Light-Cone Operator Expansions in Perturbation Theory. I*

John M. Cornwall† and George Tiktopoulos
Department of Physics, University of California, Los Angeles, California 90024
(Received 7 March 1973)

With the aid of a general algorithm for expanding a given Heisenberg operator product in terms of n-particle irreducible operator products integrated over c-number Bethe-Salpeter kernels, we discuss the expansion of bilocal products near the light cone. It is shown that, to any finite order in perturbation theory, only a small number of terms in the operator-product expansion can be singular on the light cone. In particular, some of the bilocal operators which determine the scaling properties of deep-inelastic electroproduction have potential singularities determined by operator-valued Bethe-Salpeter equations involving only these bilocals themselves (plus local fields). A generalization of the light-cone Bjorken-Johnson-Low (BJL) limit is given which reveals logarithmic singularities which occur in T products but not in commutators; this BJL limit is applied systematically to Feynman graphs in another paper. As a special case, the Wilson expansion is recovered.

I. INTRODUCTION

This is one of two companion papers (the other is Ref. 1) dealing with the light-cone expansion in perturbation theory; the present paper concentrates on operator properties.

The Wilson[2] expansion of operator products at short distances ($x^{\mu}\simeq 0$) is by now a fairly well-understood subject in perturbation theory. For example, Zimmerman[3] has given a systematic exposition and review of the Wilson expansion in perturbation theory, including its relation to renormalization techniques.[4] The generalization of the operator-product expansion to the neighborhood of the light cone ($x^2\simeq 0$) is less well studied in perturbation theory, in spite of the intense interest in light-cone physics.[5,6] (We are not referring here to works[7] on specific matrix elements—usually two-particle forward—of this operator-product expansion; our interest is in the operator nature of the expansion.) Brandt and Preparata[8] have shown that the expansion involves an infinite number of fields of arbitrary spin, but the relation of these fields to the primitive fields of the theory has not been explored in much detail.

In this paper, we develop techniques for expanding any multilocal product of Heisenberg operators near the light cone. The coefficients of the expansion are themselves multilocal operator products of a special character, integrated over c-number kernels. The full expansion, with an infinite number of terms, holds identically for any values of the coordinates. Needless to say, it is not very useful to have an infinite sum of operator products as an expansion of one operator product. But when the coordinates of the operator product which is being expanded are restricted to the neighborhood of the light cone, only a small number of terms in the sum can contribute to the light-cone singularities, if any. We shall show that the contributing terms are those which have Feynman graphs with a small number of intermediate states; those graphs all of whose intermediate states have more than a certain number of particles are nonsingular (but not necessarily vanishing) on the light cone. Analogous statements for operator expansions at short distances are well known, but it is not trivial to go from the short-distance regime to the light cone, and the precise characterization of the intermediate states differs in the two regimes (except for a theory of all spinless particles, where the light-cone and short-distance expansions coincide).

The simplest example of an expansion of one bilocal[9] operator in terms of other bilocals is very well known: Products of two currents are given as singular c-number functions times fermion bilocals.[10-12] The question of Bjorken scaling[13] in electroproduction and neutrino scattering then reduces to the question of singularities in the fermion bilocals. In the perturbation theory of two-particle forward matrix elements, scaling is broken by logarithmic terms[14,15] and this breaking can be directly traced to light-cone singularities in fermion bilocals.[16,17]

The techniques presented here do not shed any light on the scaling properties of two-particle forward matrix elements, but they do allow us to draw one conclusion: If two-particle matrix elements of fermion bilocals are nonsingular on the light cone, then arbitrary matrix elements of the same bilocals are nonsingular. This feature has some peripheral practical interest, since it may be that light-cone singularities are important in such processes as massive lepton-pair production,[7,18] which depend on four-particle matrix elements of bilocal operators. It could even be argued that electroproduction amplitudes are not simple two-particle matrix elements, because the external nucleons are composites of the basic quark fields which occur in the fermion bilocal.

In a more general way, the singular structure on the light cone of any field theory is entirely characterized by a small number of multilocal operators and associated Bethe-Salpeter kernels: The singular structure of an arbitrary matrix element of an arbitrary operator product depends on this smaller set. We shall see that, for most practical purposes, this set consists simply of local fields and bilocals, and that the singular structure (if any) of certain fermion bilocal operators is governed by an operator-valued integral equation involving local and bilocal operators only.

Our result, that only a small number of operator products contribute to light-cone singularities, is mirrored in the canonical commutators of field theory quantized on the light cone.[10] The fermion anticommutator at $x^2=0$ depends only on local fields and bilocals (both boson and fermion). Unfortunately, the explicit expression in Ref. 10 for the fermion anticommutator at $x^2=0$ is not well defined, since it refers to a product of operators at the same space-time point. If it were well defined, it would be a simple matter to prove that canonical field theory and Bjorken scaling are incompatible. As it is, all we can do is give an expression for the fermion bilocal which is well defined, and from which the anticommutator on the light cone can be extracted, if it exists (Bjorken scaling requires that this anticommutator vanish, as we shall show).

One way in which the present approach differs from the canonical treatment is that canonical theory only deals with singularities in commutators or anticommutators. In Ref. 10, it was shown how to isolate such singularities in momentum space with a generalization of the Bjorken-Johnson-Low[19] (BJL) expansion, in which the momentum variable q^- was taken to infinity in a lightlike direction and the coefficient of $1/q^-$ was determined.[20] However, this is not enough to isolate all singularities of T products near the light cone, since functions such as $\ln(x^2-i\epsilon)$ do not appear as singularities in the light-cone commutator. This must be studied, since it is precisely terms like $\ln x^2$ which break scaling in perturbation theory. We show in the Appendix that the criterion for light-cone singularities in a T product is this: A Feynman graph is not singular on the light cone if its Fourier transform (less any polynomials) decreases more rapidly than $1/(q^-)^2$, with other components of q held fixed.

It is a curious fact that there exist Feynman graphs which decrease with an arbitrary large power of q when all four components of q are taken to be large, yet which are singular on the light cone. Such graphs decrease no more rapidly than $1/(q^-)^2$ when only q^- is taken large, with q^+ and $\vec{q}_\perp$ held fixed. The space-time version of this remark is that there exist functions [e.g., $(x\cdot p)^N \ln x^2$ where p is a fixed vector] which approach zero with any power of x^μ as $x^\mu \to 0$, but which are singular on the light cone. In general, nothing can be learned about light-cone singularities from knowledge of behavior near the tip of the light cone. There is one exception: In a field theory containing only scalar particles, factors like $(x\cdot p)^N$ do not appear, so light-cone behavior and small-x^μ behavior coincide, graph by graph. In such field theories, we may immediately identify singular Feynman graphs from their naive degree of divergence. The situation is somewhat more complicated, but still tractable, in field theories with spin. Our analysis of singular graphs in such theories is similar in spirit to that of Polkinghorne,[21] who has given a graphical analysis of light-cone commutators and shown how spin-matrix identities limit the number of skeleton graphs which can contribute to the commutator. This limitation is expressed, in the canonical approach, in the finite number of powers of the coupling constant g which explicitly appear in the commutator.[10]

Observe that our ability to identify singular graphs does not guarantee that we can extract the leading light-cone singularity of the full field theory; nonsingular terms like $x^2 \ln^N x^2$ may add up to a singularity. We do not discuss this possibility further here, but it clearly merits a separate inquiry.

For the sake of readability, the hard work behind this paper is relegated to a companion paper,[1] referred to as II, in which rules are given for expressing an arbitrary Feynman graph with both fermion lines and boson lines as an integral over Feynman parameters. This generalizes the work of Shimamoto,[22] who has done the work for Feynman graphs without spin. The $q^- \to \infty$ limit of these graphs is studied, and singularities extracted according to the prescription of the Appendix. The results of this labor are summarized in Sec. II, of the present paper where a simple scaling argument is advanced to show what the results ought to be. Section II also contains the general operator expansion referred to above.

In Sec. III, we make some brief remarks about the relation to the canonical formalism and about the unphysical phenomena which appear in low orders of perturbation theory, supplementing the calculations already given by Schnitzer.[17] In particular, we point out that there can be parts of the electroproduction amplitude which exhibit

Bjorken scaling, and which come from light-cone behavior which is more singular than conventionally assumed. Conversely, certain pieces of the imaginary parts which fall below scaling can contribute to the leading light-cone singularity. In the former case the Bjorken scaling function has a pole as $\omega = -q^2/2\nu$ approaches unity, and in the second case there is an effective addition to the scaling function with support at $\omega = 1$ only. Both pathologies occur in perturbation theory. We also show in this section how to recover the Wilson expansion.

