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Singular Cores in the Three-Body Problem. II. Numerical Solution*

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The formalism for singular cores previously introduced is investigated numerically for a simple model. Calculations of three-particle binding energies and the analog of N-d scattering for this model demonstrate the practicality of the earlier theoretical development.

I. INTRODUCTION

The first paper of this series discussed a new generalization of the three-particle formalism to include two-body interactions characterized by a hard core or by a boundary condition on the wave function (BCM).[1] Specific questions such as uniqueness and three-body unitarity were investigated in some detail at that time. The present article is concerned with a numerical investigation of the formalism for a simple model. Calculations of three-particle binding energies and the analog of N-d elastic scattering within the context of this model demonstrate the practicality of the earlier theoretical development for the treatment of singular cores.

The principal motivation for this development is the versatility afforded by being able to utilize this additional class of interactions in the three-body problem. For example, calculations to date in the three-nucleon system with realistic interactions have been almost exclusively restricted to soft-core models, the single exception being the long and difficult variational calculation on the Hamada-Johnston hard core by Delves, *et al.*[2] The results of these computations have generated some doubt as to the ability of such models to fit the experimental data. For example, it appears that any soft-core model which fits the two-nucleon phase shifts reasonably well will underbind the triton by about 2 MeV. A significant discrepancy also appears to exist in the case of the ^{3}He charge form factor[3] (for a more complete discussion see SCI). More recently, the present author has generalized the boundary-condition approach to provide a complete phenomenology of three-particle final states.[4] The model discussed below may also be regarded as a first approximation to this general scheme, and hence has a significance quite apart from singular cores per se.

We begin in Sec. II with a description of our model and the relevant integral equation to be solved. We also take this opportunity to present simplified equations for the general case considered in SCI; these formulas are germane to applications of the formalism with realistic interactions. In Sec. III we introduce a numerical technique which is particularly advantageous for solving integral equations of a certain class; the method is illustrated by means of an exactly soluble example. Numerical results for our model are presented in Sec. IV, which concludes with a discussion of these results and implications for future calculations with our formalism.

II. A SIMPLE MODEL

We shall consider a model in which three identical spinless particles of mass M interact via the BCM alone (no potentials external to the core) and only in relative s waves. If a denotes the core

radius, the two-particle wave function then has the form

$$\psi_k(r)=0, \quad r<a$$
$$=\left(\frac{2}{\pi}\right)^{1/2}[j_0(kr)-i\pi M\tfrac{1}{2}kt_0(k)h_0(kr)], \quad r>a. \tag{1}$$

The on-shell t matrix $t_0(k)$ is determined by requiring that

$$\lim_{\epsilon\to 0}\frac{\psi_k'(a+\epsilon)}{\psi_k(a+\epsilon)}=\frac{1}{a}(f_0-1), \tag{2}$$

where f_0 is a constant. Thus

$$t_0(k)=\frac{N_0(k)}{D_0(k)},$$
$$N_0(k)=f_0j_0(ak)-\cos ak, \tag{3}$$
$$D_0(k)=\frac{\pi M}{2a}(f_0-iak)e^{iak}.$$

If M is taken to be the average nucleon mass and f_0, a are adjusted to produce the deuteron binding energy and the triplet scattering length, one obtains the numerical values $a=1.095$ F, $f_0=-0.253$.

The three-particle integral equation for this model was discussed in SCI in considerable detail. For our present purposes we shall specialize still further to the case of total angular momentum $L=0$, and to the form of the equation which pertains to elastic scattering from an initial two-particle bound state. We are thus led to the equation[5]

$$X(q')=\alpha N(q',\omega)+\int_0^\infty dqq^2\frac{N(q',q)}{D_0(\kappa)}X(q), \tag{4}$$

where all quantities depend parametrically on W, the total energy in the three-particle c.m. system, and

$$\alpha=\frac{4f_0\exp(-f_0)}{\pi M^2a},$$
$$\kappa=(MW-\tfrac{3}{4}q^2)^{1/2}, \tag{5}$$
$$\omega^2=\tfrac{4}{3}[MW+(f_0/a)^2].$$

