*Research supported by the National Research Council of Canada.

[1] I. O. Moen and W. J. Zakrzewski, Nucl. Phys. B (to be published), hereafter referred to as I.
[2] P. Goddard and A. R. White, Nuovo Cimento 1A, 645 (1971); A. R. White, Nucl. Phys. B39, 432 (1972); B39, 461 (1972); I. T. Drummond, P. V. Landshoff, and W. J. Zakrzewski, Nucl. Phys. B11, 383 (1969); Phys. Lett. 28B, 676 (1969).
[3] J. H. Weis, Phys. Rev. D 6, 2823 (1972); H. D. I. Abarbanel and A. Schwimmer, Phys. Rev. D 6, 3018 (1972).
[4] C. E. DeTar and J. H. Weis, Phys. Rev. D 4, 3141 (1971).
[5] O. Steinmann, Helv. Phys. Acta 33, 257 (1960); 33, 347 (1960); H. Araki, J. Math. Phys. 2, 163 (1961); H. P. Stapp, Phys. Rev. D 3, 3177 (1971).
[6] C. E. Jones, F. E. Low, S.-H. Tye, G. Veneziano, and J. E. Young, Phys. Rev. D 6, 1033 (1972).
[7] I. O. Moen and A. R. White, Phys. Lett. 42B, 75 (1972).
[8] I. O. Moen and W. J. Zakrzewski, DAMTP Report No. 73/6, 1973 (unpublished).
[9] J. H. Weis, Phys. Rev. D 5, 1053 (1972).
[10] D. Z. Freedman and J.-M. Wang, Phys. Rev. 160, 1560 (1967).
[11] See, for example, P. D. B. Collins and E. J. Squires, *Springer Tracts in Modern Physics,* edited by G. Höhler (Springer, New York, 1968), Vol. 45.
[12] J. H. Weis, Phys. Rev. D 4, 1777 (1971).
[13] See, for example, A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton Univ. Press, Princeton, New Jersey, 1957).

PHYSICAL REVIEW D VOLUME 8, NUMBER 8 15 OCTOBER 1973

Potential Scattering in the Eikonal Approximation*

F. W. Byron, Jr
Department of Physics and Astronomy, University of Massachusetts, Amherst, Massachusetts 01002

C. J. Joachain
Physique Théorique et Mathématique, Faculté des Sciences, Université Libre de Bruxelles, Brussels, Belgium

E. H. Mund†
Service de Métrologie Nucléaire, Faculté des Sciences Appliquées, Université Libre de Bruxelles, Brussels, Belgium

(Received 26 June 1972; revised manuscript received 5 February 1973)

A systematic study is made of the limits of validity of the eikonal approximation in nonrelativistic potential scattering theory. We suggest that for a large class of potentials, and for all momentum transfers, each term of the eikonal multiple-scattering series gives the asymptotic value (for large incident wave numbers) of the corresponding term in the Born series. This property, together with the requirement of unitarity, implies that in weak-coupling situations the eikonal approximation is consistently worse than the second Born approximation. For intermediate couplings we find that the eikonal method is remarkably good at all angles for potentials of the Yukawa type. For the case of strong coupling ($|V_0|>E$) we find that for all potentials studied there is good agreement between exact and eikonal results at small angles. Analytical and numerical results are given for a variety of interaction potentials.

I. INTRODUCTION

Originally introduced more than twenty years ago in quantum scattering theory,[1] the eikonal approximation has attracted a considerable amount of interest, particularly in recent years after the work of Glauber,[2] who devised a very convenient many-body generalization of the method. The eikonal approximation has been used extensively and with considerable success in the analysis of intermediate and high-energy hadronic collisions.[3] More recently, atomic scattering processes have also been studied by means of the eikonal approximation.[4]

Although several derivations and modifications of the eikonal scattering amplitude have been proposed, there is at the present time no systematic, detailed study of the range of validity of the eikonal approximation. Since the numerous applications of the eikonal method encompass atomic, nuclear and high-energy collisions, i.e., the whole range of microphysics, it seems highly desirable that the fundamental limitations of the method be well understood.

The present work is precisely an attempt to pave the way for such an understanding of the limits of applicability of the eikonal approximation. Although some of our results are also valid in more general situations,[5] we shall confine our attention to nonrelativistic scattering by a real, central potential, where "exact" solutions are readily obtained to check the accuracy of our statements.

We begin in Sec. II by an analysis of the basic formulas relevant to the eikonal approximation. Section III is devoted to a detailed comparison between the Born series and the eikonal multiple-scattering series obtained by expanding the eikonal scattering amplitude in powers of the interaction potential. We suggest that in the limit of high incident wave numbers a remarkable set of relationships hold between the corresponding terms of the eikonal and Born series *for all momentum transfers* and a large class of scattering potentials. These relations, together with the requirement of unitarity allow us to show that in the *weak coupling case* (i.e., when the Born series is rapidly convergent), the eikonal amplitude gives a consistently poorer approximation to the exact amplitude than does the second Born approximation (although the eikonal results are nevertheless fairly good at *all* angles). We also prove that as the coupling increases the eikonal method improves steadily. In Sec. IV we make a detailed numerical study of the accuracy of the eikonal approximation for *intermediate and strong* coupling cases and for a variety of interaction potentials. It is shown that in the intermediate coupling case, $|V_0|/E \lesssim 1$, where V_0 is a typical strength of the potential and E the incident particle energy, the agreement between the eikonal and the exact scattering amplitude is excellent *at all angles* for potentials of the Yukawa type. For strong couplings, $|V_0|/E > 1$, this agreement persists at *small angles* for all interactions studied. These results are of course quite unexpected in view of the conventional criteria of applicability of the eikonal approximation.[2] We also present in Sec. IV an analytical study of the scattering by a potential of the form $V(r) = \alpha/r^s$ $(s > 2)$, where an evaluation of the eikonal scattering amplitude by the method of stationary phases leads to a result for low-energy scattering which agrees with the semiclassical formula for small angles,[6] even though $|V_0|/E \gg 1$. The main results and conclusions of our work are summarized in Sec. V.

II. THE EIKONAL SCATTERING AMPLITUDE

Let us consider the nonrelativistic scattering of a spinless particle of mass m by a potential $V(\vec{r})$ of range a. We denote by $\vec{k}_i$ and $\vec{k}_f$ the initial and final wave vectors and by $U(\vec{r}) = 2m\,V(\vec{r})/\hbar^2$ the reduced potential. We shall also call V_0 a typical strength of the potential $V(\vec{r})$ while U_0 is the corresponding strength of the reduced potential. The energy of the particle which undergoes the scattering is given by $E = \hbar^2 k^2/2m$, where $k = |\vec{k}_i| = |\vec{k}_f|$ is its wave number. The stationary scattering wave function $\Psi^{(+)}_{\vec{k}_i}(\vec{r})$ which corresponds to an incident plane wave of momentum $\hbar\vec{k}_i$ and exhibits the behavior of an outgoing spherical wave then satisfies the Lippmann-Schwinger equation

$$\Psi^{(+)}_{\vec{k}_i}(\vec{r}) = \Phi_{\vec{k}_i}(\vec{r}) + \int G_0^{(+)}(\vec{r}, \vec{r}\,')\,U(\vec{r}\,') \times \Psi^{(+)}_{\vec{k}_i}(\vec{r}\,')\,d\vec{r}\,', \tag{2.1}$$

where

$$\Phi_{\vec{k}_i}(\vec{r}) = \langle \vec{r}\,|\vec{k}_i\rangle = (2\pi)^{-3/2}\,e^{i\vec{k}_i\cdot\vec{r}} \tag{2.2}$$

is the incident plane wave "normalized" in such a way that in momentum space

$$\langle \vec{k}_i'|\vec{k}_i\rangle = \delta(\vec{k}_i' - \vec{k}_i). \tag{2.3}$$

Furthermore, the free Green's function appearing in Eq. (2.1) is given by

$$G_0^{(+)}(\vec{r}, \vec{r}\,') = -(2\pi)^{-3} \lim_{\epsilon\to 0^+} \int \frac{e^{i\vec{K}\cdot(\vec{r}-\vec{r}\,')}}{K^2 - k^2 - i\epsilon}\,d\vec{K} \tag{2.4a}$$

$$= -\frac{1}{4\pi}\,\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \tag{2.4b}$$

and satisfies the equation

$$(\nabla_{\vec{r}}^2 + k^2)G_0^{(+)}(\vec{r}, \vec{r}\,') = \delta(\vec{r} - \vec{r}\,'). \tag{2.5}$$

From Eqs. (2.1) and (2.4a) one readily obtains the asymptotic behavior

$$\Psi^{(+)}_{\vec{k}_i}(\vec{r}) \underset{r\to\infty}{\sim} (2\pi)^{-3/2}\left(e^{i\vec{k}_i\cdot\vec{r}} + f\,\frac{e^{ikr}}{r}\right), \tag{2.6}$$

where the scattering amplitude is given by

$$f = -2\pi^2\,\langle \Phi_{\vec{k}_f}|\,U\,|\,\Psi^{(+)}_{\vec{k}_i}\rangle. \tag{2.7}$$

Here

$$\Phi_{\vec{k}_f}(\vec{r}) = (2\pi)^{-3/2}\,e^{i\vec{k}_f\cdot\vec{r}} \tag{2.8}$$

is a plane wave corresponding to the final wave vector $\vec{k}_f$; we shall denote by θ the scattering angle between $\vec{k}_i$ and $\vec{k}_f$.

Let us now assume that the de Broglie wavelength of the incident particle is small with respect to the range of the potential or, in other words, that

$$ka \gg 1. \tag{2.9}$$

It is then natural to factor out the free incident plane wave from the total wave function $\Psi^{(+)}_{\vec{k}_i}$ and to write[2]

$$\Psi^{(+)}_{\vec{k}_i}(\vec{r}) = (2\pi)^{-3/2}\,e^{i\vec{k}_i\cdot\vec{r}}\,\varphi(\vec{r}), \tag{2.10}$$

where $\varphi(\vec{r})$ is a slowly varying function when ka is large. Substituting the ansatz (2.10) in Eq. (2.1) we find that the function $\varphi(\vec{r})$ satisfies the equation

$$\varphi(\vec{r}) = 1 - (2\pi)^{-3} \int d\vec{R} \int d\vec{K} \frac{e^{i(\vec{K}-\vec{k}_i)\cdot\vec{R}}}{K^2 - k^2 - i\epsilon} \times U(\vec{r}-\vec{R})\,\varphi(\vec{r}-\vec{R}), \tag{2.11}$$

where we have set $\vec{R} = \vec{r} - \vec{r}\,'$. This equation is still exact. It is an easy matter, however, to obtain an approximate form of $\varphi(\vec{r})$ by using the fact that the product $U\varphi$ is slowly varying on the scale of the incident wavelength. To do so we concentrate first our attention on the free propagator $G_0^{(+)}$. Using its momentum space representation (2.4a) and introducing the new variable

$$\vec{p} = \vec{K} - \vec{k}_i ,$$

we have

$$G_0^{(+)}(\vec{R}) = -(2\pi)^{-3} e^{i\vec{k}_i\cdot\vec{R}} \int d\vec{p} \frac{e^{i\vec{p}\cdot\vec{R}}}{2\vec{k}_i\cdot\vec{p} + p^2 - i\epsilon}, \tag{2.12}$$

where the limiting process $\epsilon \to 0^+$ is always implied. Hence Eq. (2.11) becomes

$$\varphi(\vec{r}) = 1 - (2\pi)^{-3} I(\vec{r}), \tag{2.13}$$

where

$$I(\vec{r}) = \int d\vec{R} \int d\vec{p} \frac{e^{i\vec{p}\cdot\vec{R}}}{2\vec{k}_i\cdot\vec{p} + p^2 - i\epsilon} \times U(\vec{r}-\vec{R})\,\varphi(\vec{r}-\vec{R}). \tag{2.14}$$

