe-Salpeter kernel:

$$I + KI + K^2 I + \cdots + K^n I + \cdots .$$

According to our asymptotic rule for $N=2$ $(K=1)$ we have $K^n I \sim (\ln q_-)^n/q_-$, whereas for $N>2$ the asymptotic behavior is $K^n I \sim (\ln q_-)^{n-1}/q_-$ for $n=1, 2, \ldots$, but I itself behaves like $1/q_-^2$ or weaker. Thus the $1/q_-$ behavior for *any* matrix element of $\bar{\psi}\gamma_-\psi$ is traced to the irreducible kernel K.

The above phenomenon persists as we consider terms of order $1/q_-^2$, etc.: For a given n the asymptotic contributions of order $1/q_-^n$ or higher to *all* matrix elements of the operator product are, in effect, contained in a finite number of irreducible Bethe-Salpeter kernels.

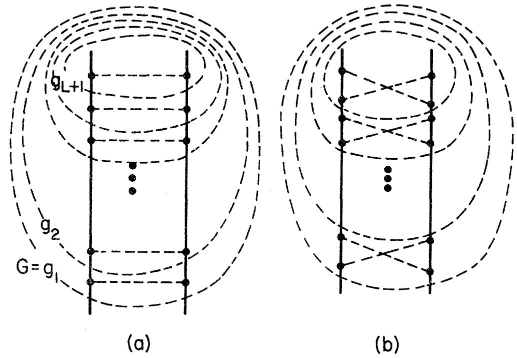

FIG. 10. (a) Sequence of m subgraphs for the straight ladder. (b) Sequence of m subgraphs for the crossed ladder.

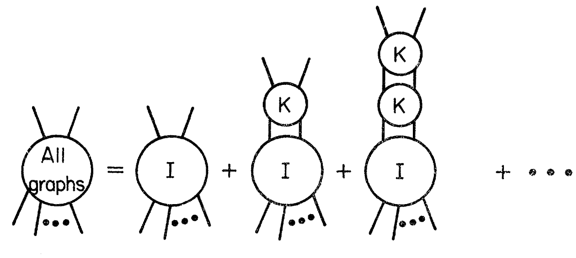

FIG. 11. Expansion of an N-particle matrix element in terms of a two-particle irreducible matrix element I and a two-particle irreducible Bethe-Salpeter kernel K.

In the companion paper we organize and discuss this property of perturbation expansions in terms of an expansion of time-ordered products in terms of N-particle "irreducible" operator products integrated over appropriate Bethe-Salpeter kernels. The important feature of the expansion is that only a *finite* number of terms are singular on the light cone.

V. SUMMARY

For the benefit of the reader who finds the foregoing reasoning highly involved we offer in simplified terms a summary of the main features emerging from the application of the general rule of Sec. III to a specific theory like $\bar{\psi}\psi\phi$.

1. Let the matrix element $\langle \alpha_2 | \phi_1(x)\phi_2(0) | \alpha_1 \rangle$ in the generalized BJL limit behave like q_-^{β} in the tree approximation. The integer β depends, of course, on the type of the fields ϕ_1 and ϕ_2 and the number and kind of particles in the states α_1, α_2. We found, in fact, that β *decreases* as the number of particles in α_1 and α_2 increases.

2. A *general* Feynman graph for the above matrix element, which we may think of as describing a process $\phi_1, \phi_2 \to (\alpha_1, \alpha_2)$, behaves asymptotically like $q_-^{\bar{\beta}}(\ln q_-)^{\gamma}$. The integer $\bar{\beta}$ is equal to the maximum β associated (in the sense described above) with matrix elements of the type $\phi_1\phi_2 \to (\eta)$, where (η) runs over *all* intermediate states of the graph in the operator-product channel

$$\phi_1\phi_2 \to (\eta) \to (\alpha_1, \alpha_2).$$

3. To determine the integer γ we consider the *longest* chains (there may exist several of the same length) of successive intermediate states in the graph

$$\phi_1\phi_2 \to (\eta_1) \to (\eta_2) \to \cdots \to (\eta_k),$$

such that the β of each process $\phi_1\phi_2 \to (\eta_i)$ is equal to $\bar{\beta}$. Then $\gamma = k-1$. [The last state (η_k) may actually coincide with (α_1, α_2).]