Section IV contains a summary and conclusions.

II. OPERATOR EXPANSIONS NEAR THE LIGHT CONE

In this section we restate the results of II in operator language, and give a plausibility argument, based on asymptotic conformal invariance, for these results. The problem is to characterize, in perturbation theory, the light-cone singularities of an arbitrary matrix element of any operator product. Let us contrast this problem to the Wilson expansion at short distances.[2-4] There an operator product is expanded as a sum of local fields multiplied by c-number functions. In the present case, an operator product is expanded near the light cone as a sum over other operator products, integrated over c-number kernels. The virtue of going to the light cone is that only a small number of operator products can contribute singularities to the operator being expanded, no matter how complex that operator may be.

The essence of the development here and in II is that if a matrix element of a particular multilocal operator has a sufficiently large number of particles in the external states, and a sufficiently large number of particles in all intermediate states, then that matrix element is nonsingular on the light cone, except for contributions from multilocal operators of lower rank. This is analogous to the corresponding short-distance result (in essence, that only the vertex, the propagator, and possibly the four-point function are infinite in perturbation theory) but is by no means a trivial consequence of it. To implement the organization of matrix elements according to the minimum number of particles in intermediate states, we introduce N-particle irreducible operators;[23] these operators have only matrix elements with N or more particles in intermediate states. We begin with notation and definitions.

Consider a field theory containing a fermion field $\psi(x)$ and a neutral scalar boson field $B(x)$, with a Yukawa coupling $g\overline{\psi}\psi B$. This is the only theory with which we deal explicitly, although with trivial modifications the results hold for fermion-vector-boson theory in the light-cone gauge.[24,10] To avoid cumbersome notation in the definition below, we use the symbol $\phi(x)$ to indicate a field, whether ψ, $\overline{\psi}$, or B; the reader will easily be able to supply specific labels if needed.

First we define recursively a generalized Wick product of fields, denoted by colons $(:\cdots:)$, as follows:

$$T(\phi(x_1)\cdots\phi(x_N)) = \sum_{\text{part.}} :\phi(x_{i_1})\cdots\phi(x_{i_j}): \times\langle 0|T\phi(x_{i_{j+1}})\cdots\phi(x_{i_N})|0\rangle \tag{2.1}$$

where the sum is over all partitions of $\{1\cdots N\}$ into two sets, either of which may be empty (in which case the corresponding operator product or matrix element is replaced by unity). A generalized Wick product has a vanishing vacuum expectation value (VEV), and its matrix elements contain no disconnected VEVs constructed from proper subsets of $\{\phi(x_i)\}$. We assume that $\langle 0|\phi(x)|0\rangle = 0$, so $:\phi: = \phi$. It is useful to deal with these Wick products, as they contain no disconnected c-number singularities.

Second, we define N-particle-irreducible connected kernels. Consider any connected skeleton graph (i.e., with no self-energy corrections) corresponding to the matrix element of Heisenberg operators

$$\langle 0|T(\phi(x_1)\cdots\phi(x_M)\phi(y_1)\cdots\phi(y_N))|0\rangle_c .$$

A *contour* is any continuous, closed line which does not cross itself and which divides the graph (cutting any line at most once) so that (1) all the x_1 lie inside the contour and (2) the part inside the contour is connected. In the set of all contours there is a subset of contours which cross the fewest lines; call this the minimum set. Define the *N-particle-irreducible (abbreviated NPI) connected kernel* $K_c(x_1, \ldots, x_M; y_1, \ldots, y_N)_{N\text{PI}}$ corresponding to the above matrix element as the sum of all connected skeleton graphs whose minimum set consists solely of contours which cut only the lines terminating in the y_i, and no other lines. The significance of connected NPI kernels is that every connected graph with external lines labeled $x_1, \ldots, x_M$, $z_1, \ldots, z_K$ can be *uniquely* divided into such a kernel and a remainder (in general, disconnected). An example is shown in Fig. 1. Observe that irreducibility refers only to the channel consisting of the legs x_i, and to no other channel.

Now we turn to the properties of this remainder. Since it is, so to speak, one half of a graph which

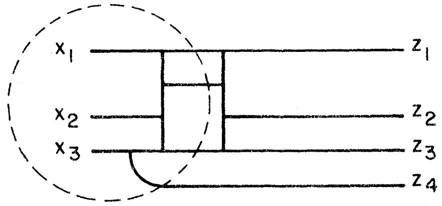

FIG. 1. Unique reduction of a Feynman graph by a minimum contour (the dashed line) into a connected kernel (inside the dashed line) and a remainder. The kernel is 4PI; the remainder is 3PI and (in this case) disconnected.

has been split across the *minimum* number (call it N) of lines, the remainder cannot be further subdivided by cutting fewer than N lines. (When we speak of cutting this remainder with a contour, we drop all requirements of connectivity imposed earlier.) In order to implement this requirement, we must construct N-point operators whose matrix elements are $(N-1)$PI; in particular, N-point operators with the property that all matrix elements of such an operator with fewer than N particles in the external states vanish. A completely general notation for such operators is rather cumbersome, so we proceed by giving the full definitions for $N=1,2,3$ only.

Since we have already assumed that $\langle 0|\phi(x)0\rangle=0$, the 1-point operator whose 0-particle matrix element vanishes is just ϕ itself. The 2-point operator which is 1PI is

$$:\phi(x_1)\phi(x_2):_{1\text{PI}} = :\phi(x_1)\phi(x_2): - \int dz\, K_c(x_1x_2;z)_{1\text{PI}}\, \phi(z)\,. \quad (2.2)$$

The graphical depiction (Fig. 2) of an arbitrary matrix element of the second term on the right-hand side of (2.2) makes it clear that $K_{c1\text{P}}$, is the full 3-point function, truncated on the leg labeled z. (This truncation of all legs in kernels which bear the same coordinate label as an operator in the same expression will be understood from now on.) With this choice for the kernel, $:\phi(x_1)\phi(x_2):_{1\text{PI}}$ has no matrix elements between the vacuum and a one-particle state, nor any matrix elements which are one-particle reducible.

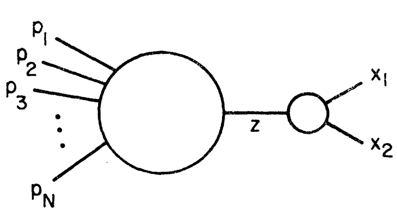

FIG. 2. Matrix element of the one-point operator contribution to $:\phi(x_1)\phi(x_2):_{1\text{PI}}$.

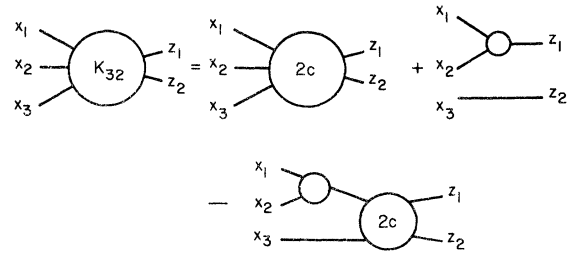

FIG. 3. The kernel K_{32} appearing in the expansion of $:\phi(x_1)\phi(x_2)\phi(x_3):_{2\text{PI}}$. The notation $2c$ means the corresponding graph is 2PI and connected. A sum over distinct permutations is understood.

Our final example is

$$\begin{aligned} :\phi(x_1)\phi(x_2)\phi(x_3):_{2\text{PI}} = {} & :\phi(x_1)\phi(x_2)\phi(x_3): \\ & - \int dz\, K_c(x_i;z)_{1\text{PI}}\, \phi(z) \\ & - \int dz_1 dz_2 K_{32}(x_i;z_j) \\ & \times :\phi(z_1)\phi(z_2):_{1\text{PI}}\,. \end{aligned} \quad (2.3)$$

Clearly, $K_c(x_i;z)$ is the connected 1PI kernel taking three fields into one. The kernel K_{32} is fairly complicated, and is shown in Fig. 3. Obviously, (2.3) has no matrix elements between the vacuum and a one-particle state, but some discussion is needed to show the vanishing of two-particle matrix elements. The difficulty comes from the fact that the two-point operator on the right-hand side of (2.3) has disconnected matrix elements, as shown in Fig. 4. To verify that the last two terms on the right-hand side of (2.3) have two-particle matrix elements which exactly cancel those of the first term on the right, use the Bethe-Salpeter equations shown in Figs. 5 and 6. The idea is to use the equation of Fig. 5 with the totally disconnected part of K_{32} (see Fig. 3) to effect a cancellation with the part of K_{32} appearing with a negative sign, and thus to arrive at Fig. 6.