Here $X(\omega)$ is the amplitude for elastic scattering of the third particle from an initial two-particle bound state (the analog of N-d scattering), and $N(q',q)$ can be represented in the form

$$N(q',q)=\theta_0\left(q',q;\frac{\sqrt{3}}{2}a\right)[D_0(\kappa)-D_0(\kappa_0)]$$
$$+\overline{N}(q',q)+\int_{\sqrt{3}\,a/2}^{b}\frac{dyy^2j_0(yq')}{z_0(y)}N(y,q). \tag{6}$$

As in SCI, we have introduced a negative-energy parameter[6] W_0 and have taken $\kappa_0=\kappa(W_0)$, while θ_0 is the s-wave momentum-space representation of the unit step function $\theta(\frac{1}{2}\sqrt{3}\,a-y)$:

$$\theta_0(q',q;R)=\frac{R}{\pi qq'}[j_0(R(q'-q))-j_0(R(q'+q))]. \tag{7}$$

The function $\overline{N}$ is analogous to the kernels which arise in the usual three-body formalism in the case of separable interactions:

$$\overline{N}(q',q)$$
$$=-M\int_{-1}^{1}dz\frac{Q}{\beta}\frac{N_0(K)[\cos bq'-ibQj_0(bq')]e^{ibQ}}{q'^2-Q^2-i\epsilon},$$
$$\beta=[MW-(1-\tfrac{1}{4}z^2)q^2]^{1/2}, \tag{8}$$
$$Q=-\tfrac{1}{2}zq+\beta,$$
$$K=(MW-\tfrac{3}{4}Q^2)^{1/2}.$$

We observe that although Q is complex, $\overline{N}$ is real for W below the break-up threshold.

The integral term in Eq. (6) is defined in terms of the function

$$N(y,q)=-M\int_{-z_0}^{z_0}dz\frac{e^{ix_0\kappa}}{x_0}I(y,q,z),$$
$$z_0=\min[1,(y^2-\tfrac{3}{4}a^2)/ay],$$
$$x_0=(a^2/4-ayz+y^2)^{1/2}, \tag{9}$$
$$y_0=(9a^2/16+\tfrac{3}{4}ayz+y^2/4)^{1/2},$$
$$I(y,q,z)=\frac{(9a+6yz)}{16y_0}aqj_1(y_0q)$$
$$+\left[f_0-1+\frac{a(a-2yz)}{4x_0^2}(1-ix_0\kappa)\right]j_0(y_0q).$$

The upper limit of the y integration is the real number $b>\frac{3}{2}a$; the point here is simply that if we took $b\to\infty$, the integral term would contribute all of $\overline{N}$(which would vanish). However, the numerical evaluation of the integral in that limit would be quite difficult. This problem is avoided by employing the explicit form for $\overline{N}$ given above.

Although one can explicitly verify that the kernel of Eq. (4) is square-integrable (and hence compact), rapid variations of sign and a relatively slow decrease in magnitude for large q,q' complicate numerical solution. In particular, a straightforward application of Gaussian quadrature is not adequate for this purpose. It is therefore necessary to employ somewhat more sophisticated techniques, one example of which is discussed in Sec. III.

In order to perform calculations in the three-nucleon system with "realistic" N-N interactions, such as the Feshbach-Lomon model,[7] one must take into account the presence of potentials ex-

ternal to the core region. Explicit formulas for this purpose were presented in SCI. However, more recent work indicates that these expressions can be considerably simplified. In terms of the operator notation defined in that article, Eqs. (106) and (108) can be replaced by the single equation

$$\begin{aligned} &Y=\Omega+\overline{K}Y, \\ &\Omega=(1-\theta)\hat{t}(H-1)[1+G_0\rho U(IQ-1)] \\ &\quad-\theta\hat{t}G_0(W_0)\rho U(IQ-1). \end{aligned} \tag{10}$$