Because the product $U\varphi$ is slowly varying, we expect this integral to be dominated by small values of p/k, a point to which we will return below. We may therefore expand[7,8] the quantity $(2\vec{k}_i\cdot\vec{p} + p^2 - i\epsilon)^{-1}$ in powers of p/k. Choosing the z axis (in $\vec{p}$ space) in the direction of the vector $\vec{k}_i$, we have

$$\frac{1}{2\vec{k}_i\cdot\vec{p} + p^2 - i\epsilon} = \frac{1}{2kp_z + p^2 - i\epsilon} = \frac{1}{2kp_z - i\epsilon}\left(1 - \frac{1}{2kp_z - i\epsilon}\,p^2 + \cdots\right). \tag{2.15}$$

Hence we can also develop

$$G_0^{(+)}(\vec{R}) = G_0^{(1)}(\vec{R}) + G_0^{(2)}(\vec{R}) + \cdots \tag{2.16}$$

and

$$I(\vec{r}) = I^{(1)}(\vec{r}) + I^{(2)}(\vec{r}) + \cdots, \tag{2.17}$$

where

$$G_0^{(1)}(\vec{R}) = -(2\pi)^{-3} e^{i\vec{k}_i\cdot\vec{R}} \int d\vec{p} \frac{e^{i\vec{p}\cdot\vec{R}}}{2kp_z - i\epsilon} \tag{2.18}$$

is the Green's function corresponding to a *linearized propagator*, obtained by omitting the term p^2 in the expression $(2\vec{k}_i\cdot\vec{p} + p^2 - i\epsilon)^{-1}$. The quantity $G_0^{(2)}(\vec{R})$ is given by

$$G_0^{(2)}(\vec{R}) = (2\pi)^{-3} e^{i\vec{k}_i\cdot\vec{R}} \int d\vec{p} \frac{e^{i\vec{p}\cdot\vec{R}}}{(2kp_z - i\epsilon)^2}\,p^2 \tag{2.19}$$

while the expressions of $I^{(1)}(\vec{r})$ and $I^{(2)}(\vec{r})$ are given by

$$I^{(1)}(\vec{r}) = \int d\vec{R}\, d\vec{p} \frac{e^{i\vec{p}\cdot\vec{R}}}{2kp_z - i\epsilon} U(\vec{r}-\vec{R})\varphi(\vec{r}-\vec{R}) \tag{2.20}$$

and

$$I^{(2)}(\vec{r}) = \int d\vec{R}\, d\vec{p} \frac{e^{i\vec{p}\cdot\vec{R}}}{(2kp_z - i\epsilon)^2} p^2 U(\vec{r}-\vec{R})\varphi(\vec{r}-\vec{R}). \tag{2.21}$$

It should be noted that the Green's functions $G_0^{(1)}$ and $G_0^{(2)}$ are singular functions which always act upon "smooth" functions [see, for example, Eq. (2.14)] in order to be well defined. It is in this sense that $G_0^{(2)}$ may be shown to be "small" compared to $G_0^{(1)}$ when $ka \gg 1$. In fact, it is clear by looking at Eqs. (2.13)–(2.21) that the contribution of $G_0^{(2)}$ in evaluating $I(\vec{r})$ will be of order U_0/k^2, where U_0 is a typical reduced potential strength and we confine our attention to distances of the order of magnitude of the range of the potential. [Clearly, if we compute the scattering amplitude according to the integral representation (2.7) only such distances are important.] From Eqs. (2.13)–(2.21), we see that it is not enough that the contribution of $G_0^{(2)}$ be small compared to that of $G_0^{(1)}$; it must be small compared to unity. Thus we require $U_0/k^2 \ll 1$ for the validity of this method.

It is now a simple matter to show that the linearized propagator (2.18) directly leads to the eikonal scattering wave function. Indeed, performing the $\vec{p}$ integral in cartesian coordinates and returning to the original variables $\vec{r}(x, y, z)$ and $\vec{r}\,'(x', y', z')$ one readily obtains

$$G_0^{(1)}(\vec{r}, \vec{r}\,') = -\frac{i}{2k} e^{ik_i(z-z')} \times \delta(x-x')\delta(y-y')\Theta(z-z'), \tag{2.22}$$

where

$$\Theta(z-z') = \begin{cases} 1, & z > z' \\ 0, & z < z'. \end{cases} \tag{2.23}$$

Then

$$I^{(1)}(x, y, z) = (2\pi)^3 \frac{i}{2k} \int_0^\infty dZ\, U(x, y, z-Z) \times \varphi(x, y, z-Z). \tag{2.24}$$

Hence the function $\varphi^{(1)}(\vec{r})$, obtained by retaining only the term $I^{(1)}$ on the right of Eq. (2.13), satisfies the equation

$$\varphi^{(1)}(x, y, z) = 1 - \frac{i}{2k} \int_0^\infty dZ\, U(x, y, z - Z) \times \varphi^{(1)}(x, y, z - Z) \quad (2.25)$$

so that

$$\varphi^{(1)}(x, y, z) = \exp\left[- \frac{i}{2k} \int_{-\infty}^{z} U(x, y, z')dz' \right]. \quad (2.26)$$

Notice that $\varphi^{(1)}$ varies negligibly over distances of order k^{-1} since U_0/k^2 is small. Thus for distances which are large compared with k^{-1} (but not necessarily as large as a) the product $U\varphi^{(1)}$ varies slowly. Hence the important values of p in the Fourier transform of $U\varphi$ appearing in Eq. (2.14) are small compared to k (but not necessarily of order a^{-1}). Thus the expansion (2.15) of the propagator in powers of p/k is justified.

Let us now return to the original ansatz (2.10). Using Eq. (2.26), we deduce that the approximate scattering wave function $\Psi_E(\vec{r})$ which we have obtained, namely the *eikonal wave function*, is such that

$$\Psi_E(\vec{r}) = (2\pi)^{-3/2} \exp\left[i\vec{k}_i\cdot\vec{r} - \frac{i}{2k} \int_{-\infty}^{z} U(x, y, z')\, dz' \right] \quad (2.27a)$$

or, in terms of the potential $V(\vec{r})$,

$$\Psi_E(\vec{r}) = (2\pi)^{-3/2} \exp\left[i\vec{k}_i\cdot\vec{r} - \frac{i}{\hbar v_i} \int_{-\infty}^{z} V(x, y, z')\, dz' \right], \quad (2.27b)$$

where $\vec{v}_i = \hbar\vec{k}_i/m$ is the incident velocity and the integral is evaluated along a straight line parallel to $\vec{k}_i$.

The eikonal scattering amplitude may now be obtained in the usual way[2] by using the integral representation (2.7) together with the eikonal scattering wave function (2.27). Thus

$$f_E(\vec{\Delta}) = - \frac{1}{4\pi} \int d\vec{r}\, e^{i\vec{\Delta}\cdot\vec{r}}\, U(\vec{r}) \times \exp\left[- \frac{i}{2k} \int_{-\infty}^{z} U(\vec{b}, z')\, dz' \right], \quad (2.28)$$

where

$$\vec{\Delta} = \vec{k}_i - \vec{k}_f \quad (2.29)$$

is the momentum-transfer vector, of length $\Delta = 2k\sin(\frac{1}{2}\theta)$. Since the actual phase of the scattering wave function should be evaluated in the semiclassical limit along a curved trajectory, it is not unreasonable to expect that an improvement on Eq. (2.28) may be achieved by performing the z integration in the phase along a direction perpendicular to the momentum transfer.[2] We shall discuss the importance of this modification in Sec. III. We then obtain the well-known result

$$f_E = \frac{k}{2\pi i} \int d^2b\, e^{i\vec{\Delta}\cdot\vec{b}} (e^{i\chi(\vec{b})} - 1), \quad (2.30)$$

where we work in a cylindrical coordinate system such that

$$\vec{r} = \vec{b} + z\hat{n} \quad (2.31)$$

and $\hat{n}$ is perpendicular to $\vec{\Delta}$. Furthermore, the eikonal phase-shift function is given by

$$\chi(\vec{b}) = - \frac{1}{2k} \int_{-\infty}^{+\infty} U(\vec{b}, z)\, dz. \quad (2.32)$$

For potentials which possess azimuthal symmetry Eq. (2.30) simply reduces to the Fourier-Bessel transform

$$f_E(\Delta) = \frac{k}{i} \int_0^\infty db\, bJ_0(\Delta b)(e^{i\chi(b)} - 1) \quad (2.33)$$

which is the form we shall use in subsequent sections.

Before leaving this section let us comment briefly on the angular range of validity of the eikonal approximation. At first sight, it might seem that the angular range of validity of Eq. (2.28) is unrestricted, since we have made no mention of angular restrictions in the derivations of this section. However, if the reader will look back at the discussion following Eq. (2.21) he will see that we have neglected quantities of order U_0/k^2 coming from higher terms in the expansion of the propagator. What we are doing here is neglecting a function of $\vec{r}$ because it is small compared to another function of $\vec{r}$. It must be remembered, however, that even though $f_1(\vec{r}) < f_2(\vec{r})$ for all $\vec{r}$, it does not follow that $\hat{f}_1(\vec{\Delta}) < \hat{f}_2(\vec{\Delta})$ when $|\vec{\Delta}|$ becomes large. (Here we use $\hat{f}$ to denote the Fourier transform of f.) This is precisely the situation which we have to deal with here: The functions involved in the solution of Eq. (2.11) must be multiplied by $U(\vec{r})$ and Fourier-transformed with respect to the transform variable $\vec{\Delta} = \vec{k}_i - \vec{k}_f$ in order to obtain the scattering amplitude which is the quantity of physical interest. For example, a function of relative order U_0/k^2 might have a Fourier transform of relative order $U_0\Delta^2/k^2$ which is not necessarily negligible when $\Delta \simeq k$ (i.e., for scattering angles of the order of 60° or greater). Thus, although the requirement that $\vec{\Delta}$ be nearly perpendicular to $\vec{k}_i$ suggests an angular validity criterion of the form[2] $\theta \lesssim (ka)^{-1/2}$; the situation is in fact considerably more complicated than simple kinematical arguments of this sort would suggest. It is to this question that we now turn our attention.

III. THE EIKONAL MULTIPLE SCATTERING EXPANSION AND THE BORN SERIES

Let us return to the eikonal scattering amplitude (2.33) and define the *eikonal multiple scattering expansion*

$$f_E=\sum_{n=1}^{\infty}\bar{f}_{En}, \tag{3.1}$$

where

$$\bar{f}_{En}=-\frac{ik}{n!}\,i^n\int_0^\infty J_0(\Delta b)\,[\chi(b)]^n\,b\,db. \tag{3.2}$$

It is worth noting that the quantities $\bar{f}_{En}$ are alternatively real and imaginary. The exact scattering amplitude $f_{\rm ex}$ has a similar expansion in powers of the interaction potential, namely the *Born series*

$$f_{\rm ex}=\sum_{n=1}^{\infty}\bar{f}_{Bn}, \tag{3.3}$$

where

$$\bar{f}_{Bn}=-2\pi^2\,\langle\Phi_{\vec{k}_f}|\,UG_0^{(+)}U\cdots G_0^{(+)}U|\,\Phi_{\vec{k}_i}\rangle\,. \tag{3.4}$$

In this last expression the potential U appears n times and the free Green's function $G_0^{(+)}$, $(n-1)$ times. We shall also define f_{En} and f_{Bn}, respectively, as the sum of the first n terms of Eqs. (3.1) and (3.3). Thus

$$f_{En}=\sum_{j=1}^{n}\bar{f}_{Ej} \tag{3.5}$$

and

$$f_{Bn}=\sum_{j=1}^{n}\bar{f}_{Bj}\,. \tag{3.6}$$

We now investigate the relationships between the quantities $\bar{f}_{En}$ and $\bar{f}_{Bn}$ when $ka\gg1$. First of all, we recall that

$$f_{E1}=f_{B1} \tag{3.7}$$

for all momentum transfers.[2] It is important to remark that this result obtains for all angles only when the z axis of the coordinate system in position space is chosen to be perpendicular to the momentum transfer $\vec{\Delta}$, so that the vector $\vec{\Delta}$, of length $\Delta=2k\sin(\frac{1}{2}\theta)$ entirely lies in the plane of impact parameters $\vec{b}$. In what follows we shall always adopt this choice of coordinate system.