The obvious generalization is that one may construct operators

$$:\phi(x_1)\cdots\phi(x_N):_{(N-1)\text{PI}}$$

in terms of

$$\phi,\ \ldots,\ :\phi(x_1)\ldots\phi(x_{N-1}):_{(N-2)\text{PI}}$$

and c-number kernels so that these operators

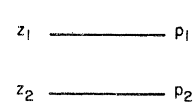

FIG. 4. A disconnected two-particle matrix element of a two-point operator; sum over permutations understood.

have no matrix elements with either intermediate states or external states containing fewer than N particles.

The combination of connected NPI kernels and NPI operators allows us to write an operator expansion for any operator which generates all the *connected* matrix elements of that operator. Such generality is of little practical use, since possible violations of Bjorken scaling[13] in electroproduction and neutrino scattering are governed by singularities in the matrix elements of the fermion bilocal $:\psi(x)\bar{\psi}(0):$.[10-12] We therefore give only the expansion of connected two-point operators:

$$:\phi(x_1)\phi(x_2):_c=\sum_{j=1}^{\infty}\int\prod^{j} dz\, K_c(x_1,x_2;z_j)_{j\mathrm{PI}} \times :\phi(z_1)\cdots\phi(z_j):_{(j-1)\mathrm{PI}}\,. \tag{2.4}$$

The subscript c on the bilocal operator means that the expansion on the right-hand side generates connected graphs only for the bilocal. It is important to note that the operator products which appear on the right-hand side of (2.4) do generate disconnected subgraphs, which yield connected graphs when integrated on the kernels. It is sufficient to have the expansion yield only connected graphs, since singularities of disconnected graphs can be completely analyzed in terms of connected operators of lower rank. (In the two-point case, disconnected graphs are not singular at all.) It is important to note that (2.4) holds when both the operators and the kernels are unrenormalized, and also when both are renormalized.

Equation (2.4) is merely another way of organizing an infinite number of Bethe-Salpeter equations if general values of x_1 and x_2 are considered. The interesting thing is that this expansion terminates after only two or three terms, if one goes to the light cone and saves only terms which perturbation theory indicates are singular there. The detailed—but not very transparent—proof of this assertion is contained in II. It is worthwhile to give a plausible argument (the outcome of which agrees in every detail with II) which more cogently illuminates the central issues.

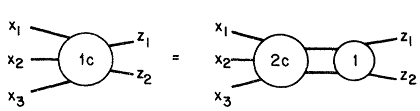

FIG. 5. A Bethe-Salpeter equation for the two-point operators. The notation $1c$, $2c$ is as in Fig. 3; the circle with a 1 and no c in its represents a two-particle matrix element of the operator $:\phi(x_1)\phi(x_2):_{1\mathrm{PI}}$, *including the disconnected part* (Fig. 4).

In what follows, we study the fermion–neutral-scalar-boson (or vector boson in the light-cone gauge) theory under the simplifying assumption that the propagator and vertex functions have canonical short-distance behavior. In other words, we pretend that the renormalization constants Z_1 and Z_2 are finite, which is most easily accomplished by considering only skeleton graphs with both vertex connections and self-energy connections removed. At several points we mention the effect of removing these restrictions, which are inessential to the main results.

Let us begin by defining the kinematical relations of light-cone current commutators and bilocal operators given in Refs. 10, 16, and 6. The covariant spin-averaged current correlation function is written

$$T^{\mu\nu}(p,q)=i\int dx\, e^{iqx}\langle p|T^*(J^{\mu}(x)J^{\nu}(0))|p\rangle\ . \tag{2.5}$$

The conventionally defined absorptive parts W_1 and W_2 appear through

$$W_2p^{\mu}p^{\nu}-W_1g^{\mu\nu}+\cdots=\frac{1}{2\pi}\,\mathrm{Im}\,T^{\mu\nu}\ , \tag{2.6}$$

$$\mathrm{Im}\,T^{\mu\nu}=\tfrac{1}{2}\int dx\, e^{iqx}\langle p|[J^{\mu}(x),J^{\nu}(0)]|p\rangle\ . \tag{2.7}$$

Next we give our first example of an operator expansion of the type in (2.4). The expansion is very well known in the context of Bjorken scaling; it is true in a certain sense even when scaling is violated by logarithmic terms, as in finite-order perturbation theory. Consider the expression

$$\langle p|[J^{\mu}(x),J^{\nu}(0)]|p\rangle=\langle p|\bar{\psi}(x)\gamma_{\alpha}\psi(0)-\mathrm{H.c.}|p\rangle \times(g^{\nu\alpha}\partial^{\mu}+g^{\mu\alpha}\partial^{\nu}-g^{\mu\nu}\partial^{\alpha})C(x)+\cdots, \tag{2.8}$$

assumed valid near $x^2=0$, where $C(x)$ is a singular c-number function and omitted terms are less singular. In canonical theory[10,16] or in the equivalent skeleton-graph theory referred to above $C(x)$ is related to the light-cone behavior of the fermion propagator, and is given by

$$C(x)=C_0(x)\equiv\frac{1}{2\pi}\,\epsilon(x^0)\delta(x^2)\,. \tag{2.9}$$

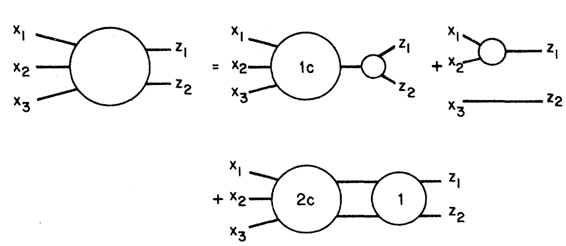

FIG. 6. Bethe-Salpeter equation for the full $3\to2$ amplitude (unlabeled circle).

However, inclusion of self-energy and vertex corrections in perturbation theory violates canonical behavior, and $C(x)$ is not given by (2.9). We return to this point in Sec. III.

In free-field theory, the matrix elements can be stripped off (2.8) and the current commutator (as an operator) can be expressed solely in terms of bilocal operators (axial-vector bilocals must be added; their spin-averaged matrix element vanishes). This is not possible in an interacting field theory, and only the commutators $[J^+,J^+]$ and $[J^+,J^-]$ may be so expressed.[10] In fact, it is not hard to see using the methods of this section that the Wightman product $J^\mu J^\nu$ needs products of up to five operators, and the commutator $[J^\mu,J^\nu]$ needs products of four operators, to account for all light-cone singularities. Fortunately, the scaling functions of electroproduction depend only on those portions of the commutator which need only bilocal operators in their expansion. In this connection, it should be noted that the formal expression for $[J^+,J^-]$ on the light cone contains a three-operator product of the form $\psi(0)\bar\psi(x)B(x)$.[8] By use of the equation of motion, this can be reduced to a fermion bilocal. In the skeleton theory, it is legitimate to use the equation of motion freely.

