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PHYSICAL REVIEW D VOLUME 8, NUMBER 8 15 OCTOBER 1973

Two-Variable Galilei-Group Expansions of Nonrelativistic Scattering Amplitudes

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(Received 21 March 1973)

Two-variable Galilei-group expansions are derived for the two-particle nonrelativistic scattering amplitude. These expansions contain the usual partial-wave and eikonal expansions and supplement them by further expansions in the remaining kinematic variable. The expansions are written both for square-integrable and for asymptotically increasing amplitudes and are shown to correspond to the nonrelativistic limit of previously considered relativistic two-variable expansions. Dynamical singularities (Breit-Wigner resonances, bound states, poles in the impact-parameter plane, etc.) are investigated and related to the asymptotic behavior of the expansion coefficients (or Galilei amplitudes). The threshold and high-energy limits of the expansions are discussed. As a mathematical by-product we give a classification of the subgroups of E(3) and also some results on the representation theory of this group; in particular we study the Clebsch-Gordan coefficients.

I. INTRODUCTION

Two-variable expansions of relativistic scattering and decay amplitudes have been proposed in previous publications[1-6] as a tool in terms of which to study elementary-particle interactions. The motivation for writing such expansions is to treat as many as possible of the general properties of scattering amplitudes (Lorentz invariance, analyticity, crossing symmetry, unitarity, etc.) once and for all as "kinematics" and thus to isolate the actual "dynamics" of interest.

This is achieved by considering the scattering amplitudes as functions over the three-dimensional hyperboloid $p^2 = m^2$ (where p is the four-momentum of one of the particles involved and m is its mass) for spinless particles, or over the homogenous Lorentz group O(3, 1) for particles with spin, and then expanding them in terms of irreducible representations of O(3, 1). The entire dependence of the amplitudes on all kinematic parameters such as energies, scattering angles, momentum transfers, etc., is displayed explicitly in known special functions—the basis functions for the representations or the matrix elements of finite transformations. The dynamics of each specific process should then be described in terms of the corresponding expansion coefficients, which we call "Lorentz amplitudes."

Relativistic two-variable expansions for reactions of the type $1+2\to 3+4$ and $1\to 2+3+4$ are thus provided by the representation theory of the Lorentz group O(3, 1). This group figures as the group of motions of the space of independent kinematic parameters, and its crucial role is of course related to the Lorentz invariance of the theory. The actual form of the expansions is not determined uniquely by the choice of the group itself, but depends also on the choice of a definite Lorentz frame of reference, in which we consider the reaction, and on the corresponding choice of basis for the representations. Two different types of bases have been considered. The first type are "subgroup-type" bases, in that they correspond to a reduction of the considered group [in this case O(3, 1)] to a definite chain of subgroups. The basis functions are eigenfunctions of a complete set of commuting operators, all of which are either Casimir operators (invariant operators) of the group itself or of one of the subgroups in the reduction chain, or discrete operators (e.g., various types of reflections). Among the different nonequivalent chains of subgroups of O(3, 1), three are of special interest in this connection. They can be written as $O(3,1)\supset G\supset O(2)$, where O(2) is the group of rotations in a plane and G is either the three-dimensional rotation group O(3), the three-dimensional Lorentz group O(2, 1), or the Euclidean group of a plane E(2). The important point is that the subgroup G simultaneously figures as a little group of the Poincaré group, leaving a certain vector invariant. The standardization of this vector corresponds to the choice of a certain frame of reference. In the case of O(3) it is the total energy-momentum and the center-of-mass frame, for O(2, 1) it is the momentum transfer p_1-p_3 and the brick-wall frame, and for E(2) it is a specially constructed lightlike vector K and the "light-velocity frame of reference".[1] This identification of the subgroup G provides an interpretation of the expansions in terms of subgroup-type bases, namely, they can be identified with the standard "little-group" expansions[7] of partial-wave analysis [the group O(3)], Regge-pole theory [group O(2, 1)], and the eikonal expansion [group E(2)]. These little-group expansions are supplemented by a further integral expansion of the corresponding partial-wave amplitude, provided by the inclusion of G into O(3, 1) and making the dependence on all variables explicit.

The second type of bases are "nonsubgroup bases," in which the basis functions are again defined as the common eigenfunctions of a complete set of commuting operators, not all of which are Casimir operators of any subgroup (some are more general second-order operators in the enveloping algebra of the corresponding Lie algebra).[2,5] The existence of such bases is related to the separation of variables in the Laplace operator on the space, where the group acts transitively, in elliptic-type coordinates. The most interesting feature of the expansions in terms of nonsubgroup basis functions is that they have very simple properties with respect to the interchange of variables and are thus very appropriate for expansions of crossing-symmetric functions.

All of the two-variable O(3, 1) expansions of scattering amplitudes $F(s,t)$ can thus be written in the form

$$F(s,t)=\sum_{\sigma,l}A(\sigma,l)\Phi_{\sigma l}(\alpha)\Psi_{\sigma l}(\beta), \quad (1)$$

where $\Phi_{\sigma l}(\alpha)$ and $\Psi_{\sigma l}(\beta)$ are definite known functions, α and β are certain combinations of the Mandelstam variables

$$s=(p_1+p_2)^2,\quad t=(p_1-p_3)^2,\quad u=(p_1-p_4)^2 \quad (2)$$

(p_i are the particle momenta), and $A(\sigma,l)$ are the Lorentz amplitudes, carrying the dynamics. The summations imply either sums or integrals, σ labels representations of O(3, 1), and l either labels those of the subgroup G or corresponds to the eigenvalues of the nonsubgroup-type operators, determining a basis. If the particles have nonzero spins then all the functions and the Lorentz amplitudes will also carry spin projection labels.

Let us for clarity and also for use below write out explicitly two of the subgroup expansions for spinless particles with arbitrary nonzero masses. For the reduction $O(3,1)\supset O(3)\supset O(2)$, we have

$$F(s,t)=\sum_{l=0}^{\infty}(2l+1)\int_{\delta-i\infty}^{\delta+i\infty}(\sigma+1)^2d\sigma\,\frac{\Gamma(\sigma+1)}{\Gamma(\sigma-l+1)}A_l(\sigma)\,\frac{1}{(\sinh a)^{1/2}}\,P_{1/2+\sigma}^{-l-1/2}(\cosh a)\,P_l(\cos\theta), \quad (3)$$

with

$$\cosh a=\frac{s+m_3^{\,2}-m_4^{\,2}}{2m_3\sqrt{s}} \quad (4)$$

and θ equal to the c.m. scattering angle [$P_\nu^\mu(z)$ are Legendre functions, the Γ functions provide a convenient normalization].

The $O(3,1)\supset O(2,1)\supset O(2)$ reduction leads to the expansion

$$F(s,t)=-\frac{1}{2i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{2l+1}{\sin\pi l}\,dl\int_{\delta-i\infty}^{\delta+i\infty}(\sigma+1)^2d\sigma\,\frac{\Gamma(\sigma-l+1)\Gamma(\sigma+l+2)}{\Gamma(\sigma+2)}\cos\pi l$$
$$\times\frac{1}{\cosh\alpha}\left[A^+(\sigma,l)\,P_l^{-\sigma-1}(-\tanh\alpha)+A^-(\sigma,l)\,P_l^{-\sigma-1}(\tanh\alpha)\right]P_l(\cosh\beta), \quad (5)$$

where

$$\sinh\alpha=\frac{t+m_1^{\,2}-m_3^{\,2}}{2m_1\sqrt{-t}} \quad (6)$$

and $\cosh\beta=\cos\theta_t$, i.e., β is the c.m. scattering angle in the t channel, analytically continued into the s channel (this is the variable of Regge-pole theory).

Expansions (3) and (5) can easily be inverted, but we shall not go into this here.

The fundamental objects in this two-variable approach to particle reactions are thus the Lorentz amplitudes like $A_l(\sigma)$ and $A^{\pm}(\sigma, l)$. It is in terms of these quantities that we must ensure that the general principles of scattering theory are satisfied, that we can formulate dynamical hypotheses, and finally that we can perform phenomenological fits to experimental data.

Let us note here that the expansion (3) has been generalized to the case of arbitrary spins[4] and also modified to describe three-body decays, rather than scattering.[3] For decays the O(3, 1) expansions are replaced by O(4) ones, so that the integral over σ in (3) is replaced by a sum. Such expansions have been directly used for fitting data on the $K^{\pm}\to 3\pi$, $\eta\to 3\pi$, and $\bar{p}n\to 3\pi$ reactions.[3,6]

For scattering it is somewhat more difficult to obtain information about the Lorentz amplitudes directly from experimental data. It is first necessary to find a convenient parametrization of the amplitudes and also to find a method of replacing the integrals in (3), (5), and other expansions by sums. In order to do this we need more information about the analyticity properties of the Lorentz amplitudes, their asymptotic behavior, their sensitivity with respect to resonances, and other important properties of the amplitude $F(s, t)$. Such information can be obtained on one hand by imposing general principles of S-matrix theory on $F(s, t)$ and studying their reflection in the properties of the Lorentz amplitudes. On the other hand a better feeling for the properties of these amplitudes can be obtained by studying various specific models.