Let us now consider the second terms $\bar{f}_{E2}$ and $\bar{f}_{B2}$ of the series (3.1) and (3.3), respectively. Since $\mathrm{Re}\bar{f}_{E2}\equiv0$ while in general $\mathrm{Re}\bar{f}_{B2}\neq0$, there is no analog of Eq. (3.7) for $\mathrm{Re}\bar{f}_{B2}$ and $\mathrm{Re}\bar{f}_{E2}$. We shall return shortly to this point while discussing the relative merits of the second Born and eikonal approximations. For the moment we concentrate our attention on $\mathrm{Im}\bar{f}_{B2}$ and $\mathrm{Im}\bar{f}_{E2}$. For a simple Yukawa potential of the form

$$U(r)=U_0\,\frac{e^{-r/a}}{r} \tag{3.8}$$

it is well known that the second term $\bar{f}_{B2}$ of the Born series is given by the expression[9]

$$\begin{aligned}\bar{f}_{B2}(k,\Delta)=&\;\frac{U_0^{\,2}}{2\Delta(a^{-4}+4k^2a^{-2}+k^2\Delta^2)^{1/2}}\\&\times\left[2\tan^{-1}\frac{\Delta a^{-1}}{2(a^{-4}+4k^2a^{-2}+k^2\Delta^2)^{1/2}}\right.\\&\left.+i\ln\left(\frac{(a^{-4}+4k^2a^{-2}+k^2\Delta^2)^{1/2}+k\Delta}{(a^{-4}+4k^2a^{-2}+k^2\Delta^2)^{1/2}-k\Delta}\right)\right].\end{aligned} \tag{3.9}$$

On the other hand, for the same interaction potentail (3.8) the eikonal phase-shift function is given by

$$\chi(b,k)=-\frac{U_0}{k}\,K_0(b/a), \tag{3.10}$$

where K_0 is a modified Bessel function of order zero. Substituting in Eq. (3.2) and evaluating the resulting integral when $n=2$, we get[10]

$$\bar{f}_{E2}=i\,\frac{U_0^2}{2\Delta(4k^2a^{-2}+k^2\Delta^2)}\times\ln\frac{(4k^2a^{-2}+k^2\Delta^2)^{1/2}+k\Delta}{(4k^2a^{-2}+k^2\Delta^2)^{1/2}-k\Delta}\,. \tag{3.11}$$

Let us now return to Eq. (3.9). For large k we see that

$$\mathrm{Im}\bar{f}_{B2}(k,\Delta)=A_{B2}(\Delta)/k\,, \tag{3.12a}$$

where $A_{B2}(\Delta)$ depends only on Δ, and terms of higher order in k^{-1} have been neglected. Moreover, from Eq. (3.11) we note that we can write

$$\mathrm{Im}\bar{f}_{E2}=A_{E2}(\Delta)/k\,, \tag{3.12b}$$

and comparing Eqs. (3.9) and (3.11) we find

$$A_{E2}(\Delta)=A_{B2}(\Delta) \tag{3.13a}$$

for all momentum transfers. Thus, when k is sufficiently large so that Eq. (3.12a) holds, we have for all momentum transfers

$$\mathrm{Im}\bar{f}_{E2}(k,\Delta)=\mathrm{Im}\bar{f}_{B2}(k,\Delta)\,. \tag{3.13b}$$

Furthermore, we show in Appendix A that Eq. (3.13a) holds for an arbitrary superposition of Yukawa potentials, namely,

$$U(r)=U_0\int_{\alpha_0>0}^{\infty}\rho(\alpha)\,\frac{e^{-\alpha r}}{r}\,d\alpha, \tag{3.14}$$

where $\rho(\alpha)$ is a weight function.

TABLE I. Comparison of $\mathrm{Im}\bar{f}_{B2}$ and $\mathrm{Im}\bar{f}_{E2}$ for a Yukawa potential $U(r) = -e^{-r}/r$ and for various values of the scattering angle θ and the wave number k. The numbers in parentheses indicate powers of 10.

k	θ (degrees)	$\mathrm{Im}\bar{f}_{B2}$	$\mathrm{Im}\bar{f}_{E2}$
1	0	2.00(−1)	2.50(−1)
	90	1.59(−1)	1.90(−1)
	180	1.34(−1)	1.56(−1)
2	0	1.18(−1)	1.25(−1)
	90	5.66(−2)	5.85(−2)
	180	3.94(−2)	4.04(−2)
3	0	8.11(−2)	8.33(−2)
	90	2.48(−2)	2.51(−2)
	180	1.58(−2)	1.60(−2)
5	0	4.95(−2)	5.00(−2)
	90	7.58(−3)	7.60(−3)
	180	4.52(−3)	4.54(−3)
10	0	2.49(−2)	2.50(−2)
	90	1.31(−3)	1.31(−3)
	180	7.46(−4)	7.46(−4)

TABLE II. Comparison of $\mathrm{Im}\bar{f}_{B2}$ and $\mathrm{Im}\bar{f}_{E2}$ for a superposition of two Yukawa potentials $U(r) = -(e^{-r} - 1.125e^{-2r})/r$ and for various values of the scattering angle θ and the wave number k.

k	θ (degrees)	$\mathrm{Im}\bar{f}_{B2}$	$\mathrm{Im}\bar{f}_{E2}$
1	0	6.77(−2)	6.92(−2)
	90	4.10(−2)	3.63(−2)
	180	2.68(−2)	2.15(−2)
2	0	3.46(−2)	3.46(−2)
	90	4.86(−3)	4.56(−3)
	180	1.07(−3)	1.09(−3)
3	0	2.30(−2)	2.31(−2)
	90	4.73(−4)	5.10(−4)
	180	−1.67(−4)	−1.13(−4)
5	0	1.38(−2)	1.38(−2)
	90	−1.15(−4)	−1.04(−4)
	180	−9.29(−5)	−8.65(−5)
10	0	6.92(−3)	6.92(−3)
	90	−2.47(−5)	−2.43(−5)
	180	−1.22(−5)	−1.20(−5)

Since the relation (3.13b) deals with the region of large k, it is interesting to ask when this asymptotic behavior actually sets in. Comparing Eqs. (3.9) and (3.11), we see that ka does not have to be much greater than one for the asymptotic relation (3.13b) to hold with fair accuracy in the case of a single Yukawa potential. This is illustrated in Table I, where $\mathrm{Im}\bar{f}_{B2}$ and $\mathrm{Im}\bar{f}_{E2}$ are compared for a potential of the form (3.8) with[11,12] $U_0 = -1$, $a = 1$ and for various values of the scattering angle θ and the wave number k. Note that the agreement between $\mathrm{Im}\bar{f}_{B2}$ and $\mathrm{Im}\bar{f}_{E2}$ is always poorest in the forward direction, a fact which is easily understood by a detailed examination of Eqs. (3.9) and (3.11). A similar comparison is made in Table II for a superposition of two Yukawa potentials of different ranges, namely,

$$U(r) = U_0(e^{-r} - 1.125\, e^{-2r})/r, \tag{3.15}$$

where we have chosen $U_0 = -1$. Interactions of the form (3.15) give a nontrivial structure in the differential cross section, even in first Born approximation and can be used to reproduce some of the features of strong interaction forces.[13] We note from Table II that the asymptotic behavior sets in more slowly with increasing wave number for the potential of Eq. (3.15) than for the simple Yukawa potential (3.8). Moreover, the angle at which the agreement is poorest now depends on the wave number.

Restricting ourselves to Yukawa-type potentials, we now investigate the relationship between $\mathrm{Re}\,\bar{f}_{B3}$ and $\bar{f}_{E3}$. Since the analytic evaluation of higher-order terms of the Born and eikonal series is extremely difficult, we proceed as follows. We first obtain $\bar{f}_{E3}$, which is purely real, by a numerical evaluation of Eq. (3.2) for $n = 3$. Then to find $\mathrm{Re}\,\bar{f}_{B3}$, we first evaluate the "exact" scattering amplitude f_{ex} by using the partial-wave method and integrating the relevant radial Schrödinger equations by means of the Numerov method. We then substract f_{B2} from f_{ex} to obtain the desired value of $\bar{f}_{B3}$. This, evidently, is an approximate procedure since

$$f_{\mathrm{ex}} - f_{B2} = \bar{f}_{B3} + \bar{f}_{B4} + \bar{f}_{B5} + \cdots, \tag{3.16}$$

but it is quite accurate for *weak coupling*, in which case the correction terms on the right of Eq. (3.16) are very small with respect to $\bar{f}_{B3}$. Table III shows the comparison of $\mathrm{Re}\,\bar{f}_{B3}$ with $\bar{f}_{E3}$ for a simple Yukawa potential (3.8) of unit range $a = 1$ and "strength" $U_0 = -1$ and an incident particle of

TABLE III. Comparison of $\mathrm{Re}\bar{f}_{B3}$ with $\bar{f}_{E3}$ for a Yukawa potential $U(r) = -e^{-r}/r$ and an incident wave number $k = 5$.

θ (degrees)	$\mathrm{Re}\bar{f}_{B3}$	$\bar{f}_{E3}$
0	−3.84(−3)	−3.91(−3)
30	−3.01(−3)	−3.02(−3)
60	−2.05(−3)	−2.04(−3)
90	−1.52(−3)	−1.51(−3)
120	−1.24(−3)	−1.23(−3)
150	−1.10(−3)	−1.09(−3)
180	−1.06(−3)	−1.05(−3)

TABLE IV. Comparison of $\mathrm{Re}\bar{f}_{B3}$ with $\bar{f}_{E3}$ for a superposition of two Yukawa potentials $U(r)=-(e^{-r}-1.125e^{-2r})/r$ and an incident wave number $k=5$.

θ (degrees)	$\mathrm{Re}\bar{f}_{B3}$	$\bar{f}_{E3}$
0	−2.79(−4)	−2.87(−4)
30	−9.41(−5)	−9.31(−5)
60	−4.52(−6)	−6.52(−6)
90	5.78(−6)	4.16(−6)
120	5.67(−6)	4.67(−6)
150	4.95(−6)	4.27(−6)
180	4.84(−6)	4.10(−6)

TABLE V. Comparison of $\mathrm{Im}\bar{f}_{B4}$ with $\mathrm{Im}\bar{f}_{E4}$ for a Yukawa potential $U(r)=-e^{-r}/r$ and an incident wave number $k=5$.

θ (degrees)	$\mathrm{Im}\bar{f}_{B4}$	$\mathrm{Im}\bar{f}_{E4}$
0	−3.13(−4)	−3.51(−4)
30	−2.85(−4)	−3.16(−4)
60	−2.37(−4)	−2.58(−4)
90	−2.00(−4)	−2.16(−4)
120	−1.76(−4)	−1.89(−4)
150	−1.63(−4)	−1.75(−4)
180	−1.59(−4)	−1.70(−4)

wave number $k=5$ (measured in units of the inverse range). The agreement between $\mathrm{Re}\,\bar{f}_{B3}$ and $\bar{f}_{E3}$ is seen to be excellent for all values of the momentum transfer. A similar comparison is made in Table IV for the potential (3.15). The agreement is still good but not as good as in Table III. This is not surprising in light of the comparison made between Tables I and II.