As is well known, the requirement of Bjorken scaling means that the operator $\bar\psi(x)\gamma^\alpha\psi(0)$ has specified behavior on the light cone. In terms of the conventional scaling-function expansion of the commutator near the light cone[6]

$$i\langle p|[J^\mu(x),J^\nu(0)]|p\rangle=[g^{\mu\nu}-\partial^\mu\partial^\nu]\frac{\epsilon(x^0)\delta(x^2)}{2\pi}\int_{-1}^{1}\frac{d\omega}{\omega^2}F_L(\omega)e^{i\omega x\cdot p}$$
$$-[p^\mu p^\nu-p\cdot\partial(p^\mu\partial^\nu+p^\nu\partial^\mu)+g^{\mu\nu}(p\cdot\partial)^2]\frac{\epsilon(x^0)\theta(x^2)}{2\pi}\int_{-1}^{1}i\frac{d\omega}{\omega}\frac{e^{i\omega x\cdot p}}{x\cdot p}F_2(\omega)+\cdots. \qquad (2.10)$$

Jackiw and Waltz[16] have combined (2.8), (2.9), and (2.10) to arrive at the matrix element (valid for $x^2\simeq 0$) (see Ref. 25)

$$\langle p|\bar\psi(x)\gamma^\alpha\psi(0)-\text{H.c.}|p\rangle=2p^\alpha\int_{-1}^{1}\frac{d\omega}{\omega}F_2(\omega)e^{i\omega x\cdot p}-\frac{ix^\alpha}{x^2}\int_{-1}^{1}\frac{d\omega}{\omega^2}F_L(\omega)e^{i\omega x\cdot p}+\cdots. \qquad (2.11)$$

Based on this result, we shall say that the operator $\bar\psi(x)\gamma^\alpha\psi(0)$ exhibits Bjorken scaling if the expression $\bar\psi(x)\gamma\cdot x\,\psi(0)$ is nonsingular at $x^2=0$. In perturbation theory, this expression has logarithmic singularities, as Schnitzer[17] has discussed. We now make it plausible that these logarithmic singularities came only from the one- and two-point functions in the operator expansion (2.4). Note that the one-point operator which occurs in the operator: $\psi(z_1)\,\psi(z_2)\!:_{1\mathrm{PI}}$ [see (2.2)] in this expansion can be combined with the explicit one-point term in (2.4) with the aid of the Bethe-Salpeter equation shown in Fig. 7 to yield a very simple one-point contribution:

$$:\psi(x_1)\,\bar\psi(x_2):_c=ig\int dz\,S_F(x_1-z)\,B(z)\,S_F(z-x_2)$$
$$+\int dz_1\,dz_2\,[K_c^{FF}(x_1x_2;\,z_1z_2)_{2\mathrm{PI}}\!:\psi(z_1)\,\psi(z_2)\!:\,+K_c^{FB}(x_1x_2;\,z_1z_2)_{2\mathrm{PI}}\!:B(z_1)\,B(z_2)\!:]+\cdots, \qquad (2.12)$$

where we use the notation F for a fermion-antifermion channel and B for a boson-pair channel; S_F is the fermion propagator. There is a similar expansion for the boson two-point function. In writing (2.12) we have, for simplicity, ignored any possible B^3 coupling (which can be forbidden by taking B to be pseudoscalar).

Suppose for the moment that scale invariance holds for these operator expansions in the sense that every operator which appears explicitly in (2.4) is assigned a canonical dimension d ($d=\frac{3}{2}$ for ψ and $\bar\psi$, $d=1$ for B) and the various kernels have dimensions consistent with over-all scale invariance, modulo possible logarithmic terms. [Thus K_c^{FF} has $d=8$, K_c^{FB} has $d=9$, etc., in (2.12).] Possible singularities at $(x_1-x_2)^2\simeq 0$ in (2.4) or (2.12) ostensibly come only from the c-number kernels, although it seems possible that singularities from the operators might be felt through the integrals over dz_j. Actually, because of the irreducibility properties of the kernels and operators in (2.4), which forbid any pair of z_j's from terminating in a common vertex either in the kernels or in the operators, the integrals over singularities of the operators are

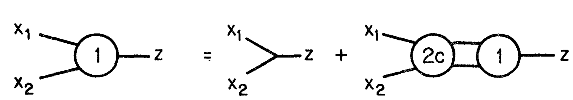

FIG. 7. Bethe-Salpeter equation for the three-point function.

only felt, after integration, as logarithmic deviations from the singularities of the kernels integrated over entirely nonsingular operators. In short, under the assumption of scale invariance, the light-cone behavior of the bilocal operator on the left-hand side of (2.4) or (2.12) is entirely governed by the light-cone behavior of the integrals

$$\int \prod^{j} dz\, K_c(x_1 x_2\,;\, z_j)_{j\mathrm{PI}} \tag{2.13}$$

(modulo logarithmic terms).

The operator $\overline{\psi}(x)\gamma^\mu \psi(0)$ is similarly governed by the c numbers

$$\int \prod^{j} dz\, \mathrm{Tr}\, \gamma^\mu K_c(x;\, z_j)_{j\mathrm{PI}}\ , \tag{2.14}$$

which, by translational invariance, depend only on the difference variable $x = x_1 - x_2$. As an example, consider the $j=1$ term in (2.12). The function $F^\mu(x)$ defined by

$$F^\mu(x) = ig\,\mathrm{Tr}\,\gamma^\mu \int dz\, S_F(x-z) S_F(z) \tag{2.15}$$

must be a vector with dimension 2. It is impossible to construct such an object unless we allow a mass M to appear explicitly, in which case the form

$$F^\mu(x) = \frac{M x^\mu}{x^2}\, D(x^2)\ , \tag{2.16}$$

where $D(x^2)$ is a dimensionless function of $\ln(x^2/M^2)$, is allowed. This is precisely the form of $F^\mu(x)$ in perturbation theory, as is readily verified. [In general, explicit mass factors can appear raised to positive powers but not negative powers in integrals of the type (2.14), in finite orders of perturbation theory.] Thus $x \cdot F$ may be logarithmically divergent in general, and the first term on the right-hand side of (2.12) may contribute a logarithmic singularity to $\overline{\psi}(x)\gamma \cdot x\, \psi(0)$ [there is no singularity in the skeleton theory, where the S_F in (2.15) are free propagators]. Similarly, one may show that the other two terms in (2.12) contribute at most logarithmic singularities to $\overline{\psi}(x)\gamma \cdot x\,\psi(0)$.

Next consider the term $:\psi(z_1)\overline{\psi}(z_2) B(z_3):_{2\mathrm{PI}}$ appearing in (2.4), and its corresponding kernel for $j=3$ in (2.14). The kernel, integrated over $\Pi^j dz$, must be a Dirac matrix and a vector, with dimension -1. The only allowed forms are $x^\mu D(x^2)$ or $\gamma^\mu \gamma \cdot x\, D(x^2)$, where $D(x^2)$ is possibly logarithmically singular, or other less singular forms with explicit positive powers of a mass M. But both of these objects, when multiplied by x^μ, are of the form $x^2 D(x^2)$ and *not* singular on the light cone. Similarly, the only allowed form for the three-boson kernel, of dimension 0 after integration, is $M x^\mu D(x^2)$, which contributes no singularities upon multiplication with x^μ to form $\overline{\psi}(x)\gamma \cdot x\,\psi(0)$. It is easy to check that these forms are all that arise in perturbation theory.

All other integrated kernels have negative dimensions, which prevents them from contributing light-cone singularities to $\overline{\psi}(x)\gamma \cdot x\,\psi(0)$, by a simple extension of the above argument. We have thus made it plausible that *the operator-valued integral equation (2.12) contains all potential light-cone singularities of the fermion bilocal operator (less, of course, c-number terms).* Similarly, all light-cone singularities (at worst logarithmic) in $:B(x_1) B(x_2):_c$ come from the fermion and boson bilocal operators.

The reader may easily verify for himself the nature of the singularities (if any) in the various kernels in the tree approximation, using the extension of the generalized BJL limit given in the Appendix. For the much harder problem of the full skeleton theory, see II.

III. IMPLICATIONS OF THE OPERATOR EXPANSION

In this section, we discuss the relation of the fermion bilocal expansion (2.12) to the canonical results[10] and to the usual Wilson[1,2] expansion at short distances. For completeness, we dwell briefly upon certain curious features encountered in perturbation theory (but presumably not in the real world): It is possible that the electroproduction structure functions W_1 and νW_2 become functions of $\omega = -q^2/2\nu$ only in the Bjorken scaling limit, and yet there are stronger singularities on the light cone than the canonical ones implied by (2.10). Conversely, there may be pieces of W_1 and νW_2 which fall far below scaling, yet contribute to the leading singularities on the light cone. Both types of anomalous behavior are connected with singularities of scaling at the threshold, $\omega = 1$.