Here $\overline{K}$ is the same kernel defined by Eq. (109) of SCI.[8]

III. INTERIOR-EXTERIOR METHOD OF SOLUTION

As an illustration of the numerical problems encountered in the solution of Eq. (4), let us consider the following integral equation:

$$\begin{aligned} &F(q',q)=\lambda^{-1}\int_0^\infty dpp^2K(q',p)F(p,q), \\ &K(q',q)=\theta_0(q',q;b)\frac{q\sin 2bq}{q^2-\left(\frac{\pi}{2b}\right)^2}. \end{aligned} \tag{11}$$

This is a simple eigenvalue problem for the unknown λ. Due to the trivial analytic properties of the kernel, one can deduce that $F(q',q)$, for fixed q, is an entire function of q', growing exponentially like $\exp(-ibq')$ as $\mathrm{Im}q'\to+\infty$. The integral can then be evaluated analytically by the method of residues,[9] and Eq. (11) becomes

$$F(q',q)=-\frac{\lambda^{-1}\pi^3}{8b^2}\theta_0(q',\pi/2b;b)F(\pi/2b,q). \tag{12}$$

Setting $q'=\pi/2b$ and invoking Eq. (7), we obtain $\lambda=-b/2$.

In order to solve this problem numerically by Gaussian quadrature, one would in general introduce a mapping, $q=f(x)$, of q onto the finite domain $-1<x<1$. The integral would then be written as

$$\begin{aligned} &\int_{-1}^{1}dx\left(\frac{df}{dx}\right)p^2K(q',p)F(p,q) \\ &\qquad\simeq\sum_{j=1}^{N}H_j\left[\frac{df}{dx}p^2K(q',p)F(p,q)\right]_{x=a_j}, \end{aligned} \tag{13}$$

where p is to be interpreted as $p=f(x)$. The N Gaussian points a_j, and weights H_j, are chosen so as to integrate a polynomial of degree $2N$ in x exactly. The problem then is reduced to a finite-dimensional eigenvalue equation ($\dim=N$), with a matrix kernel

$$K_{ij}=H_j\left[\frac{df}{dx}p^2K(f(a_i),p)\right]_{x=a_j}. \tag{14}$$

A difficulty arises in this procedure for kernels such as the above due to a slow rate convergence at large p. To see this it is sufficient to consider the diagonal element K_{ii}. For definiteness let us assume that the mapping is given by

$$\begin{aligned} &p=\alpha\frac{(1+x)}{1-x}, \\ &dp=\frac{2\alpha}{(1-x)^2}dx. \end{aligned} \tag{15}$$

The large p values are thus attained for points $x=a_j$ approaching $+1$ (say the largest of these corresponds to $j=N$). It is clear that dp/dx is effectively proportional to p^2 in that limit, so that $K_{NN}\propto p^4K(p,p)$ for large N. However, $K(p,p)$ only falls off like p^{-3}, and hence K_{NN} increases like $p(a_N)$. As a consequence, the finite matrix array K_{ij} involves increasingly large elements in the lower right-hand corner, and fails to converge as one increases the number of Gaussian points. This phenomenon is not restricted to matrix manipulations; an attempt to calculate $\mathrm{Tr}K$ by such an approach also fails, although $\mathrm{Tr}K=-b/2$ is perfectly well defined. Similar problems (although less severe) arise with kernels which decrease like p^{-4}, but involve factors like $\sin bp$ which oscillate rapidly in sign. Alternate choices of the mapping do not alter these observations, and hence one is forced to employ somewhat different numerical techniques.