The aim of this investigation is to consider the nonrelativistic limit of the O(3, 1) expansions, i.e., expansions in terms of the representations of the homogeneous Galilei group, which is isomorphic to the three-dimensional Euclidean group E(3). Presently we only study expansions in terms of the subgroup-type bases corresponding to the group reductions $E(3)\supset O(3)\supset O(2)$ and $E(3)\supset E(2)\times T_{\perp}\supset O(2)\times T_{\perp}$ [where $T_{\perp}$ is the group of translations perpendicular to the plane given by E(2)]. The first of these leads to the partial-wave expansion, supplemented by an integral representation of the partial-wave amplitude; the second leads to the eikonal expansion, again supplemented by an integral representation for the eikonal amplitude.

The main reason for our interest in this nonrelativistic limit is that it is more accessible to a theoretical study than the relativistic case. Indeed, a ready-made model for which we can study the behavior of the expansion coefficients—let us call them Galilei amplitudes—is potential scattering. In this initial article we formulate the problem, i.e., obtain the Galilei expansions for the scattering of spinless particles, study their relation to the relativistic expansions, investigate expansions in terms of unitary and nonunitary representations, and consider their convergence properties and also their behavior in various physically interesting limits. In view of future applications in high-energy physics, the aim is to be able to formulate scattering completely in terms of Galilei (or Lorentz) amplitudes, which thus in the conceptual sense replace the potential.

Section II is devoted to mathematical preliminaries. We classify all subgroups of E(3) into equivalence classes and demonstrate their relation to the subgroups of O(3, 1). Further, we construct the basis functions of E(3), corresponding to various group reductions (for general representations). For representations corresponding to spinless particles we also derive the Clebsch-Gordan coefficients in the $E(3)\supset O(3)\supset O(2)$ basis. In Sec. III we write the direct and inverse expansion formulas for spinless particles using unitary representations or a class of nonunitary ones. We investigate the convergence properties and the way in which O(3, 1) expansions contract into E(3) ones, and also compare the relativistic and nonrelativistic kinematics involved. In Sec. IV we study threshold behavior, various types of asymptotic behavior, and the occurrence of Breit-Wigner-type resonances in the $E(3)\supset O(3)$ expansion and various types of asymptotic behavior, particularly Regge poles, in the eikonal-type E(3) expansion. In the final section, V, we summarize the results and discuss the future outlook.

II. MATHEMATICAL PRELIMINARIES

A. The Euclidean Group E(3) and the Lorentz Group O(3,1)

In order to establish notation, let us review a few important facts[8,9] about the groups E(3) and O(3, 1).

The group E(3). This is the group of distance-preserving transformations of a three-dimensional Euclidean space and it is a semidirect product of the group of rotations O(3) and translations T(3). Denoting the generators of rotations by L_i and those of translations by P_i, we have

$$\begin{aligned} [L_i, L_k] &= i\epsilon_{ikl} L_l , \\ [L_i, P_k] &= i\epsilon_{ikl} P_l , \\ [P_i, P_k] &= 0 . \end{aligned} \tag{7}$$

The two Casimir operators of E(3) are

$$\vec{P}^2 \text{ and } \vec{P}\cdot\vec{L} . \tag{8}$$

All unitary continuous faithful representations of E(3) are infinite-dimensional and are labeled by the pair of real numbers $\{k, m_0\}$, where $k>0$ and

$2m_0 = 0, \pm 1, \pm 2, \ldots$. In any unitary irreducible representation we have

$$\vec{P}^2 = k^2, \quad \vec{P}\cdot\vec{L} = -km_0. \tag{9}$$

Representations of E(3) are usually considered in the canonical basis $|km_0LM\rangle$, corresponding to the group reduction $E(3)\supset O(3)\supset O(2)$, in which

$$\begin{aligned}\vec{L}^2|km_0LM\rangle &= L(L+1)|km_0LM\rangle,\\ L_3|km_0LM\rangle &= M|km_0LM\rangle,\end{aligned} \tag{10}$$

with

$$\begin{aligned}L &= |m_0|, |m_0|+1, \ldots,\\ M &= -L, -L+1, \ldots, L.\end{aligned} \tag{11}$$

Note that the case $k=0$ is exceptional in that it corresponds to finite-dimensional discrete representations for which the generators of translations are represented by $P_i \equiv 0$. We shall not be concerned with these representations in the present article.

The group O(3, 1). The homogeneous Lorentz group O(3, 1) is generated by infinitesimal rotations L_i and infinitesimal boosts (pure Lorentz transformations) K_i along each coordinate axis. The Lie algebra of O(3, 1) is given by the relations

$$\begin{aligned}[L_i, L_k] &= i\epsilon_{ikl} L_l,\\ [L_i, K_k] &= i\epsilon_{ikl} K_l,\\ [K_i, K_k] &= -i\epsilon_{ikl} L_l.\end{aligned} \tag{12}$$

The two Casimir operators of O(3, 1) are

$$\Delta = \vec{L}^2 - \vec{K}^2 \quad\text{and}\quad \Delta' = \vec{L}\cdot\vec{K}. \tag{13}$$

The unitary irreducible representations of the principal series are characterized by two real numbers $\{\rho, j_0\}$, with ρ real and $j_0 = 0, \frac{1}{2}, 1, \ldots$. In any unitary representation of the principal series we have

$$\Delta = j_0^2 - \rho^2 - 1, \quad \Delta' = -j_0\rho. \tag{14}$$

Contraction of O(3, 1) *to* E(3). It is well known[10,11] that the algebra of E(3) can be obtained from that of O(3, 1) by the process of contraction corresponding to taking the limit $c\to\infty$ (where c is the velocity of light). Indeed, putting

$$K_i = cP_i, \tag{15}$$

we rewrite (12) as

$$\begin{aligned}[L_i, L_k] &= i\epsilon_{ikl} L_l,\\ [L_i, P_k] &= i\epsilon_{ikl} P_l,\\ [P_i, P_k] &= -\frac{i}{c^2}\epsilon_{ikl} L_l,\end{aligned} \tag{16}$$

which for $c\to\infty$ reduces to the E(3) relations (7). From the physical point of view this contraction obviously corresponds to a transition from relativistic to nonrelativistic kinematics (see also below).

B. Subgroups of the Euclidean Group E(3)

We shall be interested in expansions of scattering amplitudes in terms of basis functions of the irreducible representations of E(3). Since a different type of basis corresponds to each nonequivalent chain of subgroups (having invariants), we need a classification of all continuous subgroups of E(3) into equivalence classes (the results are summarized in Table I). We perform this classification using methods previously applied to classify subgroups[12,13] of O(3, 1) and SU(2, 1). We shall actually classify subalgebras of the algebra of E(3) and then comment on the corresponding groups. Two subalgebras A_1 and A_2 of the algebra A of the Lie group G will be considered equivalent if for every $a_1\in A_1$ there exists $g\in G$ and $a_2\in A_2$ such that $ga_1g^{-1} = a_2$, i.e., if there exists an inner automorphism, transforming A_1 into A_2.

Let us consider a general element of the algebra of E(3):

$$C = a_i L_i + b_i P_i. \tag{17}$$

A general inner automorphism can be considered by setting

$$P_i = -i\frac{\partial}{\partial x_i}, \quad L_i = -i\epsilon_{ikl}x_k\frac{\partial}{\partial x_l} \tag{18}$$

and performing an E(3) transformation

$$x_i' = \alpha_{ik}x_k + \beta_i, \quad \alpha_{ik}\alpha_{il} = \delta_{kl} \tag{19}$$

(α_{ik} and β_i are real; summation over identical subscripts is understood). Under the transformation (19) the operator C transforms into C' (Ref. 14):

$$\begin{aligned}C' &= a_i\alpha_{ik}L_k + (a_i\epsilon_{iml}\beta_m\alpha_{lk} + b_i\alpha_{ik})P_k\\ &= a_k'L_k + b_k'P_k.\end{aligned} \tag{20}$$

Two quantities remain invariant under the automorphism $C\to C'$, namely,

$$a^2 = a_ia_i = a_i'a_i', \tag{21}$$

$$\vec{a}\cdot\vec{b} = a_ib_i = a_i'b_i'. \tag{22}$$

It is easy to check that by a judicious choice of the rotations α_{ik} and translations β_i we can transform the most general element C into $A = aL_1 + bP_1$, which in turn we can normalize into

$$A(\alpha) = \cos\alpha\, L_1 + \sin\alpha\, P_1, \quad 0\le\alpha<\pi. \tag{23}$$

In other words, a general one-parameter subalgebra determined by the operator (17) is equivalent to an algebra generated by $A(\alpha)$ (with $a_i^2 = N^2\cos^2\alpha$; $a_ib_i = N^2\sin\alpha\cos\alpha$, N real). Algebras correspond-

TABLE I. Continuous subgroups of the group E(3).