In the Wilson expansion, one extracts from an ill-defined operator such as $:\overline{\psi}(x)\psi(x):$ a well-defined local finite part called $N(\overline{\psi}(x)\,\psi(x))$, by an expansion in terms of local operators and singular c-number functions[2-4]:

$$:\overline{\psi}(x)\psi(y): \underset{x\to y}{\sim} C_1(x-y)\, B(x) + C_2(x-y)\,\Box B(x) + C_3(x-y)\, N(\overline{\psi}(x)\,\psi(x)) + N(\overline{\psi}(x)\,\psi(x)) + \cdots\ , \tag{3.1}$$

where the functions C_i are singular at $x = y$, and the omitted terms are finite. Let us extract the $O(g)$

terms of (3.1) from the light-cone expansion, by taking the trace:

$$:\bar{\psi}(x)\,\psi(y):\underset{x\to y}{\sim} ig\int dz\,\mathrm{Tr}\,S_F(y-z)\,S_F(z-x)\,B(z)+O(g^2)$$

$$=\frac{ig}{(2\pi)^4}\int dp\,e^{-ip(y-x)}\,\frac{\mathrm{Tr}(\not{p}+M)(\not{p}-\not{q}+M)}{(p^2-M^2)\,[(p-q)^2-M^2]}$$

$$\times\int\frac{dq}{(2\pi)^4}\,\tilde{B}\,(q)\,e^{-iqx}+O(g^2)\,, \tag{3.2}$$

where we have introduced the Fourier decomposition

$$B(x)=\frac{1}{(2\pi)^4}\int dq\,e^{-iqx}\,\tilde{B}(q) \tag{3.3}$$

for the boson operator. The integral over p in (3.2) is readily carried out; the leading terms are

$$\frac{-g}{\pi^2(x-y)^2}+\frac{4gM^2}{3}\ln(x-y)^2\frac{-g}{8\pi^2}\,q^2\ln(x-y)^2$$
$$+\text{nonsingular terms}\,. \tag{3.4}$$

By converting q^2 to $-\Box$ and comparing with (3.1), we find, to $O(g)$,

$$C_1(x)=\frac{-g}{\pi^2x^2}+\frac{4gM^2}{3}\ln x^2\,,$$
$$C_2(x)=\frac{g}{8\pi^2}\ln x^2\,. \tag{3.5}$$

The function C_3 in (3.1) is of $O(g^2)$; we shall not calculate it explicitly.

The significance of the Wilson expansion in perturbation theory is that it automatically accounts for the infinities which are ordinarily relegated to the renormalization constants (thus, using the Wilson expansion, only finite renormalizations remain). To illustrate,[4] let us demand that the equation of motion for the boson field be expressible solely in finite quantities, and write

$$(\Box+m^2)\,B(x)=gN(\bar{\psi}(x)\,\psi(x))\,, \tag{3.6}$$

with m and g finite. Thus the equation of motion for the propagator is

$$(\Box_x+m^2)\langle 0|-iT(B(x)\,B(y))\,|\,0\rangle=-\delta_4(x-y)-ig\,\langle 0|T(N(\bar{\psi}(x)\,\psi(x)))\,B(y)|\,0\rangle\,. \tag{3.7}$$

The second term on the right-hand side is, using (3.1) and (3.5),

$$-ig\,\langle 0|T(N(\bar{\psi}(x)\,\psi(x))\,B(y))\,|\,0\rangle=\lim_{x\to x'}-ig\,\langle 0|T(\bar{\psi}(x)\,\psi(x'))\,B(y)|\,0\rangle+\frac{ig^2}{\pi^2(x-x')^2}\langle 0|T(B(x)\,B(y))\,|\,0\rangle$$
$$+\frac{ig^2}{8\pi^2}\ln(x-x')^2\Box_x\langle 0|T(B(x)B(y))\,|\,0\rangle+O(g^4)\,. \tag{3.8}$$

To $O(g^2)$, the right-hand side of (3.8) is, in the limit $x=x'$,

$$\frac{ig^2}{(2\pi)^4}\int dk\,e^{-ik(x-y)}\int dp\left[\frac{\mathrm{Tr}(\not{p}-\not{k}+M)(\not{p}+M)}{(p^2-M^2)[(p-k)^2-M^2](k^2-m^2)}-\frac{4}{p^2(k^2-m^2)}+\frac{2k^2}{p^4(k^2-m^2)}-\frac{3M^2}{p^4(k^2-m^2)}\right]. \tag{3.9}$$

It will be found, using the ordinary techniques of symmetric integration, that the integrand of (3.9) behaves like p^{-6} at large p for fixed k, thus establishing the finiteness of (3.7) for general values of $x-y$. [There are infrared singularities in (3.9), but they are easily removed by a suitable definition of the *nonsingular* terms in the Wilson expansion (3.1).]

Another specialized situation which can be recovered from the general light-cone expansion of Sec. II is that of light-cone commutators. We record the canonical fermion anticommutator from Ref. 10, using scalar bosons instead of vectors:

$$\{\psi(x),\bar{\psi}(0)\}_{x^+=0}=\Big\{\tfrac{1}{4}ig\,\epsilon(x^-)\,[P_+B(0,\vec{0},x^-)+P_-B(0)]$$
$$+\tfrac{1}{16}\int_{-\infty}^{\infty}d\xi\,\epsilon(x^--\xi)\,\epsilon(\xi)\,[i\gamma^i\partial_i-g{:}B(0,\vec{x}_\perp,\xi)+M]\,[i\gamma^j\partial_j-gB(0,\vec{x}_\perp,\xi){:}-M]\gamma^+$$
$$+\frac{ig^2}{64}\int d\xi\,d\xi'\,\epsilon(x^--\xi)\,\epsilon(\xi')\,\epsilon(\xi-\xi')\,\gamma^+{:}\psi(0,\vec{0},\xi)\,\bar{\psi}(0,\vec{0},\xi'){:}\gamma^+\Big\}\,\delta(\vec{x}_\perp)\quad(P_\pm=\tfrac{1}{2}\gamma^\pm\gamma^\mp)\,. \tag{3.10}$$

[Some c-number terms in (3.10) have been omitted.] Since (3.10) depends explicitly on g only through terms of order g and g^2, it is apparent that (3.10) is the light-cone commutator corresponding to the light-cone T product of Eq. (2.12), with the lowest-order expressions for the kernels K_c^{FF} and K_c^{FB} substituted:

$$:\psi(x)\,\bar{\psi}(y):_c = ig\int dz\, S_F(x-z)\,B(z)\,S_F(z-y)$$
$$+\, ig^2\int dz_1\,dz_2\,S_F(x-z_1)\,[:\psi(z_1)\,\bar{\psi}(z_2):\,\Delta_F(z_1-z_2)+S_F(z_1-z_2):B(z_1)B(z_2):]\,S_F(z_2-y)+\cdots\,, \tag{3.11}$$

where S_F is the free fermion propagator, and Δ_F the free boson propagator. The easiest way to go from the T product of (3.11) to the light-cone commutator in (3.10) is by way of the generalized BJL limit,[10] in which one extracts the coefficient of i/q^- in the Fourier transform of (3.11) as $q^-\to\infty$, with other components of q held fixed:

$$-iq^-\int dx\, e^{iqx}:\psi(x)\,\bar{\psi}(0):_c \xrightarrow[q^-\to\infty]{} \int dx^-\,d\vec{x}_\perp\, e^{iq^+x^- -i\vec{q}_\perp\cdot\vec{x}_\perp}:\{\psi(x),\,\bar{\psi}(0)\}_{x^+=0}$$
$$=\frac{g}{4(2\pi)^4}\int dk\,\frac{\gamma^+}{q}\,\tilde{B}(k)\left[\gamma^-+\frac{\gamma^j(q+k)_j+M}{(q+k)^+}\right]+\left(\gamma^-+\frac{\gamma^j q_j+M}{q^+}\right)\tilde{B}(k)\,\frac{\gamma^+}{(q+k)^+}$$
$$+\,O(g^2)+\text{polynomials} \tag{3.12}$$

[observe that $(\gamma^+)^2=0$]. It is straightforward to check, using the convolution theorem and the basic identity

$$\epsilon(x^-)=\frac{i}{\pi}\int\frac{dq^+}{q^+}\,e^{-iq^+x^-}\,, \tag{3.13}$$

that (3.12) generates all four of the $O(g)$ terms in (3.10). A similar calculation verifies that the light-cone limit of the $O(g^2)$ terms in (3.11) also yield the rest of (3.10).