A solution to this problem is suggested by the nature of the difficulty itself. The quantities $\mathrm{Tr}K$, $\mathrm{Tr}K^2$ exist for such kernels, despite the slow fall-off of $|K|$, because of the sign oscillations characteristic of entire functions. In turn, the latter typically arise from finite cutoffs in the coordinate representation. Thus, if y is the coordinate conjugate to q', $K(y,q)$ vanishes in the above example for $y>b$. This suggests expanding the unknown function in an appropriate complete set for this domain. Thus, setting

$$\begin{aligned} &F(q',q)=\sum_n A_n(q)\phi_n(q'), \\ &\int_0^\infty dqq^2\phi_n(q)\phi_m(q)=\delta_{nm}, \end{aligned} \tag{16}$$

we deduce that

$$\begin{aligned} &A_n(q)=\lambda^{-1}\sum_m K_{nm}A_m(q), \\ &K_{nm}=\int_0^\infty dqq^2\phi_n(q)\int_0^\infty dq'q'^2\phi_m(q')K(q,q'). \end{aligned} \tag{17}$$

By truncating the expansion we again arrive at a

finite matrix equation; the trick is to pick a set ϕ_n which (1) is suited to the kernel in such a way that the expansion is rapidly convergent, and (2) enables one to evaluate the K_{nm} moments without too much difficulty.

A convenient complete set for this purpose may be obtained from the eigenfunctions of the two-particle Lippmann-Schwinger equation in the case of a unit square-well potential $\theta(b-y)$. Symbolically, we have $\theta G_0(-\mu^2)\phi_n = \lambda_n{}^{-1}\phi_n$, where $-\mu^2$ is a fixed negative-energy parameter. From the explicit analytic solution for this problem,[9] one easily obtains

$$\begin{aligned}\phi_n(q) &= \gamma_n \frac{\cos bq + b\mu j_0(bq)}{q^2 - p_n^2}, \\ \lambda_n &= \mu^2 + p_n^2, \\ \gamma_n &= \frac{2p_n}{\sqrt{\pi}}\left[\mu + b(\mu^2 + p_n^2)\right]^{-1/2},\end{aligned} \tag{18}$$

where the p_n are the zeros of the numerator. Thus $\phi_n(p_m) \propto \delta_{nm}$. This set has the following very useful property: If $F(q)$ is any even, entire function of q such that $|qe^{ib_0 q}F(q)|$ is bounded in the upper half-plane, then

$$\begin{aligned}\int_0^\infty dq q^2 \phi_n(q) F(q) &= \tau_n F(p_n), \\ \tau_n &= (-1)^n \tfrac{1}{2}\pi(p_n^2 + \mu^2)^{1/2}\gamma_n,\end{aligned} \tag{19}$$

providing that $b > b_0$.

In particular, our kernel $K(q',q)$ above satisfies this condition in the q' variable, and hence

$$\begin{aligned}K_{nm} &= \tau_n \int_0^\infty dq q^2 \phi_m(q) K(p_n, q) \\ &= -\frac{\pi^3 \tau_n}{8b^2}\theta_0(p_n, \pi/2b; b)\phi_m(\pi/2b),\end{aligned} \tag{20}$$

the last line following in similar fashion to Eq. (12).[10] If we again consider the diagonal elements in this array, we now find that $K_{nm} \propto p_n{}^{-1}$. We have therefore gained a convergence factor of p^{-2} in comparison with direct Gaussian quadrature. The results of numerical calculations of λ for $b = 1.8$, $\mu = 1.0$ are given in Table I as a function of the number of terms N in the expansion (the exact result is $\lambda = -0.90$). It is clear that the procedure works quite well for such kernels.

TABLE I. Convergence of model eigenvalue calculation.

N	λ
3	−0.898 658
4	−0.899 411
5	−0.899 693
10	−0.899 961
15	−0.899 989
20	−0.899 996

In applying this technique to Eq. (4), however, one must keep in mind that the ϕ_n are only complete on the subspace corresponding to $y < b$. In fact, taking b to be the same quantity introduced with respect to that equation, and defining

$$\begin{aligned}\langle q' | K^{\text{int}} | q\rangle &= \frac{N(q',q) - \bar{N}(q',q)}{D_0(\kappa)}, \\ \langle q' | K^{\text{ext}} | q\rangle &= \frac{\bar{N}(q',q)}{D_0(\kappa)}, \\ \langle \phi_n | q\rangle &= \phi_n(q),\end{aligned} \tag{21}$$

we have

$$\begin{aligned}\langle \phi_n | K^{\text{ext}} &= 0, \\ \langle \phi_n | K^{\text{int}} &= \tau_n \langle p_n | K^{\text{int}}.\end{aligned} \tag{22}$$