A similar procedure can be used to compare $\mathrm{Im}\bar{f}_{B4}$ with $\mathrm{Im}\bar{f}_{E4}$. In this case, subtracting $\mathrm{Im}\bar{f}_{B2}$ from $\mathrm{Im}f_{\mathrm{ex}}$ gives two terms of similar order in k^{-1}, namely $\mathrm{Im}\bar{f}_{B3}$ (which is not contained in the eikonal multiple scattering series) as well as $\mathrm{Im}\bar{f}_{B4}$. However, since the former is proportional to U_0^3 and the latter to U_0^4, we may use the subtraction procedure for two values of U_0 in order to obtain both $\mathrm{Im}\bar{f}_{B3}$ and $\mathrm{Im}\bar{f}_{B4}$. Thus we write

$$\mathrm{Im}f_{\mathrm{ex}}-\mathrm{Im}f_{B2}=\mathrm{Im}\bar{f}_{B3}+\mathrm{Im}\bar{f}_{B4}+\cdots$$
$$=U_0^3g_3+U_0^4g_4+\cdots \tag{3.17}$$

and determine g_3 and g_4 (and hence $\mathrm{Im}\bar{f}_{B3}$ and $\mathrm{Im}\bar{f}_{B4}$) by using Eq. (3.17) for two different values of U_0 (for example $U_0=-1$ and $U_0=+1$). Table V shows the comparison of $\mathrm{Im}\bar{f}_{B4}$ and $\mathrm{Im}\bar{f}_{E4}$ for a Yukawa potential of the form (3.8) with a strength parameter $U_0=-1$ and an incident wave number $k=5$. As in the case of $\mathrm{Re}\bar{f}_{B3}$ and $\bar{f}_{E3}$, the agreement between $\mathrm{Im}\bar{f}_{B4}$ and $\mathrm{Im}\bar{f}_{E4}$ is excellent at *all* momentum transfers. This strongly suggests that if we write

$$\bar{f}_{Bn}(k,\Delta)\underset{k\to\infty}{\sim}A_{Bn}(\Delta)/k^{n-1}+O(k^{-n}), \tag{3.18a}$$

then, defining $A_{En}(\Delta)$ by

$$\bar{f}_{En}(k,\Delta)=A_{En}(\Delta)/k^{n-1}, \tag{3.18b}$$

we have

$$A_{Bn}(\Delta)=A_{En}(\Delta) \tag{3.18c}$$

for all n and *all values of the momentum transfer*, for potentials of the form (3.14). We note in this connection that for a Yukawa potential (and by a straightforward extension for a superposition of Yukawa potentials) Moore[14] has in effect proved that the first term in $A_{Bn}(\Delta)$, which is proportional to $\ln^{n-1}\Delta/\Delta^2$, is equal to the first term in $A_{En}(\Delta)$. However, for Yukawa type potentials $A_{Bn}(\Delta)$ and $A_{En}(\Delta)$ are linear combinations of terms of the form $\ln^m\Delta/\Delta^2$ $(0\le m\le n-1)$, so a general proof of Eq. (3.18c) is not trivial.

Let us now examine some of the consequences of the relations (3.18). We first consider the *weak coupling* case which we define by the inequalities

$$\frac{|V_0|a}{\hbar v}=\frac{|U_0|a}{2k}\ll 1, \tag{3.19}$$

where $v=\hbar k/m$ is the particle velocity and

$$\frac{|V_0|}{E}=\frac{|U_0|}{k^2}\ll 1. \tag{3.20}$$

In this case the Born series converges and Eq. (3.3) may be rewritten for $ka\gg 1$ as[14]

$$f_{\mathrm{ex}}(k,\Delta)=f_{B1}(\Delta)+\left[\frac{A(\Delta)}{k^2}+i\,\frac{B(\Delta)}{k}\right]+\left[\frac{C(\Delta)}{k^2}+i\,\frac{D(\Delta)}{k^3}\right]+\cdots$$
$$=f_{B1}(\Delta)+\bar{f}_{B2}+\bar{f}_{B3}+\cdots, \tag{3.21}$$

where we denote the expressions in square brackets by $\bar{f}_{B2}$ and $\bar{f}_{B3}$, respectively. We note that the quantities A and B coming from $\bar{f}_{B2}$ are proportional to U_0^2, while C and D, arising from $\bar{f}_{B3}$, are proportional to U_0^3. The k dependence of the various terms has been exposed and is easily checked by requiring the scattering amplitude (3.21) to satisfy the optical theorem order by order in powers of U_0 and k^{-1}. Now using the relations (3.18), the corresponding eikonal amplitude is given by

$$f_E(k,\Delta)=f_{B1}(\Delta)+i\,\frac{B(\Delta)}{k}+\frac{C(\Delta)}{k^2}+\cdots$$
$$=f_{B1}(\Delta)+\bar{f}_{E2}+\bar{f}_{E3}+\cdots. \tag{3.22}$$

Thus we see that the eikonal amplitude selects in each term of the Born series the dominant contri-

TABLE VI. The real part of the scattering amplitude for a superposition of two Yukawa potentials $U(r) = -(e^{-r} - 1.125e^{-2r})/r$ and an incident wave number $k = 5$.

θ (degrees)	f_{B1}	$\bar{f}_{B2}$	f_{B2}	f_E	$f_E + \bar{f}_{B2}$	f_{ex}
0	7.188(−1)	6.633(−4)	7.194(−1)	7.185(−1)	7.191(−1)	7.191(−1)
30	2.474(−2)	−2.061(−4)	2.453(−2)	2.465(−2)	2.444(−2)	2.444(−2)
60	−3.316(−4)	−6.606(−5)	−3.976(−4)	−3.381(−4)	−4.041(−4)	−4.021(−4)
90	−1.225(−3)	−9.408(−6)	−1.235(−3)	−1.221(−3)	−1.231(−3)	−1.229(−3)
120	−1.083(−3)	4.184(−6)	−1.078(−3)	−1.078(−3)	−1.074(−3)	−1.073(−3)
150	−9.577(−4)	7.580(−6)	−9.501(−4)	−9.534(−4)	−9.459(−4)	−9.452(−4)
180	−9.163(−4)	8.253(−6)	−9.081(−4)	−9.122(−4)	−9.039(−4)	−9.032(−4)

bution (to order k^{-1}) which is alternatively real and imaginary. It should be noted that in this way the eikonal amplitude (3.22) satisfies the optical theorem to each order in powers of U_0 and k^{-1} in a very particular way.

Let us compare in more detail Eqs. (3.21) and (3.22). We see that neither f_{B2} nor f_E are correct to order k^{-2}. Indeed, f_{B2} lacks the real term $C(\Delta)/k^2$ while the real term $A(\Delta)/k^2$ is missing in f_E. Now, since $A(\Delta)$ is proportional to U_0^2, while $C(\Delta)$ is proportional to U_0^3, it is clear that when U_0 is sufficiently small the second Born amplitude f_{B2} is actually more accurate than f_E. This is clearly illustrated in Table VI where f_{B1}, Re $\bar{f}_{B2}$, Re f_{B2}, Re f_E, and Re f_{ex} are shown for a "double Yukawa" potential (3.15) with $U_0 = -1$ and an incident wave number $k = 5$. We have also displayed in Table VI the quantity Re $(f_E + \bar{f}_{B2})$ which adds to the eikonal amplitude (3.22) the important missing term Re $\bar{f}_{B2} \simeq A(\Delta)/k^2$. The improvement of the eikonal amplitude due to this addition is seen to be spectacular. Because of Eq. (3.13b) no such situation arises for the imaginary part of the amplitude which is given in Table VII. Indeed, since Im$f_E \simeq$ Imf_{B2} because the coupling is weak, we should expect that Im$f_E \simeq$ Im$f_{B2} \simeq$ Imf_{ex}, which is seen to be the case.

The comments which we have just made about the real part of the scattering amplitude apply evidently also to the calculation of the differential cross section, since the terms involving $A(\Delta)$, $B(\Delta)$, and $C(\Delta)$ in Eq. (3.21) contribute equally in correcting the first Born differential cross section to order k^{-2}. Hence, because only $B(\Delta)$ and $C(\Delta)$ appear in the eikonal approximation, the differential cross section is given more accurately by the second Born approximation when U_0 is sufficiently small.

As a further illustration of these remarks, we present in Table VIII the results of similar calculations for the real part of the amplitude corresponding to a "double Yukawa" potential (3.15) when $U_0 = -3$ and $k = 5$. Since the coupling is now somewhat larger we expect the term $C(\Delta)/k^2$ in Eq. (3.22) to become more important. Indeed, we see from Table VIII that neither the second Born approximation—which lacks the term $C(\Delta)/k^2$—nor the eikonal method—which misses the term $A(\Delta)/k^2$—offers a significant improvement over the first Born result for the real part of the scattering amplitude. Nevertheless, by comparing the results of Tables VI and VIII we see that the eikonal method has already improved significantly (at all angles) since it includes the term $C(\Delta)/k^2$ whose importance increases with the coupling. Moreover, if we add Re $\bar{f}_{B2}$ to Re f_E, as we did in Table VI, we see from Table VIII that again a major improvement in the real part of the amplitude is obtained, as we expect from the foregoing discussion.

Before we conclude this section, we would like to emphasize that the theorem (3.13a) and the conjecture (3.18c) should only be applied to interactions having the form of an arbitrary superposition of Yukawa potentials of the form (3.14). In fact, we show in Appendix B that the relation (3.13a) holds only for *small momentum transfers* in the case of a Gaussian potential and of a polarization potential of the form

$$U(r) = \frac{U_0}{(r^2 + d^2)^2} . \tag{3.23}$$

TABLE VII. The imaginary part of the scattering amplitude for a superposition of two Yukawa potentials $U(r) = -(e^{-r} - 1.125e^{-2r})/r$ and an incident wave number $k = 5$.

θ (degrees)	f_{B2}	f_E	f_{ex}
0	1.383(−2)	1.383(−2)	1.384(−2)
30	2.380(−3)	2.363(−3)	2.364(−3)
60	4.532(−5)	5.812(−5)	3.796(−5)
90	−1.153(−4)	−1.040(−4)	−1.161(−4)
120	−1.093(−4)	−1.008(−4)	−1.085(−4)
150	−9.719(−5)	−9.019(−5)	−9.592(−5)
180	−9.288(−5)	−8.632(−5)	−9.155(−5)

TABLE VIII. The real part of the scattering amplitude for a superposition of two Yukawa potentials $U(r) = -3(e^{-r} - 1.125e^{-2r})/r$ and an incident wave number $k = 5$.

θ (degrees)	f_{B1}	$\bar{f}_{B2}$	f_{B2}	f_E	$f_E + \bar{f}_{B2}$	f_{ex}
0	2.156(0)	5.970(−3)	2.162(0)	2.149(0)	2.154(0)	2.154(0)
30	7.422(−2)	−1.854(−3)	7.236(−2)	7.172(−2)	6.986(−2)	6.986(−2)
60	−9.947(−4)	−5.945(−4)	−1.589(−3)	−1.169(−3)	−1.763(−3)	−1.689(−3)
90	−3.676(−3)	−8.467(−5)	−3.761(−3)	−3.565(−3)	−3.649(−3)	−3.604(−3)
120	−3.248(−3)	3.766(−5)	−3.210(−3)	−3.123(−3)	−3.085(−3)	−3.062(−3)
150	−2.873(−3)	6.822(−5)	−2.805(−3)	−2.758(−3)	−2.690(−3)	−2.677(−3)
180	−2.749(−3)	7.428(−5)	−2.675(−3)	−2.639(−3)	−2.565(−3)	−2.553(−3)

IV. THE INTERMEDIATE AND STRONG COUPLING CASES

We now turn to the case of "intermediate" coupling which we define by still requiring that

$$ka \gg 1 \tag{4.1}$$

together with the inequality

$$\frac{|V_0|}{E} < 1. \tag{4.2}$$

However, we relax the condition (3.19) so that we now have

$$\frac{|U_0|a}{2k} \simeq 1. \tag{4.3}$$

We show in Figs. 1–8 the real and imaginary parts of the eikonal scattering amplitudes corresponding to the following interactions:

(a) a simple Yukawa potential of the form (3.8) with $U_0 = -5$ and $a = 1$,

(b) the same interaction with $U_0 = -10$,

(c) a "double Yukawa" potential of the form (3.15) with $U_0 = -20$, and

(d) a polarization potential of the form given by Eq. (3.23) with $U_0 = -10$ and $d = 1$.