If (3.11) were the only T product which yielded the canonical light-cone commutator (3.10), we would be forced to conclude that Bjorken scaling could not be a consequence of the canonical field theory. The reason is that the simplified kernels of (3.11), which generate uncrossed ladder graphs in the Bethe-Salpeter equation, are known not to lead to scaling.[14] Fortunately (or perhaps unfortunately), one cannot draw this conclusion; for one thing, (3.10) may be ill defined, because it contains the product $:B(x)\,B(x):$ which perturbation theory indicates is logarithmically singular. Let us ignore this potential difficulty for now. It is clear from the Appendix that the bilocal terms in the light-cone commutator $:\psi(x)\bar{\psi}(0):_{x^+=0}$ must vanish or cancel, if $:\bar{\psi}(x)\gamma\cdot x\,\psi(0):$ is to be nonsingular, as is required by our scaling criterion. [Terms which involve the boson field linearly, as in (3.12), could survive; they contribute terms to the current correlation function (2.5) which violates our scaling criterion in the real part, not the imaginary part as measured in deep-inelastic electroproduction.] While the bilocal terms cannot have vanishing matrix elements (or else the scaling functions vanish), they may cancel, since the leading singular term in $:B(x)\,B(y):$ is a c-number kernel integrated over the fermion bilocal.

To conclude this section, we comment on certain pathological behavior which is realized in perturbation theory, and which violates the usually assumed connection between scaling and light-cone singularities. The first instance of such behavior comes from self-energy and vertex corrections to the current-correlation function (2.5); see Fig. 8. Cutting these graphs across the self-energy and vertex corrections yield imaginary parts which scale, in the sense that W_1 and νW_2 become functions of ω only, but which have poles at $\omega=\pm 1$. These poles come from the single-fermion lines in Fig. 8, which yield propagators of the form $(q^2\pm 2\nu)^{-1}\sim 1\pm\omega$. These poles make the expressions (2.10) and (2.11) meaningless, since the integrals over ω diverge. They are absent in the skeleton theory or in the full theory if Z_1 and Z_2 are finite. The question is, can such terms be accommodated within the general framework of operator expansions outlined in Sec. II? Without going into details, it appears that the answer is yes.[26] Clearly, the self-energy correction of Fig. 8(a) requires only a change in the c-number function $C(x)$ [see Eq. (2.8)] which multiplies the fermion bilocal in the light-cone expansion of the commutator of two currents. As for the vertex correction of Fig. 8(b), it can be taken into account in lead-

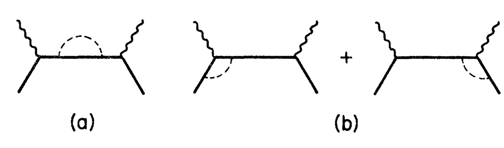

FIG. 8. (a) Lowest-order self-energy correction to the current correlation function $\langle p|T(J^\mu(x)J^\nu(y))|p\rangle$. (b) Lowest-order vertex correction. Currents indicated by wavy lines.

ing order by modifying the matrix element (2.11) of the fermion bilocal to the extent that pole terms in $F_2(\omega)$ or $F_L(\omega)$ of the type $(1-\omega)^{-1}$ should be replaced by $(1-\omega-M^2x^2)^{-1}$, thus yielding terms like $\ln x^2$ in (2.11).

Thus there are graphs which scale, but which have stronger light-cone singularities than are usually associated with scaling. (Of course, it is highly unphysical to have a pole in the scaling functions at $\omega=1$; in fact, we expect the scaling functions to vanish there.) Conversely, there may be contributions to W_1 and νW_2 which fall below scaling, but which show up on the light cone in the form of additional terms in (2.10). These terms have already been discussed in Ref. 27, but without using light-cone language. Write the current-correlation function (2.5) as

$$T_{\mu\nu} = [-q^2p_\mu p_\nu+\nu(p_\mu q_\nu+p_\nu q_\mu)-\nu^2 g_{\mu\nu}]T_2 + (g_{\mu\nu}q^2-q_\mu q_\nu)T_L \ , \tag{3.14}$$

where T_2 and T_L obey the Deser-Gilbert-Sudarshan[28] (DGS) representation. The simplest form for T_2 is

$$T_2=\int_{\sigma_0}^\infty d\sigma\int_{-1}^1 \frac{d\beta\, h(\sigma,\beta)}{q^2+2\beta\nu-\sigma+i\epsilon} \ , \tag{3.15}$$

with

$$\int_{\sigma_0}^\infty d\sigma\, h(\sigma,\beta)=0, \quad \sigma_0=\sigma_0(\beta)\geq 0 \ . \tag{3.16}$$

The restriction (3.16) is necessary for scaling to hold. It is equivalent to the statement

$$h(\sigma,\beta)=\frac{\partial}{\partial\sigma}g(\sigma,\beta), \quad g(\sigma_\sigma,\beta)=g(\infty,\beta)=0. \tag{3.17}$$

Now calculate the imaginary part νW_2 [see (2.6)] for $2\nu+q^2\geq 0$:

$$\begin{aligned}\nu W_2 &= \tfrac{1}{2}\nu q^2\int d\sigma\, d\beta\, h(\sigma,\beta)\,\delta(q^2+2\beta\nu-\sigma)\,\epsilon(q^2+2\beta\nu)\\ &= \tfrac{1}{4}q^2\int_{\sigma_0}^{q^2+2} d\sigma\left[\frac{d}{d\sigma}\, g\left(\sigma,\frac{\sigma-q^2}{2\nu}\right) - \frac{1}{2\nu}\frac{\partial}{\partial\beta}g(\sigma,\beta)\Big|_{\beta=(\sigma-q^2)/2\nu}\right]\\ &= \tfrac{1}{4}q^2 g(q^2+2\nu,1)+\tfrac{1}{4}\omega\int_{\sigma_0}^\infty d\sigma\, g_\beta(\sigma,\omega)+\cdots \ ,\end{aligned} \tag{3.18}$$

where the omitted terms do not contribute to scaling, and the subscript β indicates the partial derivative with respect to that variable. In order that scaling hold, it must be that $g(\sigma,\beta)$ decreases faster than σ^{-1} at infinity. Then the scaling function is

$$F_2(\omega)=\tfrac{1}{4}\omega\int_{\sigma_0}^\infty d\sigma\, g_\beta(\sigma,\omega)\equiv\tfrac{1}{4}\omega\, g_\beta(\omega) \ . \tag{3.19}$$

Now if $g(\sigma,1)\neq 0$, insertion of the expression (3.19) for $F_2(\omega)$ into the light-cone commutator (2.10) does not yield the correct result. The correct result is found by taking the Fourier transform of the imaginary part of (3.14) and going to the light cone:

$$\begin{aligned} i\langle p|[J^\mu(x),J^\nu(0)]|p\rangle &= -[p^\mu p^\nu - p\cdot\partial(p^\mu\partial^\nu+p^\nu\partial^\mu)+g^{\mu\nu}(p\cdot\partial)^2]\\ &\quad\times\tfrac{1}{2}[\epsilon(x^0)\,\theta(x^2)]\int_{-1}^1 d\omega\, e^{i\omega x\cdot p}\,\tfrac{1}{4}g(\omega)+\cdots\end{aligned} \tag{3.20}$$

(we omit the inessential longitudinal terms), where

$$g(\omega)=\int_{\sigma_0}^\infty d\sigma\, g(\sigma,\omega) \ . \tag{3.21}$$

Comparison of (3.19), (3.20), and (3.21) with the previously given light-cone commutator (2.10) shows that the following modification must be made in (2.10) so that it agrees with (3.20):

$$F_2(\omega)\to F_2(\omega)-\tfrac{1}{4}g(1)[\delta(\omega-1)+\delta(\omega+1)] \tag{3.22}$$

[integrate by parts, and use $g(1)=g(-1)$]. In other words, the light-cone commutator is determined not only by the scaling functions $F_2(\omega)$, $F_L(\omega)$, but also by the quantity $g(1)$, which is an integral over a piece of the imaginary part which does not contribute to scaling [see (3.18) and the remarks below this equation]. As mentioned in Ref. 27, it would be highly unphysical if $g(1)$ were not zero, since then there would be contributions to the imaginary part which, at fixed s $(=q^2+2\nu+M^2)$, would grow with ν [see (3.18)]. However, in perturbation theory one has no assurance that this will not happen.