Nevertheless, we may extend our expansion procedure to equations of this type in the following manner. Writing Eq. (4) in operator notation as $X = \Omega + KX$, we note that $X = Z^e X_0$, where

$$\begin{aligned}X_0 &= \Omega + K^{\text{int}} Z^e X_0, \\ Z^e &= (1 - K^{\text{ext}})^{-1}.\end{aligned} \tag{23}$$

No difficulty arises in constructing Z^e numerically via Gaussian quadrature; $K^{\text{ext}}(q,q)$ decreases *exponentially* for large q, providing $b > \frac{3}{2}a$. Moreover, we may expand X_0 in the form

$$X_0 = \Omega + \sum_n |\phi_n\rangle\langle X_n|, \tag{24}$$

where the $\langle X_n|$ must satisfy

$$\begin{aligned}\langle X_n| &= \langle \phi_n | K^{\text{int}} Z^e \Omega + \sum_m K_{nm}\langle X_m|, \\ K_{nm} &= \langle \phi_n | K^{\text{int}} Z^e | \phi_m\rangle.\end{aligned} \tag{25}$$

The solution to Eq. (4) can thus be obtained by solving first for the exterior operator Z^e by standard methods, constructing the K_{nm} matrix elements, and solving the resultant finite matrix equation for the $\langle X_n|$. As was noted previously, K^{ext} bears a close resemblance to the kernels which have become familiar in the three-body problem under the assumption of separable interactions; the application of Gaussian techniques to this problem is well documented.[11] The numerical results of Sec. IV demonstrate that the expansion for the interior solution converges quite rapidly (to better than 1% for $N = 6$).

In concluding this section, we note that the primary drawback to our procedure is the time required to calculate the K_{nm} moments, although the n moment is trivial as a result of Eq. (20). One can achieve far greater speed and accuracy in

this task by fully exploiting the analytic properties of the functions involved. For example, consider the following integral:

$$\Lambda_n = \int_0^\infty dq q^2 \phi_n(q) F(q) G(q), \tag{26}$$

where $F(q)$ has the properties stated prior to Eq. (17), and

$$G(q) = \int_\beta^\infty \frac{dx g(x)}{x^2+q^2}. \tag{27}$$

Here $g(x)$ may be a bounded integrable function or a δ function. Again employing the calculus of residues, it is straightforward to obtain

$$\Lambda_n = \tau_n F(p_n) G(p_n) + \frac{\pi \gamma_n}{2} \int_\beta^\infty \frac{dx g(x)}{x^2+p_n^2} (x-\mu) e^{-bx} F(ix). \tag{28}$$

In contrast to direct numerical integration of Eq. (26), which must cope with the infinite sign variations of F and ϕ_n, the integral in Eq. (28) is exponentially damped, and far more amenable to numerical treatment.

IV. NUMERICAL RESULTS

In applying the numerical techniques of Sec. III to Eq. (4), two distinct types of calculations were performed. In the first case, the determinant $|1-K|$ was computed for the $N \times N$ array $(\delta_{nm} - K_{nm})$ defined by Eq. (25). As in potential formulations of the three-body problem, this determinant approaches unity as $W \to -\infty$, decreasing with increasing W if the interaction is primarily attractive. If the interaction is sufficiently attractive, three-particle bound states manifest themselves as zeros of the determinant, corresponding to values $W = -E_B$ for which the homogeneous version of Eq. (4) has a nontrivial solution. In terms of the parameter f_0 characterizing our model, the interaction is attractive if $f_0 < 1$, achieving a two-particle bound state if $f_0 < 0$.