No.	Dimension	Generators and algebra	Invariants of algebra	Characterization of group
1	1	$A(\alpha)=L_1\cos\alpha+P_1\sin\alpha$ $0\le\alpha<\pi$	$A(\alpha)$	Compact for $\alpha=0$ (rotations in plane). Noncompact for $\alpha\ne0$. Translations along one axis for $\alpha=\frac{1}{2}\pi$.
2	2	L_1,P_1; $[L_1,P_1]=0$	L_1 and P_1	Noncompact, Abelian. Translations on a cylinder.
3	2	P_1,P_2; $[P_1,P_2]=0$	P_1 and P_2	Noncompact, Abelian. Translations on a Euclidean plane.
4	3	L_1, L_2, L_3; $[L_i,L_k]=i\epsilon_{ikl}L_l$	$L_1^2+L_2^2+L_3^2$	Compact, simple: the rotation group O(3)
5	3	P_1, P_2, $L_3+bP_3\equiv M$ $0\le b<\infty$ $[P_1,P_2]=0$, $[M,P_1]=iP_2$, $[P_2,M]=iP_3$	$P_1^2+P_2^2$	Noncompact, solvable: the Euclidean group E(2)
6	3	P_1, P_2, P_3; $[P_i,P_k]=0$	P_1, P_2, and P_3	Noncompact, Abelian: the group of translations $T_1\times T_2\times T_3$ of the Euclidean space.
7	4	P_1, P_2, L_3, P_3; $[P_i,P_k]=0$, $[P_3,L_3]=0$, $[L_3,P_1]=iP_2$, $[P_2,L_3]=iP_1$	$P_1^2+P_2^2$ and P_3	Noncompact, decomposable: the group $E(2)\times T_\perp$, where $T_\perp$ are translations perpendicular to the E(2) plane.

ing to different values of α are nonequivalent. We thus obtain a single one-parameter family of one-dimensional subalgebras $A(\alpha)$.

The classification of two-parameter, three-parameter, etc. subalgebras is now a simple matter and we proceed from lower to higher ones. We shall denote a subalgebra by $\mathfrak{L}$ and its derived algebra (algebra of commutators) by $\mathfrak{L}'$, and order the algebras according to the dimension of $\mathfrak{L}$ and $\mathfrak{L}'$.

$$\dim\mathfrak{L}=2,\ \dim\mathfrak{L}'=0:\ [A,B]=0. \tag{24}$$

We take A in the form (23), leave B general, and demand that A and B commute. This restricts the form of B, which we can further simplify, using the automorphism (19). We find that only two nonequivalent Abelian subalgebras exist and their respective bases can be given by the pairs of operators $\{P_1, P_2\}$ and $\{P_1, L_1\}$.

$$\dim\mathfrak{L}=2,\ \dim\mathfrak{L}'=1:\ [A,B]=iA. \tag{25}$$

Choosing A in the form (23), we immediately find that (25) cannot be satisfied for any B. No other two-dimensional algebras exist.

Eight types of three-dimensional Lie algebras exist over the field of real numbers.[15] All those containing a subalgebra of the type (25) may be eliminated immediately.

$$\dim\mathfrak{L}=3,\ \dim\mathfrak{L}'=0:\ [A,B]=[B,C]=[C,A]=0. \tag{26}$$

We can identify $\{A,B\}$ with one of the algebras (24) and add a general C, commuting with A and B. The only such algebra that can be obtained is $\{P_1, P_2, P_3\}$.

$\dim\mathfrak{L}=3$, $\dim\mathfrak{L}'=1$. Such an algebra must contain the subalgebra (25) and is hence ruled out.

$\dim\mathfrak{L}=3$, $\dim\mathfrak{L}'=2$. We identify $\mathfrak{L}'$ with $\{A,B\}$, hence A and B commute. In general, we can write

$$[A,B]=0,\quad [B,C]=iaA,\quad [A,C]=i(bA+dB), \tag{27}$$

where a, b, d are real. If we put $\{A,B\}=\{P_1, L_1\}$, then no C satisfying (27) exists. Choosing $A=P_1$, $B=P_2$, we find $C=L_3+bP_3$ (b real). This cannot be further simplified.

$\dim\mathfrak{L}=3$, $\dim\mathfrak{L}'=3$. Only two such algebras exist, namely that of O(3) and that of O(2, 1). The algebra of O(2, 1) contains a subalgebra of the type (25) and is hence ruled out. The algebra of O(3) is of course present. Writing $A=aL_1+bP_1$ and keeping B and C general, we find that the O(3) commutation relations can only be satisfied if $A=L_1$, $B=L_2+bP_3$, $C=L_3-bP_2$. This can be simplified by the E(3) inner automorphism to the algebra $\{L_1, L_2, L_3\}$.

Twenty types of four-dimensional algebras over the field of real numbers exist,[15] but nineteen of them can be ruled out because they contain two- or three-dimensional subalgebras that have already been ruled out. The remaining one is a direct sum of the algebra (26) and a one-parameter subalgebra. We thus have

$$\dim\mathfrak{L} = 4\,,\ \ \dim\mathfrak{L}' = 2\,.$$

Put $A = P_1$, $B = P_2$, $C = L_3 + bP_3$ and leave D general. Requiring that D commute with all other generators, we find that only one such subalgebra exists, namely that generated by $\{P_1, P_2, P_3, L_3\}$.

The algebra of E(3) has no five-parameter subalgebra. Indeed, a five-parameter subalgebra would be obtained by leaving out one element of the algebra. This can be chosen in the standard form $A(\alpha) = \cos\alpha L_1 + \sin\alpha P_1$. The subalgebra thus contains L_2, L_3, P_2, P_3, and $\cos\beta L_1 + \sin\beta P_1$ with $\beta \neq \alpha$. However, $[L_2, L_3] = iL_1$ and $[L_2, P_3] = iP_1$; hence the subalgebra also contains $A(\alpha)$. This is a contradiction; therefore no five-parameter subalgebra exists.

A complete list of all continuous nontrivial subgroups of E(3) together with some of their properties is given in Table I. A point which we wish to stress is that E(3) does not have an O(2, 1) subgroup. We know that in a relativistic theory it is the O(2, 1) subgroup of the Lorentz group that provides a group-theoretical basis for complex angular momentum theory. No such possibility exists in nonrelativistic scattering theory.

Let us note that the only chains of subgroups of E(3) that will provide us with subgroup-type bases for the representations of E(3) are $E(3) \supset O(3) \supset O(2)$, $E(3) \supset E(2) \times T_\perp \supset O(2) \times T_\perp$, and $E(3) \supset T_1 \times T_2 \times T_3$ [T_i are translations along the ith axis, $T_\perp$ are translations perpendicular to the E(2) plane].

A few words on the connection between the subgroups of O(3, 1) and E(3) are in order. A general element of the algebra of O(3, 1) is $X = \alpha_i L_i + \beta_i K_i$. Under a general O(3, 1) inner automorphism we have $X \to X'$; the invariants of the automorphism are[12]

$$\vec{\alpha}^2 - \vec{\beta}^2 = \sum_{i=1}^{3} (\alpha_i^2 - \beta_i^2) \quad \text{and} \quad \vec{\alpha} \cdot \vec{\beta} = \sum_{i=1}^{3} \alpha_i \beta_i\,. \tag{28}$$

If we set $\alpha_i = a_i$ and $\beta_i = b_i/c$ and then take the limit $c \to \infty$, we find that the O(3, 1) invariants (28) go over into the E(3) ones (21). Similarly, subalgebras of O(3, 1) contract to those of E(3). The results of this contraction procedure ($\beta_i = b_i/c$, $P_i = K_i/c$, $c \to \infty$) are summarized in Table II.

C. Basis Functions for Zero-Spin Representations of E(3) in Subgroup-Type Bases

From now on we shall only consider unitary irreducible representations of E(3), for which $m_0 = 0$ [see (9)], so that the invariant operator $\vec{P} \cdot \vec{L}$ is also zero. We shall call them zero-spin or degenerate representations. They are the only ones that can be realized in the space of functions $F(\vec{x})$

$$\int |F(\vec{x})|^2 dx_1 dx_2 dx_3 < \infty\,, \tag{29}$$

where $\vec{x}$ is a vector in the Euclidean three-space on which E(3) acts transitively. We shall return to the $m_0 \neq 0$ representations of E(3), relevant for particles with spin, in a separate study.

1. The E(3)⊃O(3)⊃O(2) *Basis (the Spherical Basis)*

The basis functions are eigenfunctions of $\vec{P}^2$, $\vec{L}^2$, and L_3 [see (9) and (10)]. Introducing spherical co-

TABLE II. Contraction of continuous subgroups of O(3, 1) into those of E(3).