For certain other anomalies with occur in perturbation theory, associated with the behavior of $F_L(\omega)$, see Schnitzer.[17]

IV. SUMMARY AND CONCLUSIONS

We have given a generalization of the Wilson expansion to the entire light cone, in which operator products are expanded as a sum of c-number kernels integrated over special operator products with certain irreducibility properties. The generalized expansion with an infinite number of terms holds as an identity, but it simplifies to only a few terms on the light cone. By specializing this truncated expansion to the short-distance regime, we have recovered the Wilson expansion; by specializing to commutators, we recovered expressions similar—but not necessarily identical—to those found canonically. It has been necessary to extend the genaralized BJL limit to find terms only logarithmically singular on the light cone, which violate scaling if they appear in fermion bilocals.

The estimation of singularities associated with specific terms in the operator expansion is merely plausible, depending as it does on assumptions of conformal quasi-invariance. In a companion paper[1] we are able to settle the question of singularities completely to any finite order in perturbation theory; in all cases, the result found this way agrees (up to logarithms) with that based on conformal quasi-invariance.

After all this work, we have nothing new to say about scaling in deep-inelastic electroproduction. Our only result in this direction is that if scaling holds for two-particle *matrix elements*, it also holds in an *operator* sense (i.e., for all matrix elements). Unfortunately, this result is of less than overwhelming practical utility. We hope to report in the near future on the kernel K_{c2PI}, whose properties govern scaling, without using the popular leading-logarithm approximation.

ACKNOWLEDGMENT

One of us (J.M.C.) thanks Professor R.E. Norton for many valuable discussions when this work was in its early stages.

APPENDIX: LIGHT-CONE SINGULARITIES OF FEYNMAN GRAPHS

Given any matrix element of the T product of two currents $J(x)$ and $K(0)$, the generalized BJL limit[19] prescribes that the light-cone commutator of these currents is found as[10]

$$i\int dx\, e^{iqx}\langle A|T(J(x)\,K(0))\,|B\rangle \xrightarrow[q^-\to\infty]{} \text{polynomials} - \frac{1}{q^-}\int dx^- d\vec{x}_\perp\, e^{iq^+x^- - i\vec{q}_\perp\cdot\vec{x}_\perp}\langle A|[J(x),K(0)]_{x^+=0}|B\rangle + \cdots . \tag{A1}$$

As long as one need deal only with *commutators* on the light cone, this result is sufficient to isolate all light-cone singularities. But deep-inelastic electroproduction is governed by the Wightman product $\psi(x)\bar{\psi}(0)$, which can have terms singular near the light cone which do not contribute to the Fourier transform of the commutator (A1); the prototypical example is $\ln(x^2 - i\epsilon x^0)$. Since it is just terms of this sort which violate scaling in perturbation theory, we must extend the BJL limit to search for terms which are singular on the light cone for T products or Wightman products, but not in commutators. We shall investigate forward single-particle matrix elements only (using the DGS[28] representation), although the results are easily generalized.

Consider the matrix element of two scalar currents:

$$T(p,q) = i\int dx\, e^{iqx}\langle p|T(J(x)\,K(0))|p\rangle . \tag{A2}$$

The properties of $T(p,q)$ will be the same as of an invariant amplitude for the case of currents with spin. The most general form of T is a sum of terms of the type

$$\int d\lambda^2\, d\beta\, \frac{h_{N,M}(\lambda^2,\beta)(p\cdot q)^N}{[(q+\beta p)^2 - \lambda^2 + i\epsilon]^M} ,$$

$$N = 0, 1, \ldots ; \quad M = 1, 2, \ldots \tag{A3}$$

(Possible terms like q^{2J} in the numerator, with $J \leq M-1$, can be eliminated by integration by parts.) The Fourier transform of such a term is

$$\int d\lambda^2\, d\beta\, h_{N,M}(\lambda^2,\beta)(ip\cdot\partial)^N \frac{e^{i\beta p\cdot\alpha}}{(M-1)!}\left(\frac{\partial}{\partial\lambda^2}\right)^{M-1}\Delta_F(x;\lambda^2). \tag{A4}$$

The Feynman propagator has the form

$$\Delta_F(x;\lambda^2) = \frac{-\lambda}{8\pi(x^2-i\epsilon)^{1/2}} H_1^{(1)}[\lambda(x^2-i\epsilon)^{1/2}] \tag{A5}$$

$$\underset{x^2\to 0}{\sim} \frac{i}{4\pi^2(x^2-i\epsilon)} - \frac{i\lambda^2}{8\pi^2}\left(1 - \frac{\lambda^2x^2}{q} + \cdots\right)\ln[\tfrac{1}{2}\gamma\lambda(-x^2+i\epsilon)^{1/2}] + \frac{i\lambda^2}{16\pi^2}\left(1 - \frac{5\lambda^2x^2}{16} + \cdots\right) , \tag{A6}$$

where $\ln\gamma = 0.5772\ldots$, and the arguments of $(-x^2+i\epsilon)^{1/2}$ and of the logarithm are chosen in the conventional way, that is,

$$\ln(-x^2+i\epsilon)^{1/2} = \ln(|x^2|^{1/2}) + i\tfrac{1}{2}\pi\,\theta(x^2) . \tag{A7}$$

It is straightforward to find that the only singular terms of (A4) obey the condition

$$N \geq M-2 . \tag{A8}$$

Terms with $N < M-2$ have no singularities on the light cone. In particular, terms with $M = 1, 2$ are singular for any allowed N. Of course, those terms for which $N \geq M-1$ are of $O(1/q^-)$ or larger in the generalized BJL limit, and contribute to the light-cone commutator (A1); they also contribute to the light-cone T product or Wightman product. We are interested in the terms with $N=M-2$ which are singular in T products on the light cone, but not contribute to the light-cone commutator. In the generalized BJL limit, these terms decrease with a power law $(q^-)^{-\alpha}$, where $1<\alpha<2$ (barring pathological oscillations in the limit; logarithmic factors are ignored). For simplicity, we discuss only the case $\alpha=2$, which corresponds to the finiteness of $\int d\lambda^2 h_{N+2,N}$. With the aid of (A4) and (A6), the light-cone singularity is easily found to be of the type $(x\cdot p)^N \times \ln x^2$. Note that we may force this function to vanish arbitrarily rapidly for small x by choosing N large enough, but it is always singular for small x^2. The equivalent momentum-space statement is that (A3) for $N=M-2$ decreases like $(q)^{-N-4}$ when all components of q are large, but only like $(q^-)^{-2}$ in the generalized BJL limit. On the other hand, a term in (A3) which vanishes as $(q^-)^{-\alpha}$ with $\alpha>2$ is definitely not singular.

It is not hard to see why $\alpha=2$ is the boundary between singular and nonsingular behavior. Barring delicate oscillating behavior, a one-dimensional Fourier transform $F(x)\int dq\, e^{-iqx}\tilde{F}(q)$ is not singular for any finite real x, if $F(0)$ exists. A similar statement can be made for a *multi-dimensional* Fourier integral only if it is known that the result is independent of the choice of integration variables and of the order in which the integrations are performed. But this is not the case for singular Feynman integrals; as an example, try to evaluate

$$\Delta_F(x;\lambda^2)=\frac{1}{(2\pi)^4}\int dq\frac{e^{-iqx}}{q^2-\lambda^2+i\epsilon} \tag{A9}$$

by setting $\vec{x}_\perp=0$, and first doing the integration over $\vec{q}_\perp$. A similarly infinite answer results if the denominator in (A9) is squared. Now consider inverting the Fourier transform in (A3) to get (A4). To check for singularities on the light cone, set $x^+=\vec{x}_\perp=0$. The integral over $\vec{q}_\perp$ is finite for $N+2<2M$, and yields an integral which behaves like $(q^-)^{N-M+1}$ for large q^-. This integral over q^- diverges if $N\geq M-2$, and converges otherwise, which is precisely the criterion [(A8)] found earlier. If neither x^+ nor x^- vanishes, so that $x^2\neq 0$, then the integrals can be carried out because of the oscillatory factor $\exp[i(q^+x^-+q^-x^+)]$.