The second type of calculation determined $X(\omega)$, the amplitude for elastic scattering from an initial two-body bound state. For this purpose parameters (a, f_0) characteristic of the n-p triplet state were employed. Inasmuch as this model has no direct physical application, our primary purpose was to explore the qualitative consequences of the formalism, and to gain experience in handling a somewhat more singular integral equation. It thus sufficed to restrict our computation to energies $W<0$ below the break-up threshold; the numerical difficulties involved for $W>0$ apply only to the $\bar{N}$ portion of our kernel and are well understood.[11]

TABLE II. Convergence of binding energy calculation.

N	E_B (F^{-1})
5	0.0931
6	0.0934
7	0.0932
8	0.0934
9	0.0933
10	0.0933

As was shown in SCI, the requirement that each two-particle subsystem satisfy Eq. (2) does not uniquely determine the three-particle wave function. The presence of singular cores thus introduces an essential ambiguity into the description of n-body systems if $n>2$. In order to arrive at a unique result, it is necessary to impose an auxiliary boundary condition, the form of which is essentially determined by three-particle unitarity. A particularly simple constraint of this type was introduced in SCI, and is manifested in the presence of the W_0-dependent term in Eq. (6). If one makes a Faddeev-type channel decomposition of the three-particle wave function,[12] $\Psi = \sum_\beta \psi_\beta$, this constraint can be visualized in terms of a boundary condition on the ψ_β of the same type as Eq. (2), *at energy* $W = W_0$. In the particular case of a hard core ($f_0 \to \infty$), this corresponds to the requirement that each of the ψ_β's vanish independently within the core at $W = W_0$, as well as their sum (which vanishes for all W).[13] Although this not unreasonable prescription guarantees unitarity and leads to a unique three-particle wave function for each W_0, the freedom to choose W_0 arbitrarily implies that the model defined in Sec. II in fact generates a one-parameter family of three-body solutions. However, the numerical results to be described display a remarkable insensitivity to this parameter, with the consequence that the two-body boundary-condition parameters imply virtually unambiguous properties for the three-body system.

Three-particle binding energies based on the n-p triplet parameters are tabulated in Table II as a function of the number of terms N in the ϕ_n expansion; W_0 was held fixed at a value of -0.10 F^{-1}.

TABLE III. Binding energies for triplet parameters.

W_0 (MeV)	E_B (MeV)
−2.3	18.41
−18.4	18.41
−98.7	18.41
−197.0	18.41
−395.0	18.42
−789.0	18.75

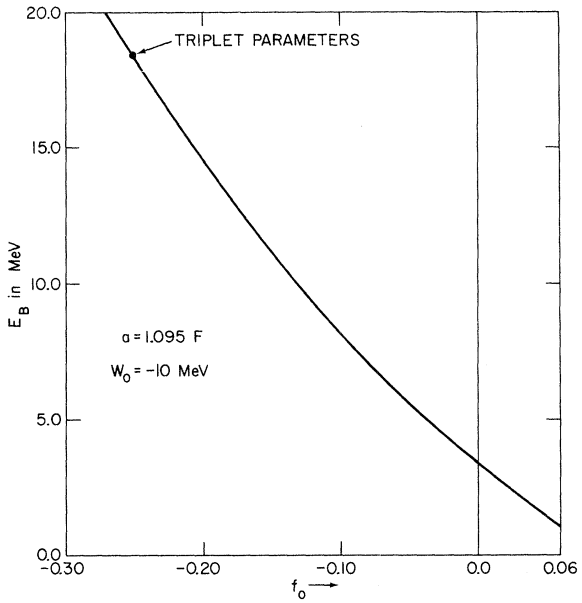

FIG. 1. Three-particle binding energy E_B as a function of the logarithmic derivative parameter f_0, for fixed $W_0 = -10$ MeV and the triplet-core radius.