No.	Dimension	Subalgebra of O(3, 1)	Subalgebra of E(3)	Comment
1	1	$L_1\cos\beta + K_1\sin\beta$	$L_1\cos\alpha + P_1\sin\alpha$	
2	1	$K_1 + L_2$	P_1	Independent for O(3, 1), contained in case 1 for E(3)
3	2	L_1, K_1	L_1, P_1	
4	2	$L_1 + K_2, L_2 - K_1$	P_1, P_2	
5	2	$L_1 + K_2, K_3$	P_2, P_3	Type $[A, B] = iA$ for O(3, 1). Abelian and equivalent to case 4 for E(3)
6	3	L_1, L_2, L_3	L_1, L_2, L_3	O(3) contracts to O(3)
7	3	K_1, K_2, L_3	P_1, P_2, L_3	O(2, 1) contracts to E(2)
8	3	$K_1 + L_2, K_2 - L_1,$ $\sin\alpha K_3 + \cos\alpha L_3$	$P_1, P_2,$ $\sin\alpha P_3 + \cos\alpha L_3$	A subalgebra of O(3, 1) with $\dim\mathfrak{L}' = 2$ contracts to E(2) for $\alpha \neq \frac{1}{2}\pi$ and to $T_1 \times T_2 \times T_3$ for $\alpha = \frac{1}{2}\pi$.
9	4	$L_3, K_3, L_1 + K_2, L_2 - K_1$	L_3, P_1, P_2, P_3	A subalgebra of O(3, 1) with $\dim\mathfrak{L}' = 2$ and no invariants contracts into $E_2 \times T_\perp$.

ordinates

$$x_1 = r\sin\theta\cos\phi\,,\quad x_2 = r\sin\theta\sin\phi\,,\quad x_3 = r\cos\theta\,, \tag{30}$$

we write $\vec{P}^2$, $\vec{L}^2$, and L_3 as differential operators in spherical coordinates and separate variables in the equations

$$\vec{P}^2|klm\rangle = k^2|klm\rangle\,,\quad \vec{L}^2|klm\rangle = l(l+1)|klm\rangle\,,\quad L_3|klm\rangle = m\,|klm\rangle\,. \tag{31}$$

Solving the ordinary differential equations obtained after separation, we find the basis functions:

$$|klm\rangle = j_l(kr)Y_{lm}(\theta,\phi) \equiv \Psi_{klm}(r,\theta,\phi)\,, \tag{32}$$

where

$$j_l(kr) = \left(\frac{\pi}{2kr}\right)^{1/2} J_{l+1/2}(kr)$$

are spherical Bessel functions[8] and $Y_{lm}(\theta,\phi)$ are spherical harmonics. The basis functions satisfy the following orthogonality and completeness relations:

$$\langle k'l'm'|klm\rangle = \int_0^\infty r^2 dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi\, j_l(kr)Y_{lm}(\theta,\phi) j_{l'}^*(k'r)Y_{l'm'}^*(\theta,\phi) = \frac{\pi}{2}\frac{\delta(k-k')}{k^2}\delta_{ll'}\delta_{mm'} \tag{33}$$

and

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_0^\infty k^2 dk\, j_l(kr)Y_{lm}(\theta,\phi) j_l^*(kr')Y_{lm}^*(\theta',\phi') = \frac{\pi}{2}\frac{\delta(r-r')}{r^2}\frac{\delta(\theta-\theta')}{\sin\theta}\delta(\phi-\phi')\,. \tag{34}$$

2. *The* $E(3)\supset E(2)\times T_\perp \supset O(2)\times T_\perp$ *Basis (the Cylindrical Basis)*

The basis functions are eigenfunctions of the operators $\vec{P}^2$, P_3, and L_3 (and hence also of $P_1^2+P_2^2$). We introduce cylindrical coordinates

$$x_1 = \rho\cos\phi\,,\quad x_2 = \rho\sin\phi\,,\quad x_3 = z\,,\quad 0\le\rho<\infty\,,\quad 0\le\phi<2\pi\,,\quad -\infty<z<\infty\,, \tag{35}$$

write $\vec{P}^2$, P_3, and L_3 as differential operators in cylindrical coordinates, and separate variables in the equations

$$\vec{P}^2|\kappa q m\rangle = k^2|\kappa q m\rangle\,,\quad (P_1^2+P_2^2)|\kappa q m\rangle = \kappa^2|\kappa q m\rangle\,,\quad P_3|\kappa q m\rangle = q|\kappa q m\rangle\,,\quad L_3|\kappa q m\rangle = m\,|\kappa q m\rangle \tag{36}$$

(we drop the symbol k in the basis functions, labeling the representations, where $k^2=\kappa^2+q^2$). Solving the obtained ordinary differential equations, we find that the basis functions are

$$|\kappa q m\rangle = \frac{1}{2\pi} J_m(\kappa\rho)e^{iqz}e^{im\phi}\,, \tag{37}$$

where $0\le\kappa<\infty$, $-\infty<q<\infty$, $m=0,\pm1,\pm2,\ldots$, and $J_m(\kappa\rho)$ are Bessel functions. These cylindrical basis functions satisfy the following orthogonality and completeness relations:

$$\langle \kappa' q' m'|\kappa q m\rangle = \frac{1}{(2\pi)^2}\int_0^\infty \rho\, d\rho \int_{-\infty}^{\infty} dz \int_0^{2\pi} d\phi\, J_m(\kappa\rho)J_{m'}(\kappa'\rho)e^{i(q-q')z}e^{i(m-m')\phi} = \frac{\delta(\kappa-\kappa')}{\kappa}\delta(q-q')\delta_{mm'} \tag{38}$$

and

$$\frac{1}{(2\pi)^2}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} dq \int_0^\infty \kappa\, d\kappa\, J_m(\kappa\rho)J_m(\kappa\rho')e^{iq(z-z')}e^{im(\phi-\phi')} = \frac{\delta(\rho-\rho')}{\rho}\delta(z-z')\delta(\phi-\phi')\,. \tag{39}$$

3. *The* $E(3)\supset T_1\times T_2\times T_3$ *Basis (the Translational Basis)*

The basis functions are eigenfunctions of $\vec{P}^2$, P_1, P_2, and P_3. We use rectangular coordinates x_i, and write

$$P_i|k_1,k_2,k_3\rangle = k_i|k_1,k_2,k_3\rangle \tag{40}$$

with $k_1^2+k_2^2+k_3^2=k^2$. The basis functions are

$$|\vec{k}\rangle = |k_1,k_2,k_3\rangle = \frac{1}{(2\pi)^{3/2}}e^{i\vec{k}\cdot\vec{x}}, \tag{41}$$

satisfying the obvious orthogonality and completeness relations

$$\langle\vec{k}'|\vec{k}\rangle = \frac{1}{(2\pi)^3}\int d\vec{x}\, e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} = \delta(\vec{k}-\vec{k}'), \tag{42}$$

$$\frac{1}{(2\pi)^3}\int d\vec{k}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} = \delta(\vec{x}-\vec{x}'). \tag{43}$$

D. Clebsch-Gordan Coefficients for Spin-Zero Representations of E(3)

We shall need the Clebsch-Gordan coefficients for irreducible unitary representations with $m_0=0$ of the group E(3) in the spherical basis. The case of general representations and other bases will be treated separately.

Consider the quasiregular representation[16] of E(3), realized in the space of functions $F(\vec{x})$, satisfying (29):

$$T_g F(\vec{x}) = F(g^{-1}\vec{x}). \tag{44}$$

The quasiregular representation can be expanded in terms of unitary ones, and the corresponding formula in the spherical system is

$$F(\vec{x}) = \sum_{L=0}^{\infty}\sum_{M=-L}^{L}\int_0^\infty k^2dk\, A_{LM}(k)\Psi_{kLM}(r,\theta,\phi), \tag{45}$$

with

$$A_{LM}(k) = \frac{2}{\pi}\int_0^\infty r^2dr\int_0^\pi \sin\theta\, d\theta\int_0^{2\pi}d\phi\, F(r,\theta,\phi) \times \Psi^*_{kLM}(r,\theta,\phi) \tag{46}$$

[see (32)–(34)]. The matrix elements for finite E(3) transformations in the $\{k,m_0\}=\{k,0\}$ representation are defined by the formulas

$$T_g\Psi_{kLM}(\vec{x}) = \Psi_{kLM}(g^{-1}\vec{x}) = \sum_{L'M'} D^{k0}_{L'M',LM}(g)\Psi_{kL'M'}(\vec{x}). \tag{47}$$

Noting that for $\vec{x}=(0,0,0)$ we have

$$\Psi_{kLM}(\vec{0}) = \delta_{L0}\,\delta_{M0}\frac{1}{(4\pi)^{1/2}}, \tag{48}$$

we obtain from (47) a relation between the basis functions and the (00) row of matrix elements:

$$\Psi_{kLM}(r,\theta,\phi) = \frac{1}{(4\pi)^{1/2}}D^{k0}_{00,LM}(g). \tag{49}$$

We can define the E(3) Clebsch-Gordan coefficients for the case of interest by the relations

$$|k_1L_1M_1\rangle|k_2L_2M_2\rangle = \sum_{M_0}\int_0^\infty K^2dK\sum_{L,M}\left\{\begin{matrix}k_1 & k_2\\ L_1M_1 & L_2M_2\end{matrix}\middle|\begin{matrix}M_0\,k\\ L\quad M\end{matrix}\right\}\left|\begin{matrix}M_0kLM\\ k_1k_2\end{matrix}\right\rangle, \tag{50}$$

$$\left|\begin{matrix}M_0kLM\\ k_1k_2\end{matrix}\right\rangle = \sum_{L_1M_1L_2M_2}\left\{\begin{matrix}k_1 & k_2\\ L_1M_1 & L_2M_2\end{matrix}\middle|\begin{matrix}M_0\,k\\ L\quad M\end{matrix}\right\}^*|k_1L_1M_1\rangle|k_2L_2M_2\rangle. \tag{51}$$