There are other cases of interest. When the integral over $q_\perp$ does not converge, there is always a short-distance singularity and *a fortiori* a light-cone singularity. Insertion of tensorial indices leads to new features, as illustrated in the following example:

$$\tilde{F}^\mu=q^{-6}(2q^\mu p\cdot q-p^\mu q^2)\ . \tag{A10}$$

One finds that, for $\mu=+$, $\tilde{F}^+\sim(q^-)^{-3}$ at infinity, yet the Fourier transform $F^+(x)$ behaves like $p^+\ln x^2$, which seems to be an exception to the general rule that amplitudes decreasing as fast as $(q^-)^{-3}$ have no light-cone singularities. However, such exceptions always have short-distance singularities, which are easily found by checking the behavior as *all* components of q become large. In any case, tensorial structure is necessary; this phenomenon cannot happen for scalar integrals.

In summary, a Feynman integral has a singular Fourier transform on the light cone if it behaves in the generalized BJL limit like $(q^-)^{-\alpha}$ with $\alpha\leq 2$, and is not singular if $\alpha>2$. In field theories with spin, the behavior in this limit is completely independent of how fast the Feynman integral converges when all components of q approach infinity. [The factors $(p\cdot q)^N$ do not appear in the Feynman integrals of ϕ^3 theory, so this remark does not apply there.] Finally, it is worth noting that all the remarks of this appendix hold equally for Wightman products, for all those singularities which do *not* appear in the light-cone commutator (A1). This is, of course, because a T product and a Wightman product differ by a (retarded) commutator.

---

*Work supported in part by the National Science Foundation.

†Alfred P. Sloan Foundation Fellow.

[1]John M. Cornwall and George Tiktopoulos, following paper, Phys. Rev. D 8, 2687 (1973); referred to as II.

[2]K. Wilson, Phys. Rev. 179, 1499 (1969). Wilson's earlier, unpublished work is reviewed by R. Brandt [Ann. Phys. (N.Y.) 44, 221 (1967)].

[3]W. Zimmermann, in *Lectures on Elementary Particles and Quantum Field Theory*, edited by S. Deser, M. Grisaru, and H. Pendleton (MIT Press, Cambridge, Mass., 1970), Vol. I, p. 395.

[4]R. Brandt, Ref. 2.

[5]Y. Frishman, Phys. Rev. Lett. 25, 966 (1970); Ann. Phys. (N.Y.) 66, 373 (1971); H. Leutwyler and J. Stern, Nucl. Phys. B20, 77 (1970).

[6]R. Jackiw, R. Van Royen, and G. B. West, Phys. Rev. D 2, 2473 (1970). Our scaling functions are $(4\pi)^{-1}$ times the scaling functions of this reference.

[7]For example, the leading-logarithm calculations of P. M. Fishbane and J. D. Sullivan, Phys. Rev. D 4 2516 (1971).

[8]R. Brandt and G. Preparata, Nucl. Phys. B27, 541 (1971); G. Altarelli, R. Brandt, and G. Preparata,

Phys. Rev. Lett. 26, 42 (1971).
[9]Throughout these papers, the term "bilocal operator" means the connected part of the product of two local operators, and not—as some authors use the term—the operator-valued residue of a c-number singularity in the connected product of two operators.
[10]J. M. Cornwall and R. Jackiw, Phys. Rev. D 4, 367 (1971).
[11]H. Fritzsch and M. Gell-Mann, in *Broken Scale Invariance and the Light Cone,* 1971 Coral Gables Conference on Fundamental Interactions at High Energy, edited by M. Dal Cin, G. J. Iverson, and A. Perlmutter (Gordon and Breach, New York, 1971), Vol. 2.
[12]D. Gross and S. B. Treiman, Phys. Rev. D 4, 1059 (1971).
[13]J. D. Bjorken, Phys. Rev. 179, 1547 (1969).
[14]R. Jackiw and G. Preparata, Phys. Rev. 185, 1748 (1969).
[15]S. L. Adler and W. K. Tung, Phys. Rev. D 1, 2846 (1970).
[16]R. Jackiw and R. E. Waltz, Phys. Rev. D 6, 702 (1972).
[17]H. Schnitzer, Phys. Rev. D 6, 2118 (1972).
[18]J. M. Cornwall, Phys. Rev. D 5, 2868 (1972).
[19]J. D. Bjorken, Phys. Rev. 148, 1467 (1966); K. Johnson and F. E. Low, Prog. Theor. Phys. Suppl. 37–38, 74 (1966). The work of these authors was generalized to the light cone in Ref. 8.
[20]We use the conventional light-cone notation; for any four-vector $a^{\mu}=(a^{+},\vec{a},a^{-})$, $a^{\pm}=(1/\sqrt{2})\,(a^{0}\pm a^{3})$, $\vec{a}_{\perp}=(a^{1},a^{2})$.
[21]J. C. Polkinghorne, Nuovo Cimento 8A, 592 (1972).
[22]Y. Shimamoto, Nuovo Cimento 25, 1292 (1962).
[23]Cf. K. Symanzik, J. Math. Phys. 1, 249 (1960).
[24]J. B. Kogut and D. E. Soper, Phys. Rev. D 1, 2901 (1970).
[25]Leading Regge poles may cause the integrals in (2.10) and (2.11) to diverge, owing to singularities at small ω. The necessary subtraction procedure to render the integrals finite has been discussed in the context of the DGS representation in Ref. 27, and in the context of the light cone by Y. Frishman, in *Proceedings of the XVI International Conference on High Energy Physics, Chicago-Batavia, Ill., 1972,* edited by J. D. Jackson and A. Roberts (NAL, Batavia, Ill., 1972), Vol. 4, p. 119.
[26]H. Schnitzer (private communication).
[27]J. M. Cornwall, D. Corrigan, and R. E. Norton, Phys. Rev. D 3, 536 (1971).
[28]S. Deser, W. Gilbert, and E. C. G. Sudharshan, Phys. Rev. 115, 731 (1959); M. Ida, Prog. Theor. Phys. 23, 1151 (1960); N. Nakanishi, *ibid.* 26, 337 (1961); Prog. Theor. Phys. Suppl. 18, 70 (1961).

PHYSICAL REVIEW D VOLUME 8, NUMBER 8 15 OCTOBER 1973

Light-Cone Operator Expansions in Perturbation Theory. II*

John M. Cornwall† and George Tiktopoulos
Department of Physics, University of California, Los Angeles, California 90024
(Received 7 March 1973)

A systematic investigation, in perturbation theory, is presented of the light-cone behavior of multiparticle matrix elements of time-ordered products of local fields: $\int d^4x\ e^{iqx}\ \langle\ \beta|T\psi_1\,(x)\psi_2(0)\ |\alpha\ \rangle$. In the limit $q_-\rightarrow\infty$, the contribution of any single Feynman graph is of the form $q_-^{\ \beta}(\ln q_-)^{\gamma}$. The main result here is a rule by means of which the integers β and γ can be read off from the topology of the graph. The implications of this investigation for local field theories are organized and discussed in operator language in a companion paper. A by-product of the special methods here developed to obtain asymptotic estimates in perturbation theory is a refinement of Weinberg's theorem for the *Euclidean* region: the determination of the logarithmic factors in the asymptotic form of Feynman amplitudes when a set of external momenta q_1, q_2, ... is allowed to approach infinity according to $q_i=\eta q'_i$, $\eta\rightarrow\infty$.

I. INTRODUCTION

This is the second of two papers[1] dealing with light-cone singularities of bilocal operator products as manifested in perturbation theory. The present paper is devoted exclusively to a systematic investigation of the asymptotic contribution of an arbitrary Feynman graph to Fourier-transformed multiparticle matrix elements of time-ordered products:

$$\int d^4x\, e^{iqx}\langle\beta\mid T\psi_1(x)\psi_2(0)\mid\alpha\rangle$$

in the generalized Bjorken-Johnson-Low (BJL) limit[2]: $q_-\rightarrow\infty$ with $q_+,\vec{q}_\perp$ fixed.

[For reasons of typographical clarity, the notation used in this paper differs from that in I. For any vector q^{μ}, the quantities $q_{\pm}$ are not defined as vector components, but rather as scalar products: $q_{\pm}=q^{\pm}=M_{\pm}\cdot q$, $M^{\mu}_{\pm}=2^{-1/2}(1,0,0,\mp 1)$. Thus q_- or q^- is numerically equal to the $-$ component of the contravariant vector.]

The asymptotic behavior of any single Feynman graph turns out to be of the form $q_-^{\ \beta}(\ln q_-)^{\gamma}$. Our main result here is the formulation of a rule according to which the integers β and γ can be read off from the topology of any given graph. The ap-