We have performed all the calculations leading to Figs. 1–8 for the value $k = 5$. For comparison with the eikonal results, we have also plotted the

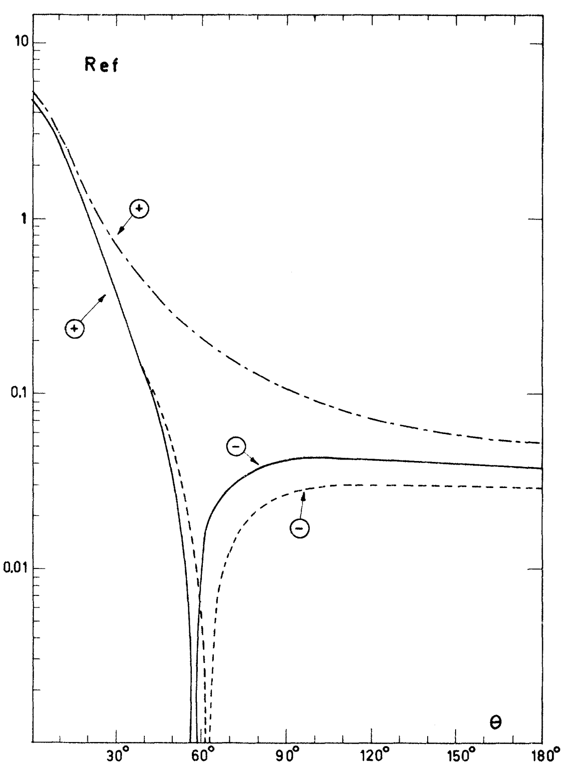

FIG. 1. The real part of the scattering amplitude for a Yukawa potential of the form given in Eq. (3.8) with $U_0 = -5$, $a = 1$, and $k = 5$. The solid curve shows the exact result, the dashed curve gives the eikonal result, and the dash-dotted curve is the second Born approximation.

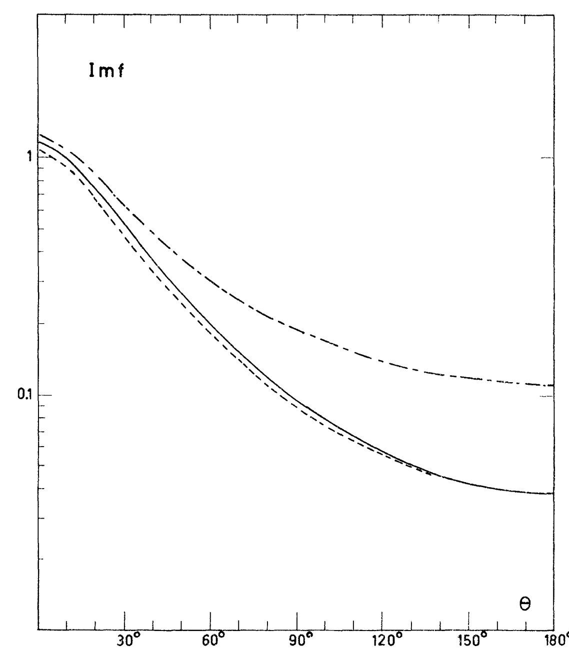

FIG. 2. Same as Fig. 1 except that the imaginary part of the amplitude is shown.

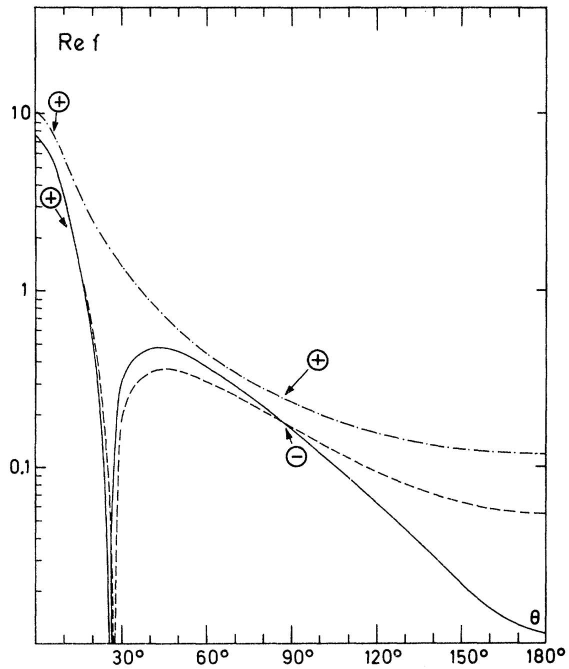

FIG. 3. The real part of the scattering amplitude for a Yukawa potential of the form given in Eq. (3.8) with $U_0 = -10$, $a = 1$, and $k = 5$. The solid curve shows the exact result, the dashed curve gives the eikonal result, and the dash-dotted curve is the second Born approximation.

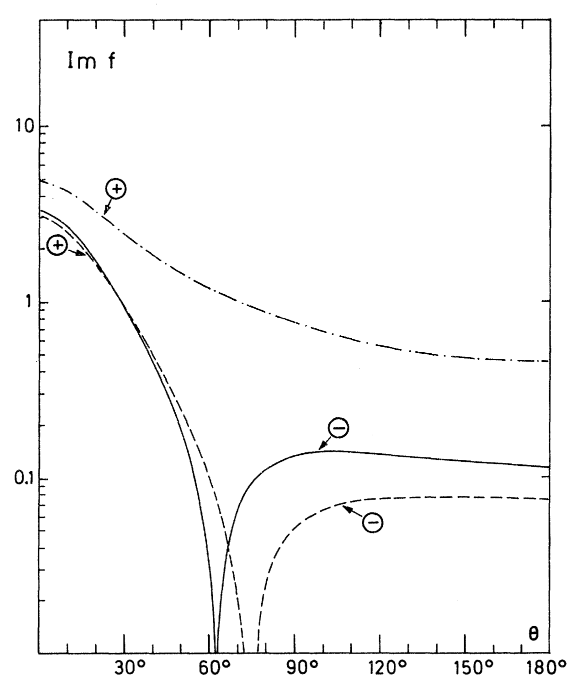

FIG. 4. Same as Fig. 3, except that the imaginary part of the amplitude is shown.

exact results. Furthermore, we display in Figs. 1–6 the values of the amplitudes given by the second Born approximation and in Fig. 7 that given by the first Born approximation. We note that for all Yukawa-type potentials *the eikonal results have the correct sign and reproduce very well the structure of the exact amplitude.* We also see that the first and second Born approximations both do very poorly in the case of the single and double Yukawa potentials (see Figs. 1–6). For example, if we analyze in detail Fig. 5, which shows the real part of the scattering amplitude for a double Yukawa potential, we see that the eikonal result follows very well the exact amplitude, with only a shift of the second zero spoiling the agreement somewhat. We emphasize that even at $\theta = 180°$ the agreement between the two curves is striking. The first Born approximation does not agree with the exact results except at very small angles. It exhibits only one zero and has the wrong sign at large angles. The second Born approximation offers no worthwhile improvement. An examination of Fig. 6, which gives the imaginary part of the scattering ampli-

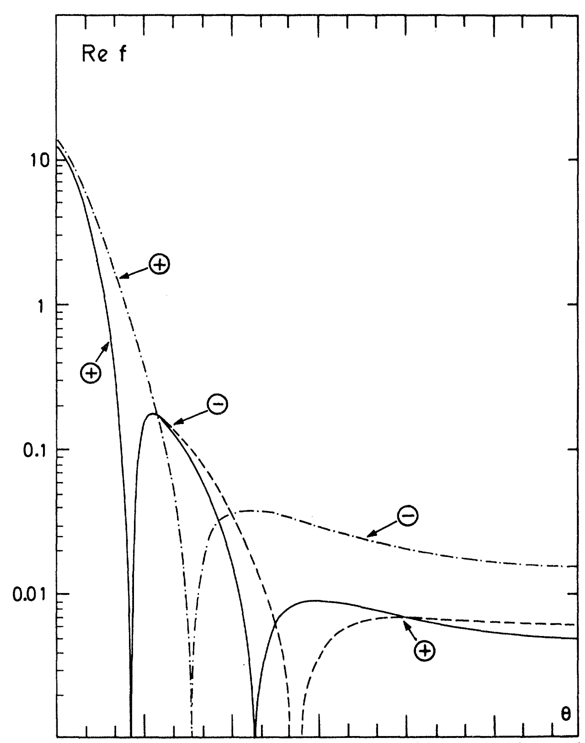

FIG. 5. The real part of the scattering amplitude for a superposition of two Yukawa potentials of the form given in Eq. (3.15), with $U_0 = -20$, $k = 5$. The solid curve shows the exact result, the dashed curve gives the eikonal result, and the dash-dotted curve is the second Born approximation.

tude for a double Yukawa potential, leads also to the conclusion that the eikonal result is excellent at all angles. Here the second Born approximation has the correct sign for all values of θ but is seen to be consistently poorer than the eikonal result at all scattering angles.

The reason for this agreement at large angles in the intermediate coupling case and for Yukawa-type potentials is undoubtedly related to the asymptotic properties which are discussed in Sec. III [see Eqs. (3.18)]. The Born series is fairly rapidly convergent and the large-angle term-by-term properties persist in the total amplitude. Also, the fact that $|U_0|$ is large means that the terms of the Born series missing from the eikonal series are unimportant compared to those included in the eikonal series.

Finally, we may remark that the wide-angle disagreement found for the simple Yukawa potential in Fig. 3 is due to the fact that the exact scattering amplitude is about to pass through zero. Clearly any expanded picture of the vicinity of a zero of the exact amplitude will show similar disagreement (this would be the case, for example, between 50° and 80° in Fig. 4). Corrections to the eikonal result play a major role at such points. The situation is particularly striking in Fig. 3

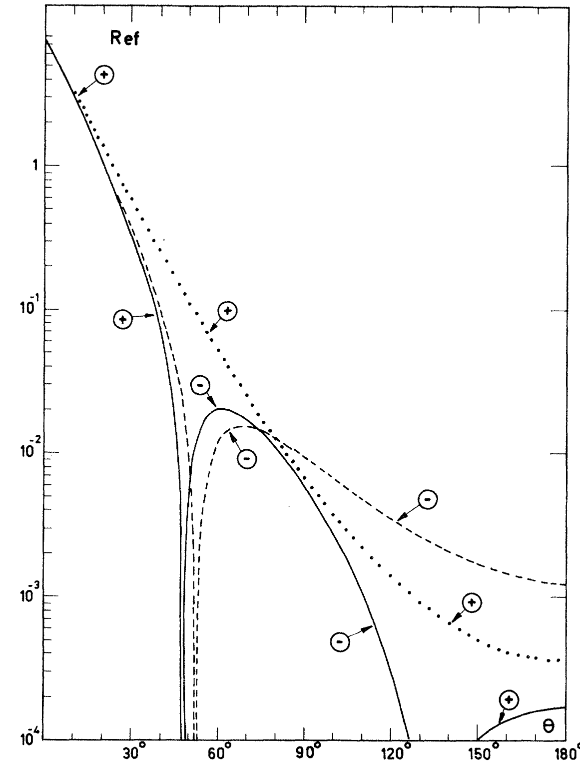

FIG. 7. The real part of the scattering amplitude for a polarization potential of the form (3.23), with $U_0 = -10$, $d = 1$, and $k = 5$. The solid curve shows the exact result, the dashed curve gives the eikonal result, and the dotted curve is the first Born approximation.

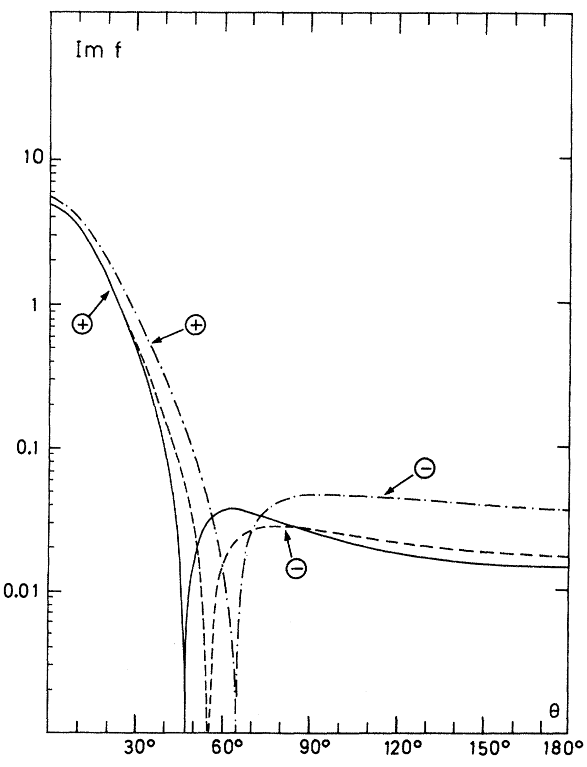

FIG. 6. Same as Fig. 5, except that the imaginary part of the amplitude is shown.