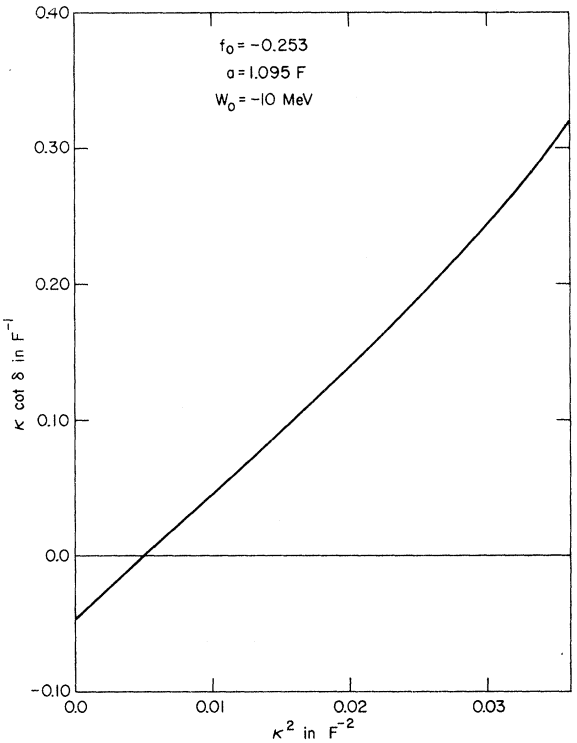

FIG. 2. Plot of $\kappa \cot\delta$ against κ^2 for triplet parameters and $W_0 = -10$ MeV. The calculation is below the break-up threshold ($\kappa^2 \simeq 0.07$ F^{-2}).

It is clear that convergence is quite rapid, with as few as six terms producing better than 1% accuracy. This was in fact the case for all of the numerical results reported. Variation of the binding energy as a function of W_0 for the same f_0, a parameters is shown in Table III for $N = 10$. As stated above, there is virtually no sensitivity to this parameter unless extremely large values are employed. The simple model under consideration thus implies an essentially unambiguous value of 18.41 MeV binding for the three-body system. This is on the order of 4–10 MeV more than simple potential models fitted to the same two-body data give for this problem.[14] In fact, E_B is extremely sensitive to variations in f_0, a; a similar calculation with the N-N singlet parameters produces a value of only 1.13 MeV binding.

As a check on the qualitative behavior of E_B with the boundary-condition parameters, f_0 was varied and a held fixed at 1.095 F; the results are plotted as Fig. 1. Aside from the large sensitivity to f_0 already noted, which may be interpreted as sensitivity to the value of the two-particle binding energy produced, there is no qualitative difference with similar curves produced by potential models.[14] In particular, the three-body system binds for a range of f_0 parameters too weakly attractive to bind the two-body systems.

Calculations of elastic scattering from the two-body bound state are plotted in Fig. 2 in terms of

$$\omega \cot\delta = i\omega - \frac{3}{2\pi MX(\omega)} \; ; \qquad (29)$$

to conform with the usual notation we have replaced ω by k. The curve shown corresponds to $W_0 = -10$ MeV, but variations of the curve for the values of W_0 listed in Table III would not be visible on the scale of the figure. Again, taking into account the unusually large binding energy produced (18.4 MeV), these results do not differ in a qualitative sense from previous computations by the present author for potential models.

Thus, although "realistic" singular-core models have yet to be confronted with experiment, prospects for such calculations within the formalism previously introduced seem quite encouraging. The numerical stability of the results obtained demonstrates the practicality of the basic approach, and the qualitative similarities with results produced by potential models provide an important sense of continuity. The quantitative differences exhibited, such as the rather large binding energy produced by the triplet parameters, offer some hope that such calculations may provide a definitive test of the various phenomenological treatments of the short-range N-N interaction. As a bonus, the unwanted degree of freedom afforded by the introduction of the W_0 parameter does not seem to have any practical consequences, thus permitting an unambiguous test of a particular singular-core model.

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[1]D. D. Brayshaw, Phys. Rev. D 7, 1835 (1973); hereafter we shall refer to this paper as SCI.

[2]L. M. Delves, J. M. Blatt, I. Pask, and B. Davies, Phys. Lett. 28B, 472 (1969), and earlier work quoted therein.