These definitions, together with the normalization of the basis functions, imply the orthogonality properties of the Clebsch-Gordan coefficients:

$$\sum_{LMM_0}\int_0^\infty K^2dK\left\{\begin{matrix}k_1 & k_2\\ L_1M_1 & L_2M_2\end{matrix}\middle|\begin{matrix}M_0K\\ L\quad M\end{matrix}\right\}\left\{\begin{matrix}k_1 & k_2\\ L_1'M_1' & L_2'M_2'\end{matrix}\middle|\begin{matrix}M_0K\\ L\quad M\end{matrix}\right\}^* = \delta_{L_1L_1'}\delta_{L_2L_2'}\delta_{M_1M_1'}\delta_{M_2M_2'}, \tag{52}$$

$$\sum_{L_1M_1L_2M_2}\left\{\begin{matrix}k_1 & k_2\\ L_1M_1 & L_2M_2\end{matrix}\middle|\begin{matrix}M_0K\\ L\quad M\end{matrix}\right\}\left\{\begin{matrix}k_1 & k_2\\ L_1M_1 & L_2M_2\end{matrix}\middle|\begin{matrix}M_0'K'\\ L'\ M'\end{matrix}\right\}^* = \frac{\delta(K-K')}{K^2}\delta_{LL'}\delta_{MM'}\delta_{M_0M_0'}. \tag{53}$$

Acting on Eq. (50) with the general group operator T_g we obtain after some manipulations

$$D^{k_10}_{L_1'M_1',L_1M_1}(g)D^{k_20}_{L_2'M_2',L_2M_2}(g) = \sum_{LML'M'M_0}\int_0^\infty K^2dK\left\{\begin{matrix}k_1 & k_2\\ L_1M_1 & L_2M_2\end{matrix}\middle|\begin{matrix}M_0K\\ L\quad M\end{matrix}\right\}\left\{\begin{matrix}k_1 & k_2\\ L_1'M_1' & L_2'M_2'\end{matrix}\middle|\begin{matrix}M_0K\\ L'\ M'\end{matrix}\right\}^* D^{KM_0}_{L'M',LM}(g). \tag{54}$$

It follows from the Wigner-Eckart theorem that the E(3) Clebsch-Gordan coefficient may be factorized as

$$\left\{\begin{matrix} k_1 & k_2 \\ L_1M_1 & L_2M_2 \end{matrix} \middle| \begin{matrix} M_0K \\ L\ \ M \end{matrix}\right\} = (L_1M_1L_2M_2|LM)\begin{bmatrix} k_1\ k_2\ KM_0 \\ L_1\ L_2\ L \end{bmatrix}, \tag{55}$$

where $(L_1M_1L_2M_2|LM)$ is an O(3) Clebsch-Gordan coefficient. Now consider the special case of (54), when $L_1' = L_2' = M_1' = M_2' = 0$. Using (55) and (49) we have

$$\Psi_{k_1L_1M_1}(r,\theta,\phi)\Psi_{k_2L_2M_2}(r,\theta,\phi) = \frac{1}{(4\pi)^{1/2}}\sum_{L,M}\int_0^\infty K^2dK \left\{\begin{matrix} k_1 & k_2 \\ L_1M_1 & L_2M_2 \end{matrix} \middle| \begin{matrix} K \\ LM \end{matrix}\right\}\left\{\begin{matrix} k_1\ k_2 \\ 00\,00 \end{matrix} \middle| \begin{matrix} K \\ 00 \end{matrix}\right\}^* \Psi_{kLM}(r,\theta,\phi). \tag{56}$$

(Here and below we are dropping the symbol $M_0 = 0$ in the Clebsch-Gordan coefficients.) Using orthogonality [(33)], relation (55), and standard angular momentum theory,[17] we have

$$\begin{bmatrix} k_1\ k_2\ K \\ L_1\ L_2\ L \end{bmatrix}\begin{bmatrix} k_1\ k_2\ K \\ 0\ \ 0\ \ 0 \end{bmatrix}^* = \frac{2}{\pi}\left\{\frac{(2L_1+1)(2L_2+1)}{2L+1}\right\}^{1/2}(L_10\,L_20|L0)\int_0^\infty r^2dr\, j_{L_1}(k_1r)j_{L_2}(k_2r)j_L^*(Kr). \tag{57}$$

Let us first consider the special case $L_1 = L_2 = L = 0$, when the integral on the right-hand side is known[18]:

$$\left|\begin{bmatrix} k_1\ k_2\ K \\ 0\ \ 0\ \ 0 \end{bmatrix}\right|^2 = \frac{2}{\pi k_1k_2K}\int_0^\infty \frac{\sin k_1r\,\sin k_2r\,\sin Kr}{r}\,dr = \begin{cases} \dfrac{1}{2k_1k_2K} & \text{for } |k_1-k_2|<K<k_1+k_2 \\ \dfrac{1}{4k_1k_2K} & \text{for } K=|k_1-k_2| \text{ or } k_1+k_2 \\ 0 & \text{otherwise} \end{cases} \tag{58}$$

(we assume $k_1k_2K \neq 0$). We can make such a phase convention for the basis functions in the direct-product space that the reduced coefficient in (58) is itself real and positive.

We now proceed to calculate the integral in (57) in the general case. To do this we first invert (56), using the orthogonality relation (53). After trivial manipulations we obtain

$$\begin{bmatrix} k_1\ k_2\ K \\ 0\ \ 0\ \ 0 \end{bmatrix} j_L(Kr) = \sum_{L_1,L_2}\left[\frac{(2L_1+1)(2L_2+1)}{2L+1}\right]^{1/2} \times (L_10\,L_20|L0)\begin{bmatrix} k_1\ k_2\ K \\ L_1\ L_2\ L \end{bmatrix}^* \times j_{L_1}(k_1r)j_{L_2}(k_2r). \tag{59}$$

A similar relation can be obtained directly by making use of the Clebsch-Gordan decomposition in the translational basis. Indeed, put $\vec{k}_1 + \vec{k}_2 = \vec{K}$ and write

$$e^{i\vec{K}\cdot\vec{r}} = e^{i\vec{k}_1\cdot\vec{r}}\,e^{i\vec{k}_2\cdot\vec{r}} \tag{60}$$

[this immediately tells us that the K spectrum is $|k_1-k_2| \leq K \leq k_1+k_2$, as indicated in (58)]. Choose a coordinate frame in which

$$\begin{aligned} \vec{K} &= K(0,0,1), \\ \vec{k}_1 &= k_1(\sin\theta_1, 0, \cos\theta_1), \\ \vec{k}_2 &= k_2(-\sin\theta_2, 0, \cos\theta_2), \\ \vec{r} &= r(\sin\theta\cos\phi,\ \sin\theta\sin\phi, \cos\theta). \end{aligned} \tag{61}$$

It is easy to see that

$$\cos\theta_1 = \frac{K^2+k_1^2-k_2^2}{2Kk_1}, \qquad \cos\theta_2 = \frac{K^2-k_1^2+k_2^2}{2Kk_2}. \tag{62}$$

Expanding all three exponentials in (60) into spherical waves we obtain

$$j_L(Kr) = \sum_{L_1L_2M} i^{L_1+L_2-L}(-1)^M \frac{(2L_1+1)(2L_2+1)}{2L+1}(L_10\,L_20|L0)(L_1M\,L_2-M|L0) \times P_{L_1}^M(\cos\theta_1)P_{L_2}^{-M}(\cos\theta_2)j_{L_1}(k_1r)j_{L_2}(k_2r). \tag{63}$$

Comparing (59) and (63) we find

$$\begin{bmatrix} k_1\ k_2\ K \\ L_1\ L_2\ L \end{bmatrix} = (-i)^{L_1+L_2-L}\left[\frac{(2L_1+1)(2L_2+1)}{2L+1}\right]^{1/2}\begin{bmatrix} k_1\ k_2\ K \\ 0\ \ 0\ \ 0 \end{bmatrix}\sum_M(-1)^M(L_1M\,L_2-M|L0)P_{L_1}^M(\cos\theta_1)P_{L_2}^{-M}(\cos\theta_2) \tag{64}$$

for L_1+L_2-L even. Equation (57) implies

$$\begin{bmatrix} k_1 k_2 K \\ L_1 L_2 L \end{bmatrix} = 0 \quad \text{for } L_1 + L_2 - L \text{ odd}. \tag{65}$$

In combination with the formula

$$\begin{bmatrix} k_1 k_2 K \\ 0\ 0\ 0 \end{bmatrix} = \frac{1}{(k_1 k_2 K)^{1/2}} \begin{cases} 1/\sqrt{2} & \text{for } |k_1 - k_2| < K < k_1 + k_2 \\ \frac{1}{2} & \text{for } K = |k_1 - k_2| \text{ or } k_1 + k_2 \\ 0 & \text{otherwise} \end{cases} \tag{66}$$

Eq. (64) determines the E(3) Clebsch-Gordan coefficients for zero-spin representations.