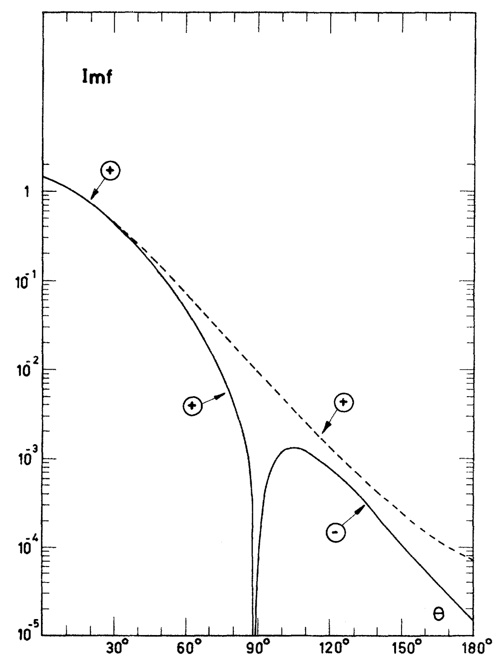

FIG. 8. Same as Fig. 7 except that the imaginary part of the amplitude is shown.

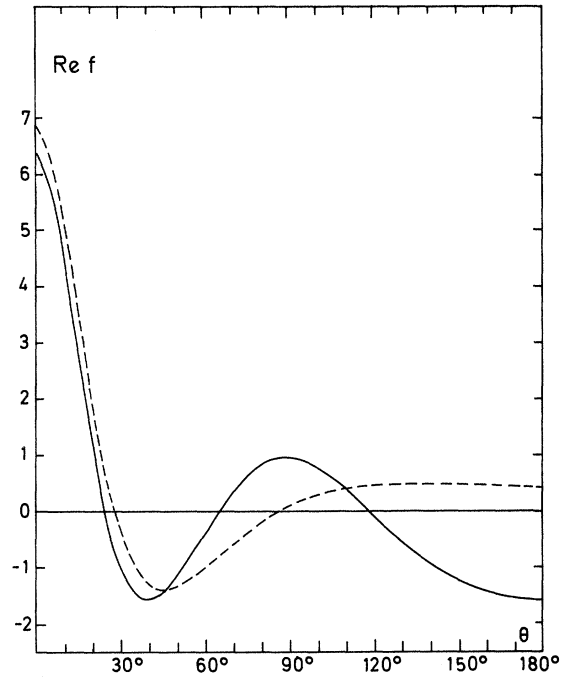

FIG. 9. The real part of the scattering amplitude for a Yukawa potential of the form given in Eq. (3.8) with $U = -20$, $a = 1$, and $k = 2$. The solid curve shows the exact result and the dashed curve gives the eikonal result. Born values are not shown since they are in gross disagreement with the exact results.

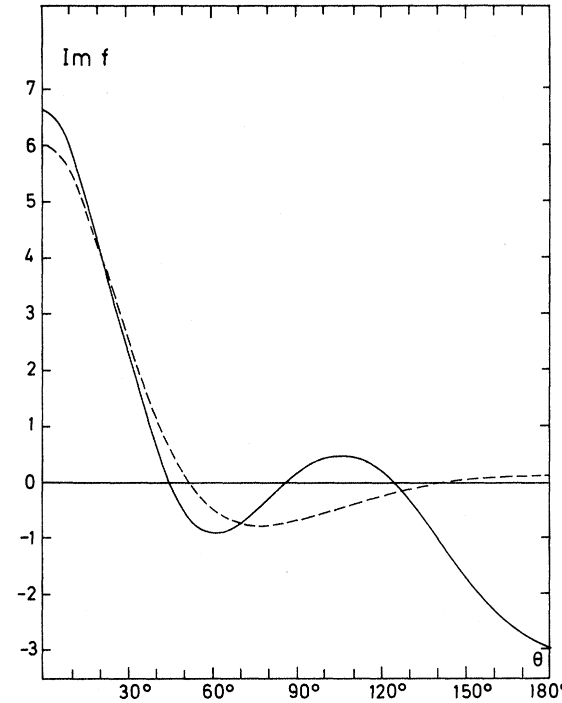

FIG. 10. Same as Fig. 9, except that the imaginary part of the amplitude is shown.

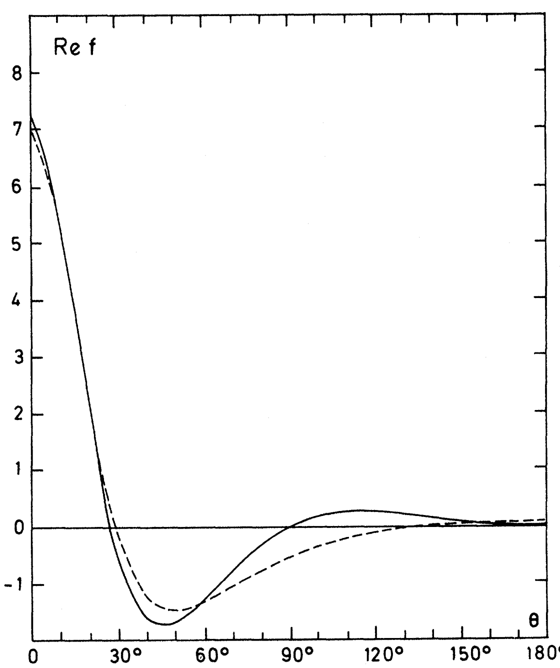

FIG. 11. The real part of the scattering amplitude for a superposition of two Yukawa potentials of the form given in Eq. (3.15), with $U_0 = -20$, $k = 2$. The solid curve shows the exact result and the dashed curve gives the eikonal result. Born values are not shown since they are in gross disagreement with the exact results.

because the amplitude really varies most significantly as a function of the momentum transfer, and the momentum transfer changes by only about 10% as one moves from, say, 120° to 180°.

Before leaving our discussion of intermediate coupling, let us comment briefly on the results for the polarization potential shown in Figs. 7 and 8. We see that at small angles the agreement between exact and eikonal results is very good, but at larger angles this agreement does not persist. This is in accord with the discussion in Appendix B where we show that for potentials of the polarization type the term-by-term properties of Eqs. (3.18) do not hold for all angles.

Finally, we come to the strong coupling case for which $|V_0|/E > 1$. We compare in Figs. 9–14 the eikonal and exact amplitudes for the single Yukawa, double Yukawa, and polarization potentials considered above, with $U_0 = -20$ and $k = 2$. For this case the Born series does not converge. However, in all cases the agreement for both the real and imaginary parts of the amplitude is good at small angles. This agreement is quite unexpected in view of the conventional criteria of validity for the eikonal approximation.

In order to gain some insight into this question,

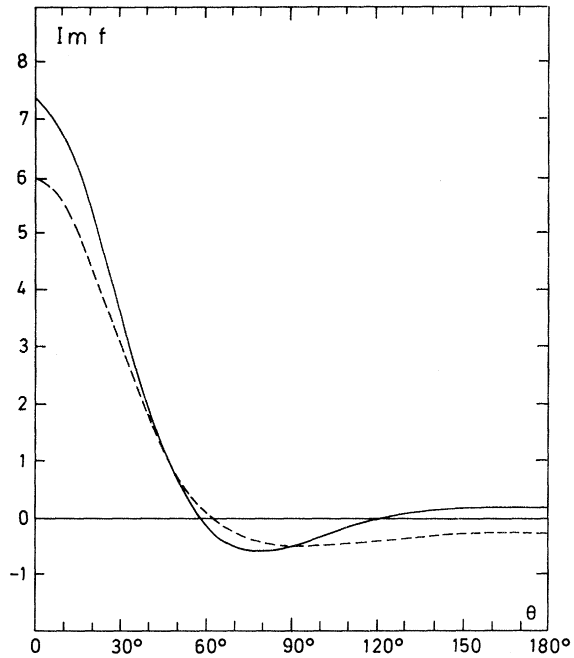

FIG. 12. Same as Fig. 11 except that the imaginary part of the amplitude is shown.

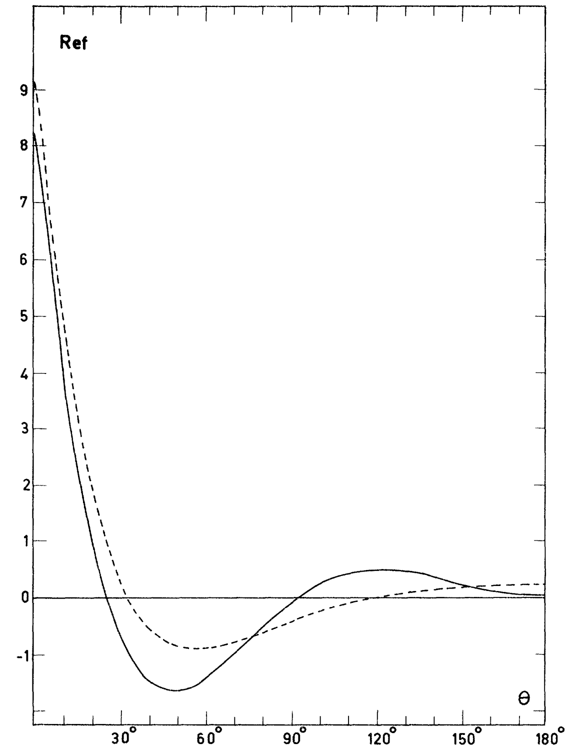

FIG. 13. The real part of the scattering amplitude for a polarization potential of the form (3.23), with $U_0=-20$, $d=1$, and $k=2$. The solid curve shows the exact result and the dashed curve gives the eikonal result. Born values are not shown since they are in gross disagreement with the exact results.

let us consider the problem of scattering by a potential of the form

$$U(r)=\frac{U_0}{r^s}, \quad s>2. \tag{4.5}$$

We choose this form for mathematical convenience; the ideas involved apply to much more general potentials. In particular, the apparent singularity of Eq. (4.5) at $r=0$ is of no consequence. If one likes, Eq. (4.4) can be replaced by $U(r)=U_0/(r^2+d^2)^{s/2}$ and under most circumstances our results will be unchanged. If we assume that the criteria for the validity of the eikonal approximation are satisfied, then the scattering amplitude is given by Eq. (2.59):

$$f(\theta)=\frac{k}{i}\int_0^\infty J_0(\Delta b)\times\left[\exp\left(-i\frac{U_0}{2k}\int_{-\infty}^{\infty}\frac{dz}{(b^2+z^2)^{s/2}}\right)-1\right]b\,db. \tag{4.6}$$

Doing the phase integral and integrating by parts once with respect to b, we find

$$f(\theta)=-(s-1)C(s)\frac{U_0}{\Delta}\int_0^\infty J_1(\Delta b)\frac{1}{b^{s-1}}\times\exp\left(-i\frac{C(s)U_0}{kb^{s-1}}\right)db, \tag{4.7}$$

where

$$C(s)=\int_0^{\pi/2}\cos^{(s-2)}\theta\,d\theta=\tfrac{1}{2}\sqrt{\pi}\,\frac{\Gamma(\frac{1}{2}(s-1))}{\Gamma(\frac{1}{2}s)}. \tag{4.8}$$

Let us assume that we are interested in sufficiently large values of Δ so that for important impact parameters we may write

$$J_1(\Delta b)=\left(\frac{2}{\pi\Delta b}\right)^{1/2}\cos(\Delta b-3\pi/4). \tag{4.9}$$

Then Eq. (4.7) becomes

$$f(\theta)=-(s-1)C(s)U_0(2\pi\Delta^3)^{-1/2}\int_0^\infty\frac{1}{b^{s-1/2}}\left\{\exp\left[i\left(\Delta b-\frac{C(s)U_0}{kb^{s-1}}-\frac{3\pi}{4}\right)\right]+\exp\left[-i\left(\Delta b+\frac{C(s)U_0}{kb^{s-1}}-\frac{3\pi}{4}\right)\right]\right\}db. \tag{4.10}$$

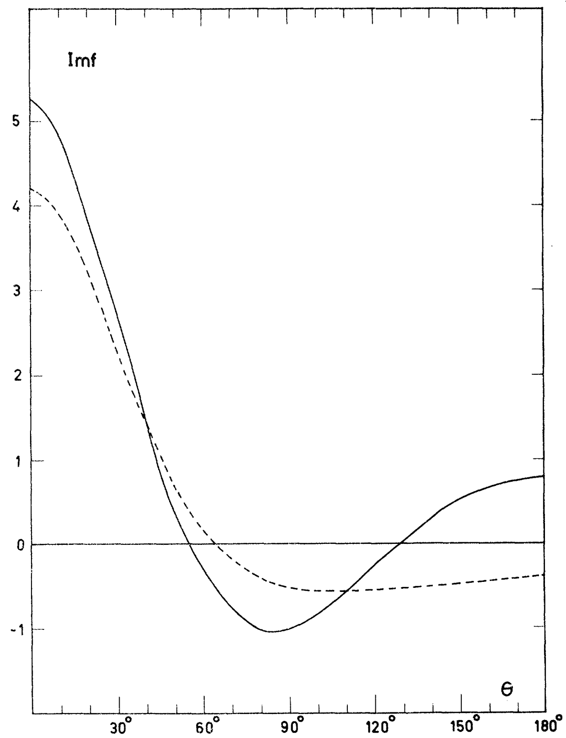

FIG. 14. Same as Fig. 13 except that the imaginary part of the amplitude is shown.