[3]The most relevant calculations have been performed with the Reid potential; see, for example, C. Gignoux and A. Laverne, Phys. Rev. Lett. 29, 436 (1972); E. P. Harper, Y. E. Kim, and A. Tubis, *ibid.* 28, 1533 (1972). More recent results add further corroboration to the conclusions of these papers (Y. E. Kim, personal communication).

[4]D. D. Brayshaw, Phys. Rev. D 8, 952 (1973).

[5]In deriving this from the expressions given in SCI, one should remove an incorrect factor of $\frac{1}{3}$ which appears in Eqs. (101) and (102).

[6]The interpretation of this parameter is discussed in Sec. IV.

[7]H. Feshbach and E. Lomon, Phys. Rev. Lett. 6, 635 (1961).

[8]For simplicity we delete the explicit definitions of these operators; qualitatively, however, U is an operator which vanishes in the case of no external potential, while ρ approaches unity in that limit.

[9]A more detailed discussion regarding the analytic solution of such integral equations can be found in D. D. Brayshaw, Phys. Rev. C 3, 35 (1971).

[10]Alternatively, we could define the ϕ_n in terms of $b' = 2b$, and employ Eq. (17) twice to obtain $K_{nm} = \tau_n \tau_m \times K(p_n, p_m)$. This is more directly analogous to the procedure employed in practice.

[11]See, for example, J. H. Hetherington and L. H. Schick, Phys. Rev. 137, B935 (1965); R. Aaron and R. D. Amado, Phys. Rev. 150, 857 (1966).

[12]For a discussion of the three-body problem in the coordinate representation see H. P. Noyes, Phys. Rev. Lett. 23, 1201 (1969).

[13]As shown in SCI, requiring the ψ_β to vanish within the core for all W is not compatible with three-body unitarity.

[14]Computations for a number of simple potentials are reported by J. W. Humberston, R. L. Hall, and T. A. Osborn [Phys. Lett. 27B, 195 (1968)]. A previous calculation with the same BCM parameters within the standard Faddeev formalism produced a value of 12.7 MeV; see Y. E. Kim and A. Tubis, *ibid.* 38B, 354 (1972).

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Schwinger Terms and the Light-Cone Expansion

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A singularity on the light cone is conjectured which accounts for the Schwinger term required for the consistency of the Lee–Dashen–Gell-Mann sum rule with that of Cabibbo and Radicati. This singularity requires W_3 to scale, rather than (as conventionally assumed) νW_3; because of positivity constraints, such behavior forbids in principle the usual scaling law for W_1 and νW_2.

I. INTRODUCTION

The scaling behavior of deep-inelastic electron scattering successfully predicted by Bjorken[1] has led to the study of current commutators at almost lightlike distances.[2] A simple form for the light-cone expansion of the commutator of two electromagnetic currents was abstracted from the observed scaling behavior of the structure functions. It was subsequently observed that the SLAC-MIT experiments seem to indicate that the operators appearing in the light-cone expansion have canonical dimensions, and that the expansion could have a structure which is compatible with that given by free-field theory. These observations led Gell-Mann and Fritzsch to the assumption that the light-cone expansion of commutators involving SU(3)×SU(3) currents can be abstracted from the free-quark model, making the connection between the light-cone and parton approaches very clear.[3]

Some years ago, Buccella, Gatto, Okubo, and Veneziano[4] showed that in order for the Jacobi identity between three space-current components to be satisfied, a q-number Schwinger term (ST), antisymmetric with respect to interchange of unitary indices, must be present in the equal-time commutator (ETC) of two space components of currents. The quark model, however, suggests the commutation relation

$$[J_a^i(x), J_b^k(0)]_{x_0=0} = i\delta_{ik} f_{abc} J_c^0(x)\delta^3(x) + \text{terms symmetric in } a, b \,. \tag{1}$$

Therefore, we do not expect the Jacobi identity to be satisfied in the free-quark model.

Motivated by these remarks, we study here the