Let us mention that as a by-product we have evaluated the integral over three Bessel functions:

$$(L_1 0\, L_2 0\,|\,L0) \int_0^\infty r^2 dr\, j_{L_1}(k_1 r) j_{L_2}(k_2 r) j_L(kr) = \tfrac{1}{2}\pi(-i)^{L_1+L_2-L} \begin{bmatrix} k_1 k_2 K \\ 0\ 0\ 0 \end{bmatrix}^2 \times \sum_M (-1)^M (L_1 M\, L_2 - M\,|\,L0) P_{L_1}^M(\cos\theta_1) P_{L_2}^{-M}(\cos\theta_2), \tag{67}$$

where $\cos\theta_1$ and $\cos\theta_2$ are given by (62) and the formula is valid for $L_1 + L_2 + L$ even (both sides vanish for $L_1 + L_2 + L$ odd).

In all the above considerations we have completely ignored any multiplicity problem that may be present in the reduction of the direct product of two E(3) representations. Similarly as in the case of the group E(2),[19] a twofold degeneracy exists in this reduction for E(3), related to the fact that the two equivalent representations (k, m_0) and $(-k, -m_0)$ are both present. This degeneracy must be removed by introducing a supplementary label. The method used for the E(2) group[19] was to consider the extended group including a reflection operator, which is then simply reducible. Note that in our realization (32) the basis functions do have definite properties with respect to the reflection operator P, namely,

$$P|kLM\rangle = (-1)^L |kLM\rangle .$$

Since the reduced Clebsch-Gordan coefficient which we have calculated in (64) vanishes for $L_1 + L_2 + L$ odd we find from (51) that the basis vectors in the direct product space also have definite parity:

$$P \left| \begin{matrix} KLM \\ k_1 k_2 \end{matrix} \right\rangle = \eta(-1)^L \left| \begin{matrix} KLM \\ k_1 k_2 \end{matrix} \right\rangle, \qquad \eta = 1 .$$

States transforming identically under proper E(3) transformations, but with $\eta = -1$ under reflection, must be constructed differently. We shall return to this problem in a subsequent publication.

III. EXPANSIONS OF AMPLITUDES FOR THE SCATTERING OF NONRELATIVISTIC SPINLESS PARTICLES

Let us now consider the scattering of spinless particles

$$1 + 2 \to 3 + 4 . \tag{68}$$

Similarly as in the relativistic case,[1] we can consider the scattering amplitude to be a function of the momentum of one of the particles only (in a chosen frame of reference). This function $F(\vec{k})$, where $\vec{k}$ is a vector in Euclidean three-space, can now be decomposed into irreducible components with respect to the Euclidean group E(3) [instead of O(3, 1), as in the relativistic case]. In this manner we obtain formulas of "generalized" or "two-variable" nonrelativistic partial wave analysis. The actual expansions, as in the relativistic case, depend on the frame of reference, the parametrization of the E(3) space, and the choice of a basis for the representations.

We shall first discuss the nonrelativistic kinematics that enables us to write the scattering amplitude as a function of a point in momentum space, then derive and discuss the expansions. For simplicity we restrict ourselves to elastic scattering, when the particle masses satisfy

$$m_1 = m_3, \quad m_2 = m_4, \tag{69}$$

since this is the case of interest for potential scattering. Note, however, that the kinematics would be only slightly more complicated for general (positive) masses.

A. Scattering Amplitude as a Function of a Point in Momentum Space

By analogy with the relativistically invariant Mandelstam variables s, t, and u of Eq. (2), let us introduce the Galilei-invariant variables

$$\begin{aligned} s_E &= (\vec{p}_1 - \mu\vec{p}_2)^2, \\ t_E &= -(\vec{p}_1 - \vec{p}_3)^2, \quad \mu = m_1/m_2, \\ u_E &= -(\vec{p}_1 - \mu\vec{p}_4)^2 \end{aligned} \tag{70}$$

(the subscript E stand for "Euclidean"). The factors μ in (70) are necessary, since Galilei transformations conserve differences between velocities rather than between momenta. It may be appropriate at this point to stress that the E(3) group generating our expansions consists of rotations and translations in momentum space, i.e., it is actually the "homogeneous Galilei group" [and thus the nonrelativistic limit of O(3, 1)]. The variables (70) satisfy

$$s_E + \mu t_E + u_E = 0\,. \tag{71}$$

The scattering amplitude, irrespectively of the choice of a frame of reference, can be considered as a function of any two of these variables. Let us now specify the frame of reference.

1. *Center-of-Mass System* [*Fig. 1(a)*]

We introduce spherical coordinates in momentum space and write each of the momenta as $p_i = (k_i \sin\theta_i \cos\phi_i, k_i \sin\theta_i \sin\phi_i, k_i \cos\theta_i)$. Taking $\vec{p}_1 \parallel -\vec{p}_2$ parallel to the third axis and identifying the scattering plane with the xz plane, we have

$$\begin{aligned} \vec{p}_1 &= (0, 0, k)\,, \\ \vec{p}_2 &= (0, 0, -k)\,, \\ \vec{p}_3 &= (k\sin\theta,\ 0,\ k\cos\theta)\,, \\ \vec{p}_4 &= (-k\sin\theta,\ 0,\ -k\cos\theta)\,. \end{aligned} \tag{72}$$

The relation between the invariant variables (70) and the c.m. ones is

$$\begin{aligned} s_E &= k^2(1+\mu)^2\,, \\ t_E &= -2k^2(1-\cos\theta)\,, \\ u_E &= -k^2(1+\mu^2+2\mu\cos\theta)\,, \end{aligned} \tag{73}$$

i.e.,

$$\begin{aligned} k &= \frac{(s_E)^{1/2}}{1+\mu}\,, \\ \cos\theta &= 1+(1+\mu)^2\,\frac{t_E}{2s_E}\,. \end{aligned} \tag{74}$$

Thus, we can write

$$F(s_E, t_E) = F(k, \cos\theta) = F(\vec{p}_3)\,, \tag{75}$$

where $0 \le k < \infty$ and $0 \le \theta \le \pi$ are the usual c.m. energy and scattering angle, which are the spherical coordinates of the momentum $\vec{p}_3$ in (72) (the azimuthal angle $\phi_3 = \phi$ is redundant for spinless particles, i.e., the amplitude does not depend on it).

2. *Brick-Wall System of Particles 2 and 4* [*Fig. 1(b)*]

We introduce cylindrical coordinates (35), writing each of the momenta as

$$\vec{p}_i = (p_{\parallel i}\cos\phi_i,\ p_{\parallel i}\sin\phi_i,\ p_{\perp i})\,.$$

Taking $-\vec{p}_2 \parallel \vec{p}_4$ parallel to the z axis, taking $0xz$ as the scattering plane, and putting $E_1 = E_3$, $E_2 = E_4$, we have

$$\begin{aligned} \vec{p}_1 &= (q_\parallel, 0, q_\perp)\,, \quad \vec{p}_3 = (q_\parallel, 0, -q_\perp)\,, \\ \vec{p}_2 &= (0, 0, -q_\perp)\,, \quad \vec{p}_4 = (0, 0, q_\perp)\,. \end{aligned} \tag{76}$$

The subscripts of q indicate whether the component is parallel or perpendicular to the "brick wall" (b.w.) identified with the xy plane. The relation to the invariants (70) is

$$\begin{aligned} s_E &= q_\parallel^{\,2} + q_\perp^{\,2}(1+\mu)^2\,, \\ t_E &= -4q_\perp^{\,2}\,, \\ u_E &= -q_\parallel^{\,2} - q_\perp^{\,2}(1-\mu)^2\,, \end{aligned} \tag{77}$$

$$\begin{aligned} q_\perp &= \tfrac{1}{2}(-t_E)^{1/2}\,, \\ q_\parallel &= [\,s_E + t_E\tfrac{1}{4}(1+\mu)^2]^{1/2}\,. \end{aligned} \tag{78}$$

Thus, we can write the scattering amplitude as a function of the cylindrical coordinates $q_\parallel$, $q_\perp$ of the

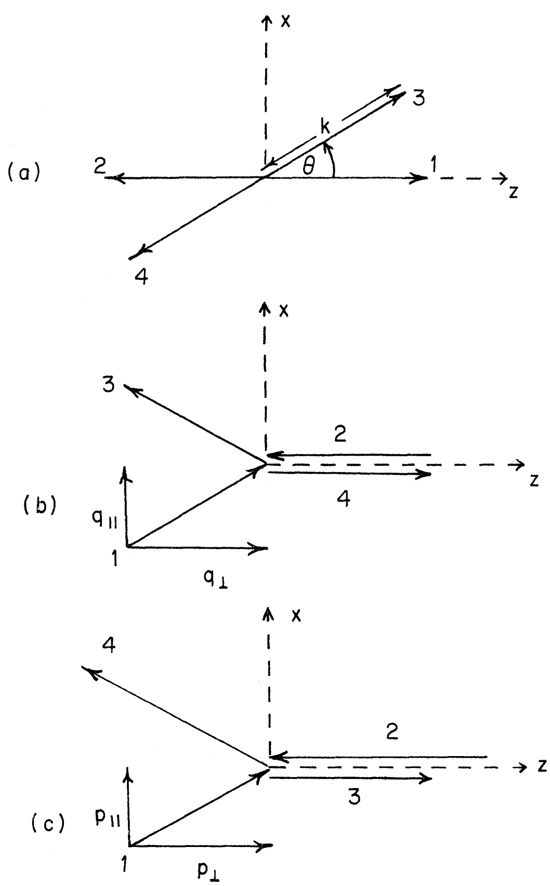

FIG. 1. Frames of reference. (a) Center-of-mass frame, (b) brick-wall frame of particles 2 and 4, (c) brick-wall frame of particles 2 and 3.

momentum $\vec{p}_1$ (the azimuthal angle ϕ is redundant):

$$F(s_E, t_E) = F(q_\perp, q_\parallel) = F(\vec{p}_1). \tag{79}$$

Formulas (73) and (77) also relate the b.w. variables to the c.m. ones, namely,

$$q_\perp = k \sin\tfrac{1}{2}\theta, \quad q_\parallel = (1+\mu)k\cos\tfrac{1}{2}\theta. \tag{80}$$

Note that forward scattering corresponds to $\theta \sim 0$, i.e., $q_\perp \sim 0$, $q_\parallel \sim (1+\mu)k$, and backward scattering to $\theta \sim \pi$, i.e., $q_\perp \sim k$, $q_\parallel \sim 0$.