If we make the change of variable $y=b/b_0$, where

$$b_0=[C(s)U_0/k\Delta]^{1/s}$$
$$=\{C(s)V_0/[2E\sin(\tfrac{1}{2}\theta)]\}^{1/s}, \tag{4.11}$$

the two phases, $\chi_\pm$, in Eq. (4.8) take the form

$$\chi_\pm=\Delta b_0(y\pm 1/y^{s-1})-3\pi/4. \tag{4.12}$$

Thus, if $b_0\gg 1$ we can evaluate the integrals in Eq. (4.10) by the method of stationary phases. Assuming U_0 positive only the second phase occurring in Eq. (4.11) has a stationary point, located at

$$b_s=\left(\frac{(s-1)C(s)U_0}{k\Delta}\right)^{1/s}$$
$$=\left(\frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}(s+1))V_0}{2E\Gamma(\frac{1}{2}s)\sin(\frac{1}{2}\theta)}\right)^{1/s} \tag{4.13}$$

The main contribution to Eq. (4.10) comes from the vicinity of $b=b_s$. The evaluation of Eq. (4.10) is now straightforward. One finds

$$f(\theta)=-\frac{b_s}{2\sqrt{s}\,\sin(\frac{1}{2}\theta)}\exp\left[-i\left(\frac{s\Delta b_s}{s-1}-\frac{\pi}{2}\right)\right], \tag{4.14}$$

with b_s determined via Eq. (4.13). This result agrees with the semiclassical result[6] for angles such that $\sin\frac{1}{2}\theta\simeq\frac{1}{2}\theta$, which suggests that the modifications made in going from Eq. (2.28) to Eq. (2.30) are useful in extending the angular validity of the eikonal approximation. The differential scattering cross section is then given by

$$\frac{d\sigma}{d\Omega}=\frac{1}{4s}\left(\frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}(s+1))}{2\Gamma(\frac{1}{2}s)}\frac{V_0}{E}\right)^{2/s}\frac{1}{[\sin(\frac{1}{2}\theta)]^{2+2/s}}. \tag{4.15}$$

There are several points to be made concerning these results. First, note that if the potential of Eq. (4.5) is cut off at distances much smaller than b_s, then the result of Eq. (4.14) is still valid. Second, the singularity at $\theta=0$ in Eq. (4.14) is spurious; it results from the fact that we have assumed Δb_0 large, whereas for θ small Δb_0 becomes small. Thus, Eqs. (4.14) and (4.15) cannot be used for very-small-angle scattering. Finally, let us ask what is the value of $|V(r)|/E$ in the important regions of space, i.e., in the vicinity of $r=b_s$. We have

$$\left|\frac{V(b_s)}{E}\right|=\frac{2\Gamma(\frac{1}{2}s)\sin(\theta/2)}{\sqrt{\pi}\,\Gamma(\frac{1}{2}(s+1))}$$
$$\simeq 2\left(\frac{2}{\pi s}\right)^{1/2}\sin(\tfrac{1}{2}\theta). \tag{4.16}$$

Thus, for $\theta\lesssim 60°$, $|V(b_s)/E|<1$. For still smaller angles, we would expect $|V(r)/E|$ to become sufficiently small *in the important regions of space* so that the eikonal result Eq. (2.33) will be valid, even though $|V_0|/E$ is not small. Thus, it is plausible that for more general potentials the range of validity of the eikonal method may be much greater than is suggested by the crude criterion $|V_0|/E<1$.

V. SUMMARY AND CONCLUSIONS

We have presented in this work a systematic study of the eikonal approximation in the simple case of nonrelativistic potential scattering. In particular, we suggest that the eikonal amplitude has the following remarkable properties:

(1) For a large class of potentials, given by Eq. (3.14), each term of the eikonal multiple scattering series gives the asymptotic value (as $k\to\infty$) of the corresponding term in the Born series for *all* momentum transfers.

(2) In the weak coupling case, i.e., when the Born series is rapidly convergent, the eikonal amplitude gives a consistently poorer approximation to the exact amplitude than does the second Born approximation $f_{B2}=\bar{f}_{B1}+\bar{f}_{B2}$. However, by adding Re $\bar{f}_{B2}$ to the eikonal result one obtains an extremely good value of the scattering amplitude. Moreover, the agreement between the eikonal and exact results improves as the coupling increases.

(3) For intermediate couplings such that $|V_0|/E$

$\lesssim 1$, ka large and $U_0 a/2k \simeq 1$ we find that the eikonal amplitude reproduces very well the exact result *for all scattering angles* and Yukawa-type potentials.

(4) Finally, even for strong coupling cases, i.e., when $|V_0|/E > 1$, we still find that $f_E \simeq f_{ex}$ for small momentum transfers and ka somewhat larger than unity.

These results strongly suggest that the traditional criteria for the validity of the eikonal approximation are only sufficient conditions which are often unnecessarily stringent.

APPENDIX A

We want to prove that for an arbitrary superposition of Yukawa potentials of the form

$$U(r) = U_0 \int_{\alpha_0 > 0}^{\infty} \rho(\alpha) \frac{e^{-\alpha r}}{r} d\alpha, \tag{A1}$$

one has for all values of Δ

$$A_{B2}(\Delta) = A_{E2}(\Delta), \tag{A2}$$

where $A_{B2}(\Delta)$ and $A_{E2}(\Delta)$ are given, respectively, by Eqs. (3.12a) and (3.12b).

Let us first calculate $\mathrm{Im}\bar{f}_{B2}(k, \Delta)$. We start from the general expression

$$\bar{f}_{B2} = 2\pi^2 \int d\vec{q}\, \langle \vec{k}_f | U | \vec{q} \rangle \frac{1}{q^2 - k^2 - i\epsilon} \langle \vec{q} | U | \vec{k}_i \rangle, \tag{A3}$$

where

$$\langle \vec{K}' | U | \vec{K} \rangle = (2\pi)^{-3} \int e^{i(\vec{K} - \vec{K}')\cdot\vec{r}}\, U(\vec{r}) d\vec{r}, \tag{A4}$$

and the limit $\epsilon \to 0^+$ is always understood. For potentials of the type (A1) we have

$$\langle \vec{K}' | U | \vec{K} \rangle = (2\pi^2)^{-1} U_0 \int d\alpha \frac{\rho(\alpha)}{\Delta^2 + \alpha^2} \tag{A5}$$

and therefore

$$\bar{f}_{B2} = \frac{U_0^2}{2\pi^2} \int d\alpha\, \rho(\alpha) \int d\beta\, \rho(\beta) \int d\vec{q}\, \frac{1}{(q^2 - k^2 - i\epsilon)[(\vec{q} - \vec{k}_i)^2 + \alpha^2][(\vec{q} - \vec{k}_f)^2 + \beta^2]}. \tag{A6}$$

Using the Feynman integral representation

$$a^{-1} b^{-1} = \int_0^1 \frac{dt}{[at + b(1-t)]^2}, \tag{A7}$$

we may rewrite Eq. (A6) as

$$\bar{f}_{B2} = \frac{U_0^2}{2\pi^2} \int d\alpha\, \rho(\alpha) \int d\beta\, \rho(\beta) \int_0^1 dt \int d\vec{q}\, \frac{1}{(q^2 - k^2 - i\epsilon)[(\vec{q} - \vec{\Lambda})^2 + \Gamma^2]^2}, \tag{A8}$$

where we have set

$$\Gamma^2 = \alpha^2 t + \beta^2(1-t) + t(1-t)\Delta^2 \tag{A9a}$$

and

$$\vec{\Lambda} = t\vec{k}_i + (1-t)\vec{k}_f. \tag{A9b}$$

The $\vec{q}$ integral may now be readily evaluated to yield[9]

$$\int d\vec{q}\, \frac{1}{(q^2 - k^2 - i\epsilon)[(\vec{q} - \vec{\Lambda})^2 + \Gamma^2]^2} = -\frac{\pi^2}{\Gamma(k^2 - \Gamma^2 - \Lambda^2 + 2ik\Gamma)} \tag{A10}$$

so that

$$\bar{f}_{B2} = -\frac{U_0^2}{2} \int d\alpha\, \rho(\alpha) \int d\beta\, \rho(\beta) \int_0^1 \frac{dt}{\Gamma(2ik\Gamma - \alpha^2 t - \beta^2(1-t))}, \tag{A11}$$

where we have used the fact that

$$\Gamma^2 + \Lambda^2 = k^2 + \alpha^2 t + \beta^2(1-t). \tag{A12}$$

We now consider the particular case $k \to \infty$, so that we may write

$$\lim_{k\to\infty} \bar{f}_{B2} = i \lim_{k\to\infty} \mathrm{Im}\,\bar{f}_{B2} = i\frac{U_0^2}{4k} \int d\alpha\, \rho(\alpha) \int d\beta\, \rho(\beta) \int_0^1 \frac{dt}{\Gamma^2} \tag{A13}$$

or, using Eq. (A9a),

$$\lim_{k\to\infty} \operatorname{Im} \bar{f}_{B2}(k, \Delta) = \frac{U_0^2}{4k} \int d\alpha\,\rho(\alpha) \int d\beta\,\rho(\beta) \int_0^1 \frac{dt}{\beta^2 + (\alpha^2 - \beta^2 + \Delta^2)t - \Delta^2 t^2} . \tag{A14}$$

The integral on the variable t is easily evaluated, and yields

$$\int_0^1 dt \frac{1}{\beta^2 + (\alpha^2 - \beta^2 + \Delta^2)t - \Delta^2 t^2} = \frac{1}{\alpha\beta(u^2-1)^{1/2}} \ln[u + (u^2-1)^{1/2}], \tag{A15}$$

where

$$u = \frac{\alpha^2 + \beta^2 + \Delta^2}{2\alpha\beta} . \tag{A16}$$

Hence

$$\lim_{k\to\infty} \operatorname{Im} \bar{f}_{B2}(k, \Delta) = \frac{U_0^2}{2k} \int d\alpha\,\rho(\alpha) \int d\beta\,\rho(\beta) \frac{1}{\alpha\beta(u^2-1)^{1/2}} \ln[u + (u^2-1)^{1/2}]. \tag{A17}$$

We now evaluate $\operatorname{Im} \bar{f}_{E2}(k, \Delta)$, which is given by

$$\operatorname{Im} \bar{f}_{E2}(k, \Delta) = \frac{k}{2} \int_0^\infty J_0(\Delta b)\, \chi^2(b, k)\, b\, db, \tag{A18}$$

with

$$\chi(b, k) = -\frac{1}{2k} \int_{-\infty}^{+\infty} U(b, z) dz. \tag{A19}$$

For a potential of the form (A1), we have

$$\chi(b, k) = \frac{1}{k} U_0 \int \rho(\alpha) K_0(\alpha b) d\alpha, \tag{A20}$$

where K_0 is the modified Bessel function of order zero. Hence, substituting Eq. (A20) into Eq. (A18), we obtain

$$\operatorname{Im} \bar{f}_{E2}(k, \Delta) = \frac{U_0^2}{2k} \int d\alpha\,\rho(\alpha) \int d\beta\,\rho(\beta) \int_0^\infty J_0(\Delta b) K_0(\alpha b) K_0(\beta b) b\, db. \tag{A21}$$

The integral on the b variable yields[10]

$$\int_0^\infty J_0(\Delta b) K_0(\alpha b) K_0(\beta b) b\, db = \frac{1}{2\alpha\beta(u^2-1)^{1/2}} \ln[u + (u^2-1)^{1/2}], \tag{A22}$$

where the quantity u is given by Eq. (A16). Hence

$$\operatorname{Im} \bar{f}_{E2}(k, \Delta) = \frac{U_0^2}{4k} \int d\alpha\,\rho(\alpha) \int d\beta\,\rho(\beta) \frac{1}{\alpha\beta(u^2-1)^{1/2}} \ln[u + (u^2-1)^{1/2}] \tag{A23}$$

and Eq. (A2) follows by comparing Eqs. (A17) and (A23). We also remark that it is a simple matter to extend the validity of Eq. (A2) to a superposition of exponential potentials.