The simplicity of this result is essentially due to the absence of "pinch" contributions as shown in Sec. III. Our methods also led to a similar rule for the precise logarithmic power of the asymptotic behavior of an arbitrary Feynman integral in the *Euclidean* regime, thus improving the estimates of Weinberg's theorem.[4]

4. In I, it is shown that Bjorken scaling implies that the operator $\bar{\psi}(x)\gamma\cdot x\psi(0)$ is nonsingular at $x^2=0$. The corresponding statement in momentum space is that $(\partial/\partial q^\mu)F_\mu(q)$, where $F_\mu(q)$ is the Fourier transform of any matrix element of $\bar{\psi}(x)\gamma_\mu\psi(0)$, have an index $\bar{\beta}$ which is less than two (see the Appendix of I). Application of the results of Sec. III shows that this is true for every one- and two-particle irreducible matrix element of the fermion bilocal, as long as there are three or more particles in the external states; reducible graphs or those with less than three external legs have $\bar{\beta}\leq 2$ and are singular in perturbation theory. These results are the basis for the truncated operator expansion given in I, and valid near the light cone.

5. As long as one is not concerned with logarithmic singularities, summary statement 2 has the interpretation that concepts of naive scale invariance and canonical dimensionality ($d=\frac{3}{2}$ for ψ, $\bar{\psi}$, $d=1$ for ϕ) hold, in perturbation theory, near the light cone. (These concepts are clearly known to be valid at short distances, according to Weinberg's theorem.[4])

These results, in themselves, do not solve the problem of scaling in canonical field theory, but merely help to organize our knowledge of how scaling is violated in perturbation theory. Yet the techniques used seem to be sufficiently powerful to give a precise answer to the problem; we hope to return to this question in a future publication.

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†Alfred P. Sloan Foundation Fellow.

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[8]Our definitions are somewhat different from those of Shimamoto. Our graphs G are connected but not necessarily "nonseparable," i.e., some of their lines may belong to no loop.

[9]Ordinarily, one does not include, in the Feynman identity, propagators which are not integrated over. We shall, nevertheless, include them to simplify the discussion.

[10]The usual factor of $(2\pi)^4 i$ times the over-all momentum-conservation δ function has been removed in the definition of $F(G)$.

[11]See J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965), Chapter 18.

[12]Note that d_i/x_i and r_{ij}/x_ix_j are polynomials because all cut-set products in d_i contain x_i as a factor and all cut-set products in r_{ij} contain x_ix_j as a factor.

[13]We have in mind the cancellations of leading asymptotic terms between different graphs of the same order yielding the "eikonal" approximation in field theory; see G. Tiktopoulos and S. B. Treiman, Phys. Rev. D 2, 805 (1970), and *ibid.* 3, 1037 (1971).

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[16]Arguments analogous to those for f/U show that no "pinch" contributions occur in $\tilde{f}/\tilde{U}$ either, namely, the only zeros of $\tilde{f}/\tilde{U}$ that cannot be avoided by contour distortion correspond to the vanishing of the parameters of some m subgraph.

[17]Recall that we presumed that no cancellations of leading terms occur between different numerator patterns. In any case, our final result for the behavior of $F(G)$ will never be an underestimate.

[18]Unless, of course, there are cancellations of the leading behavior between different *regions* in parameter space.

[19]With the exception of the external lines carrying the momenta $q_1, q_2, \ldots, q_n$ if their propagators are included in $F(G)$.

[20]The singularities of tree graphs were studied by H. Schnitzer, Phys. Rev. D 6, 2118 (1972).

[21]These identities played a similar role in the work of J. Polkinghorne, Nuovo Cimento 8A, 592 (1972).

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