3. *Brick-Wall System of Particles 2 and 3 [Fig. 1(c)]*

We again introduce cylindrical coordinates and $0xz$ as the scattering plane (so that $\sin\phi_i = 0$, $i = 1, \ldots, 4$), but put $-\vec{p}_2 \parallel \vec{p}_3$ along the third axis and $E_1 = E_4$, $E_2 = E_3$. We have

$$\begin{aligned} &\vec{p}_1 = (p_\parallel, 0, p_\perp), \quad \vec{p}_3 = (0, 0, \sqrt{\mu}\, q), \\ &\vec{p}_2 = (0, 0, -q), \quad \vec{p}_4 = (p_\parallel, 0, -\tilde{p}_\perp), \end{aligned} \tag{81}$$

where

$$\tilde{p}_\perp = \frac{1}{\sqrt{\mu}}[p_\perp^2 + p_\parallel^2(1-\mu)]^{1/2},$$

$$q = \frac{1}{1+\sqrt{\mu}}\left\{p_\perp + \frac{1}{\sqrt{\mu}}[p_\perp^2 + p_\parallel^2(1-\mu)]^{1/2}\right\}.$$

The relation between the invariants, the b.w. variables $p_\parallel$ and $p_\perp$, and the c.m. variables is

$$\begin{aligned} -t_E &= \frac{2}{(1+\sqrt{\mu})^2} \\ &\quad \times \{p_\perp^2 + p_\parallel^2(1+\sqrt{\mu}) - p_\perp[p_\perp^2 + p_\parallel^2(1-\mu)]^{1/2}\} \\ &= 4k^2\sin^2(\tfrac{1}{2}\theta), \\ -u_E &= p_\parallel^2(1-\mu) + p_\perp^2(1+\mu) \\ &\quad + 2p_\perp\sqrt{\mu}[p_\perp^2 + p_\parallel^2(1-\mu)]^{1/2} \\ &= k^2(1+\mu^2+2\mu\cos\theta). \end{aligned} \tag{82}$$

Thus, we again find that the amplitude is a function of the cylindrical coordinates $p_\perp, p_\parallel$ of a vector, namely $\vec{p}_1$:

$$F(s_E, t_E) = F(p_\perp, p_\parallel) = F(\vec{p}_1). \tag{83}$$

Let us note that formulas (82) simplify greatly for equal masses;

$$\begin{aligned} &(-t_E)^{1/2} = p_\parallel = 2k\sin\tfrac{1}{2}\theta, \quad \mu = 1 \\ &\tfrac{1}{2}(-u_E)^{1/2} = p_\perp = k\cos\tfrac{1}{2}\theta. \end{aligned} \tag{84}$$

Formulas (82) simplify also in the forward direction (for arbitrary masses):

$$\begin{aligned} &p_\parallel \sim 2k\sin\tfrac{1}{2}\theta, \quad p_\parallel/p_\perp \ll 1, \\ &p_\perp \sim \frac{k}{1+\sqrt{\mu}}(1+\mu^2+2\mu\cos\theta)^{1/2}, \quad \sin\theta \ll 1. \end{aligned} \tag{85}$$

The three frames of reference considered above are the $c \to \infty$ limit of the relativistic center-of-mass system and the t- and u-channel brick-wall systems, respectively.

B. Expansions of Square-Integrable Scattering Amplitudes

The scattering amplitude is now a function $F(\vec{p})$ of one of the particle three-momenta. Assuming first that it belongs to the Hilbert space of square-integrable functions (with respect to the invariant measure), satisfying

$$\int |F(\vec{p})|^2 dp_x\, dp_y\, dp_z < \infty, \tag{86}$$

we can expand it in terms of unitary irreducible representations of E(3), making use of the results of Sec. II and of the reduction of the quasiregular representation.[16]

Let us consider the individual bases separately.

1. *Spherical Expansion and the* c.m. *System*

We parametrize $\vec{p}_3$ in spherical coordinates, remember that the azimuthal angle ϕ_3 is redundant [see (75)], and expand $F(\vec{p})$ in terms of the spherical basis (32) (setting $m=0$). Using the orthogonality relations (33), we have

$$F(k, \cos\theta) = \sum_{l=0}^{\infty}(2l+1)\int_0^\infty r^2 dr\, A_l(r) j_l(kr) P_l(\cos\theta), \tag{87}$$

with

$$A_l(r) = \frac{1}{\pi}\int_0^\infty k^2 dk \int_0^\pi \sin\theta\, d\theta\, F(k,\cos\theta) j_l(kr) P_l(\cos\theta). \tag{88}$$

Setting

$$a_l(k) = \int_0^\infty r^2 dr\, A_l(r) j_l(kr), \tag{89}$$

we immediately obtain an interpretation of the expansion (87). It is simply the usual partial-wave expansion, supplemented by the representation (89) for the O(3) partial-wave amplitude.

The square-integrability condition (86) imposes a severe restriction on the asymptotic behavior of the total cross section. Indeed

$$\int_0^\infty\int_0^\pi |F(k,\cos\theta)|^2 k^2 dk \sin\theta\, d\theta = \int_0^\infty \sigma(k) k^2 dk, \tag{90}$$

so that the total cross section must vanish more strongly than

$$\sigma(k) \underset{k\to\infty}{\sim} \frac{1}{k^3}. \tag{91}$$

2. Cylindrical Expansion and the Brick-Wall Systems

We parametrize the amplitude $F(\vec{p})$ using cylindrical coordinates as in (79) and (83) and again remember that the azimuthal angle ϕ is redundant. We can now expand in terms of the cylindrical basis (37). Using the orthogonality relations (38), we have

$$F(\kappa_\perp,\kappa_\parallel)=\int_0^\infty b\,db\int_{-\infty}^\infty dz\,A(b,z)J_0(b\kappa_\parallel)e^{i\kappa_\perp z}, \tag{92}$$

with

$$A(b,z)=\frac{1}{2\pi}\int_0^\infty \kappa_\parallel d\kappa_\parallel \times\int_{-\infty}^\infty d\kappa_\perp F(\kappa_\perp,\kappa_\parallel)J_0(b\kappa_\parallel)e^{-i\kappa_\perp z}. \tag{93}$$

The variables $\kappa_\parallel$, $\kappa_\perp$ are to be identified either with $q_\parallel$, $q_\perp$ of (80) or with $p_\parallel$, $p_\perp$ of (82). Setting

$$a(\kappa_\perp,b)=\int_{-\infty}^\infty dz\,A(b,z)e^{i\kappa_\perp z}, \tag{94}$$

we find that (92) is simply an eikonal expansion, supplemented by a Fourier transform of the eikonal amplitude (94). The eikonal expansion is usually written for high-energy forward scattering. Expansion (92) provides a generalization to arbitrary angles and energies. Formulas (85) show that the variable $p_\parallel$ corresponds to the usual variable figuring in the forward eikonal expansion.[20-22] By the same token $q_\parallel$ in (80) resembles an eikonal-type variable for backward scattering.

The square-integrability condition (86) again restricts the high-energy behavior. Indeed, we have

$$\int_0^\infty \kappa_\parallel d\kappa_\parallel\int_{-\infty}^\infty d\kappa_\perp|F(\kappa_\parallel,\kappa_\perp)|^2<\infty, \tag{95}$$

which is satisfied if, e.g.,

$$F(\kappa_\parallel,\kappa_\perp)\underset{\substack{\kappa_\perp\to\infty\\ \kappa_\parallel\ \text{fixed}}}{\sim}\frac{1}{\kappa_\perp^{1/2+\epsilon}} \tag{96}$$

and

$$\int_{-\infty}^\infty d\kappa_\perp|F(\kappa_\parallel,\kappa_\perp)|^2\underset{\kappa_\parallel\to\infty}{\sim}\frac{1}{\kappa_\parallel^{2+\epsilon}}. \tag{97}$$

A more complete discussion of the convergence conditions on the Fourier and Bessel transforms is found in the literature.[23-25]

We shall not discuss the expansions obtained using the translational basis (41) since they amount simply to Fourier transforms of the amplitude.