APPENDIX B

We now turn to two cases, the polarization potential and the Gaussian potential, for which the results (3.13) do not hold. Taking the polarization potential to be of the form

$$U(r) = \frac{U_0}{(r^2 + d^2)^2}, \tag{B1}$$

we have

$$f_E = \frac{k}{i} \int_0^\infty J_0(\Delta b) \left[\exp\left(-\frac{i}{k} \frac{\pi U_0}{4(b^2 + d^2)^{3/2}} \right) - 1 \right] b\, db$$

so that

$$\bar{f}_{En} = \frac{1}{n!}\left(\frac{i}{k}\right)^{n-1}\left(-\frac{\pi U_0}{4}\right)^n \int_0^\infty J_0(\Delta b)\frac{b}{(b^2+d^2)^{3n/2}}\,db\,.$$

This last integral is of the Hankel-Nicholson type and can be evaluated for all n. We find

$$\bar{f}_{En} = i^{n-1} k\left(-\frac{\pi U_0}{4k}\right)^n \left(\frac{\Delta}{2d}\right)^{(3n-2)/2} \frac{K_{(3n-2)/2}(\Delta d)}{n!\,\Gamma(3n/2)}\,. \tag{B2}$$

In particular for the case $n=1$ we recover the familiar first Born approximation

$$\bar{f}_{E1} = -\frac{\pi U_0}{4d} e^{-\Delta d} = \bar{f}_{B1}\,. \tag{B3}$$

For the case $n=2$ we find

$$\bar{f}_{E2} = i\,\frac{\pi^2 U_0^2}{64k}\left(\frac{\Delta}{2d}\right)^2 K_2(\Delta d). \tag{B4}$$

As far as the second Born approximation is concerned, we have

$$\bar{f}_{B2} = \frac{U_0^2}{32d^2}\int \frac{e^{-|\vec{k}_i-\vec{q}|d}\, e^{-|\vec{q}-\vec{k}_f|d}}{q^2-k^2-i\epsilon}\, d\vec{q}\,.$$

The imaginary part of this expression is easily evaluated when the scattering angle θ is such that $\theta=0$ or $\theta=\pi$. In the former case, $\vec{k}_f=\vec{k}_i$, and we have immediately

$$\bar{f}_{B2}(\theta=0) = \frac{U_0^2}{32d^2}\int \frac{e^{-2d|\vec{k}_i-\vec{q}|}}{q^2-k^2-i\epsilon}\, d\vec{q}\,.$$

Using $(x-i\epsilon)^{-1} = Px^{-1} + i\pi\delta(x)$, we obtain

$$\mathrm{Im}\bar{f}_{B2}(\theta=0) = \frac{\pi U_0^2}{64d^2}\,k\int \exp[-4kd\sin(\tfrac{1}{2}\theta_q)]\sin\theta_q\, d\theta_q\, d\phi_q\,,$$

where θ_q and ϕ_q are the polar angles of the vector $\vec{q}$. This leads to

$$\mathrm{Im}\bar{f}_{B2}(\theta=0) = \frac{\pi^2 U_0^2}{128 d^4 k}\left[1-(4kd+1)e^{-4kd}\right]. \tag{B5}$$

Comparing with Eq. (B4) and assuming $kd \gg 1$ so that the exponential term in Eq. (B5) can be neglected, we see that $\mathrm{Im}\bar{f}_{B2}(\theta=0) = \mathrm{Im}\bar{f}_{E2}(\theta=0)$.

For $\theta=\pi$, however, the situation is different. Proceeding as before, we find

$$\mathrm{Im}\bar{f}_{B2}(\theta=\pi) = \frac{\pi^2 U_0^2}{32d^2}\,k\int_0^\pi \exp\{-2kd[\sin(\tfrac{1}{2}\theta_q)+\cos(\tfrac{1}{2}\theta_q)]\}\sin\theta_q\, d\theta_q\,.$$

Clearly, if $ka \gg 1$, the main contribution comes from the regions near $\theta_q=0$ and $\theta_q=\pi$, and the contributions are equal. Thus

$$\mathrm{Im}\bar{f}_{B2}(\theta=\pi) \simeq \frac{\pi^2 U_0^2}{16d^2}\, k e^{-2kd}\int_0^\pi \exp[-2kd\sin(\theta_q/2)]\sin\theta_q\, d\theta_q\,.$$

Neglecting terms of order e^{-4kd}, we have

$$\mathrm{Im}\bar{f}_{B2}(\theta=\pi) \simeq \frac{\pi^2 U_0^2}{16 d^4 k} e^{-2kd}\,. \tag{B6}$$

Comparing this result with Eq. (B4) we see that

$$\mathrm{Im}\bar{f}_{E2}(\theta=\pi) \neq \mathrm{Im}\bar{f}_{B2}(\theta=\pi),$$

the two terms differing by a factor $\pi^{1/2}(kd)^{3/2}/8$ when $kd \gg 1$.

For the case of a Gaussian potential of the form

$$U(r) = U_0 e^{-r^2/a^2} \tag{B7}$$

the eikonal phase is readily evaluated, leading to

$$f_E = \frac{k}{i}\int_0^\infty J_0(\Delta b)\left[\exp\left(-i\frac{\sqrt{\pi}\,U_0 a}{2k} e^{-b^2/a^2}\right)-1\right] b\, db.$$

Hence

$$\bar{f}_{En} = i^{n-1} k \frac{1}{n!}\left(-\frac{\sqrt{\pi}\,U_0 a}{2k}\right)^n \int_0^\infty J_0(\Delta b) e^{-nb^2/a^2}\, b\, db.$$

The integral is straightforward and yields

$$\bar{f}_{En} = \frac{i^{n-1} a^2 k}{(2n)n!}\left(-\frac{\sqrt{\pi}\,U_0 a}{2k}\right)^n e^{-a^2\Delta^2/4n}. \tag{B8}$$

We recover the familiar first Born approximation when $n = 1$, namely

$$\bar{f}_{E1} = -\tfrac{1}{4}\sqrt{\pi}\, a^3 U_0 e^{-a^2\Delta^2/4} = \bar{f}_{B1}\,. \tag{B9}$$

The second Born approximation is given by

$$\bar{f}_{B2} = \frac{U_0^{\,2} a^6}{32\pi}\int e^{-a^2|\vec{k}_i-\vec{q}|^2/4}\, e^{-a^2|\vec{q}-\vec{k}_f|^2/4}\,\frac{d\vec{q}}{q^2-k^2-i\epsilon} \tag{B10}$$

or

$$\bar{f}_{B2} = \frac{U_0^{\,2} a^2}{8|\vec{k}_i+\vec{k}_f|}\int_0^\infty \left(e^{-a^2(k_i^{\,2}+q^2-|\vec{k}_i+\vec{k}_f|q)/2} - e^{-a^2(k_i^{\,2}+q^2+|\vec{k}_i+\vec{k}_f|q)/2}\right)\frac{q\,dq}{q^2-k^2-i\epsilon}\,.$$

If we are interested only in[15] $\mathrm{Im}\bar{f}_{B2}$ the remaining integration is trivial, giving

$$\mathrm{Im}\bar{f}_{B2} = \frac{\pi U_0^{\,2} a^4}{32k\cos(\frac{1}{2}\theta)}\left(e^{-a^2k^2[1-\cos(\theta/2)]} - e^{-a^2k^2[1+\cos(\theta/2)]}\right). \tag{B11}$$

On the other hand, $\bar{f}_{E2}$ is pure imaginary, and

$$\mathrm{Im}\bar{f}_{E2} = \frac{\pi U_0^{\,2} a^4}{32k}\, e^{-a^2k^2\sin^2(\theta/2)/2}, \tag{B12}$$

according to Eq. (B8). It is obvious that for small values of θ [in fact, for $\theta \ll 1/(ka)^{1/2}$], $\mathrm{Im}\bar{f}_{B2}$ and $\mathrm{Im}\bar{f}_{E2}$ agree. For larger values of the scattering angle, however, this agreement is lost. When $\theta=\pi$, for example, the two expressions (B10) and (B12) differ by a factor $(ka)^2 e^{-k^2a^2/2}$.

---

*Supported in part by the NATO Scientific Affairs Division, Research Grant No. 586.

†Chargé de Recherches du Fonds National de la Recherche Scientifique, Belgium.

[1]G. Molière, Z. Naturforsch. 2A, 133 (1947).

[2]R. J. Glauber, in *High-Energy Collision Theory*, edited by W. E. Brittin (Interscience, New York, 1959).

[3]See, for example, R. J. Glauber, in *High-Energy Physics and Nuclear Structure*, edited by S. Devons (Plenum, New York, 1970); K. Gottfried and J. D. Jackson, Nuovo Cimento 34, 375 (1964); A. S. Goldhaber and C. J. Joachain, Phys. Rev. 171, 1566 (1968).

[4]See, for example, J. C. Y. Chen and K. M. Watson, Phys. Rev. 174, 152 (1968); V. Franco, Phys. Rev. Lett. 20, 109 (1969); H. Tai, R. H. Bassel, E. Gerjuoy, and V. Franco, Phys. Rev. A 1, 1819 (1970); C. J. Joachain and M. H. Mittleman, Phys. Rev. A 4, 1492 (1971); F. W. Byron, Jr., Phys. Rev. A 4, 1907 (1971); F. W. Byron, Jr. and C. J. Joachain, Phys. Letters 38A, 185 (1972).

[5]F. W. Byron, Jr. and C. J. Joachain, Phys. Rev. A 8, 1267 (1973).

[6]See, for example, R. B. Bernstein, in *Advances in Chemical Physics*, (Interscience, New York, 1966), Vol. 10.

[7]B. J. Malenka, Phys. Rev. 95, 522 (1954).

[8]L. I. Schiff, Phys. Rev. 103, 443 (1956).

[9]R. H. Dalitz, Proc. R. Soc. A206, 509 (1951).

[10]I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1971).

[11]Clearly the values of $\mathrm{Im}\bar{f}_{B2}$ and $\mathrm{Im}\bar{f}_{E2}$ for any other choice of U_0 are obtained by multiplying the results of Table I by $U_0^{\,2}$.

[12]Note that the choice $a = 1$ is not restrictive since the range a can always be scaled out of the Schrödinger equation by a change of variable. Thus in all tables and figures k is really to be measured in units of a^{-1} while f is in units of a.

[13]N. F. Bali, Shu-Yuan Chu, R. W. Haymaker, and Chung-I Tan, Phys. Rev. 161, 1450 (1967).

[14]R. J. Moore, Phys. Rev. D 2, 313 (1970).

[15]A more detailed study of the second Born term (B10) including the expression of $\mathrm{Re}f_{B2}$, may be found in C. Joachain, Nucl. Phys. 64, 529 (1965).