C. Expansions of Non-Square-Integrable Amplitudes

The square-integrability condition (86) imposes conditions on the high-energy behavior of the scattering amplitude that are too restrictive for most cases of physical interest. In order to relax this restriction we must write expansions in terms of a class of nonunitary irreducible representations. Although nonunitary representations of noncompact groups, in particular E(3), have received little attention, the tools for generalizing the expansions (87) and (92) (as well as the translational-basis expansion) are readily available. They are provided by the possibility of generalizing a Fourier transform to a complex Laplace transform and a Fourier-Bessel transform to a Meijer transform.[23-25]

The Meijer transform of a function $f(t)$ of a real variable t, with $0\le t_1\le t\le t_2$, for which the integral

$$\int_0^\infty e^{-\beta' t}|f(t)|dt<\infty\quad(\beta'>0) \tag{98}$$

converges, can be written as

$$f(t)=\frac{1}{i(2\pi)^{1/2}}\int_{\beta-i\infty}^{\beta+i\infty}I_\nu(ts)(ts)^{1/2}a(s)ds\quad(\beta\ge\beta'). \tag{99}$$

The inverse formula is

$$a(s)=\left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty K_\nu(st)(st)^{1/2}f(t)dt. \tag{100}$$

Above $K_\nu(z)$ is a Macdonald cylindrical function and

$$I_\nu(z)=e^{-(\pi/2)\nu i}J_\nu(e^{i\pi/2}z), \tag{101}$$

where $J_\nu(z)$ is a Bessel function.

Similarly, a Laplace transform[23-25] of a function $f(t)$ satisfying (98) can be written as

$$f(t)=\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}a(s)e^{st}ds\quad(\beta\ge\beta'), \tag{102}$$

$$a(s)=\int_0^\infty f(t)e^{-ts}dt. \tag{103}$$

Let us now rewrite the two-variable expansions of scattering amplitudes, using the Meijer and Laplace transforms.

1. The Spherical Expansion

Using (99) we can generalize (87) to the case of amplitudes that may increase for $k\to\infty$ as fast as exponentially. We have

$$F(k,\cos\theta)=\sum_{l=0}^\infty(2l+1)\int_{\beta-i\infty}^{\beta+i\infty}r^2dr\,B_l(r)\zeta_l(kr)P_l(\cos\theta), \tag{104}$$

with the inversion formula

$$B_l(r)=\frac{1}{\pi^2 i}\int_0^\infty k^2dk\int_0^\pi \sin\theta\, d\theta\, F(k,\cos\theta)\kappa_l(kr)P_l(\cos\theta), \tag{105}$$

where

$$\zeta_l(kr)=\left(\frac{\pi}{2kr}\right)^{1/2} I_{l+1/2}(kr),$$
$$\kappa_l(kr)=\left(\frac{\pi}{2kr}\right)^{1/2} K_{l+1/2}(kr). \tag{106}$$

Setting $\beta=0$ and $r=e^{-i\pi/2}r'$, we find that (104) and (105) do indeed reduce to (87) and (88), with

$$A_l(r)=B_l(e^{-i\pi/2}r)e^{-i(\pi/2)(l+1)} - B_l(e^{i\pi/2}r)e^{i(\pi/2)(l+1)}. \tag{107}$$

2. The Cylindrical Expansion

The generalization of (92) and (93) to amplitudes that may increase as fast as exponentially when $\kappa_\perp\to\infty$ or $\kappa_\parallel\to\infty$ is

$$F(\kappa_\perp,\kappa_\parallel)=\int_{\beta-i\infty}^{\beta+i\infty} b\,db\int_{\gamma-i\infty}^{\gamma+i\infty} dz\, B(b,z)I_0(b\kappa_\parallel)e^{\kappa_\perp z}, \tag{108}$$

where

$$B(b,z)=-\frac{1}{2\pi^2}\int_0^\infty \kappa_\parallel d\kappa_\parallel \int_0^\infty d\kappa_\perp F(\kappa_\perp,\kappa_\parallel) \times K_0(b\kappa_\parallel)e^{-\kappa_\perp z}. \tag{109}$$

Setting $\beta=\gamma=0$ we recover the expansion in terms of unitary representations, with

$$A(b,z)=-i[B(ib,iz)-B(-ib,iz)]. \tag{110}$$

3. The Translational Expansion

For completeness let us also give the general expansion of a scattering amplitude (that may increase exponentially as a function of any component of the relevant momentum) in terms of the translational basis. This amounts to a three-dimensional Laplace transform

$$F(\vec{k})=\int_{\vec{\beta}-i\infty}^{\vec{\beta}+i\infty} B(\vec{x})e^{\vec{k}\cdot\vec{x}}d\vec{x}, \tag{111}$$

with

$$B(\vec{x})=\frac{1}{(2\pi i)^3}\int_0^\infty F(\vec{k})e^{-\vec{k}\cdot\vec{x}}d\vec{k}. \tag{112}$$

Thus, we have derived expansions of a very general class of scattering amplitudes, in which all the dependence on the kinematic parameters is contained in known special functions, provided by the representation theory of E(3). The dynamics are in the expansion coefficients and the expansions themselves reflect the Galileian invariance of the theory. Let us now demonstrate the connection between the nonrelativistic expansions of this paper and the previously considered expansions of relativistic amplitudes [based on the O(3, 1) group].

D. Contraction of Relativistic Expansions to Nonrelativistic Ones

Let us now demonstrate that the E(3)-subgroup-type expansions obtained above are indeed the nonrelativistic limits of O(3, 1)-subgroup-type expansions (obtained when properly taking the velocity of light $c\to\infty$). First of all, let us note that there are seven nonequivalent chains of subgroups of O(3, 1) (counting only subgroups that have invariants).[12] If we use the prescription $K_i=cp_i$, $c\to\infty$ [see (15)], we can contract the O(3, 1)-subgroup chains to E(3) ones. Doing this it is easy to check that the $O(3,1)\supset O(3)\supset O(2)$ chain contracts to $E(3)\supset O(3)\supset O(2)$, i.e., the relativistic spherical system expansion (3) should go over into the nonrelativistic one (87). Three different relativistic expansions contract into the nonrelativistic cylindrical system expansion (92), namely the hyperbolic expansion (5), corresponding to $O(3,1)\supset O(2,1)\supset O(2)$; the relativistic cylindrical expansion,[26] corresponding to the reduction $O(3,1)\supset O(2)\times O(1,1)$; and finally the horospheric expansion[1,26] corresponding to the reduction $O(3,1)\supset E(2)\supset O(2)$. The translational-basis expansion (111) is also obtained by contraction from three different O(3, 1) expansions, corresponding to the reductions $O(3,1)\supset O(2,1)\supset O(1,1)$, $O(3,1)\supset O(2,1)\supset E(1)$, and $O(3,1)\supset E(2)\supset E(1)\times E(1)$.

We shall investigate the general connection between the various O(3, 1) and E(3) basis functions and matrix elements in a separate article. Here let us look only at the spinless case [$j_0=0$ in (14)] and the spherical and hyperbolic expansions (3) and (5). We shall consider unitary representations only, setting $\sigma=-1+i\rho$ (ρ real) in (3) and (5) and $l=-\frac{1}{2}+iq$ (q real) in (5). Comparing Eqs. (14) and (9) for the Casimir operators of O(3, 1) and E(3), we see that (14) goes into (9) if we set $j_0=m_0$ and $\rho=cr$ ($c\to\infty$) (we also set $\vec{K}=c\vec{P}$, as usual).

Let us now establish the connection between the basis functions. For previous investigations along similar series see Refs. 11, 27, and 28.

1. The Spherical System

The relativistic momentum is parametrized as

$$p=c^2(\cosh a, \sinh a\sin\theta\cos\phi, \sinh a\sin\theta\sin\phi, \sinh a\cos\theta) =(E_R, c\vec{p}_R) \tag{113}$$

(we take the mass $m=1$). The nonrelativistic limit is obtained by setting $a=k/c$ ($c\to\infty$); indeed we have

$$|\vec{p}_R| = c\sinh a \to ck/c = k$$

and

$$E_R = c^2\cosh a \to c^2(1+k^2/2c^2) = c^2 + \tfrac{1}{2}k^2 .$$

Thus, the O(3, 1) basis functions must go into the E(3) ones, if we set

$$\rho = cr, \quad a = k/c, \quad \rho a = kr, \quad c\to\infty, \quad 0\le k<\infty . \tag{114}$$

We now write the energy-dependent part of the basis function in (3) as

$$\frac{\Gamma(i\rho)}{\Gamma(i\rho-l)}\frac{1}{(\sinh a)^{1/2}}P^{-l-1/2}_{-1/2+i\rho}(\cosh a) = \frac{\Gamma(i\rho)}{\Gamma(i\rho-l)}\frac{(\sinh a)^l}{2^l(2\pi)^{1/2}\Gamma(l+1)}\int_{-1}^{1}\frac{(1-t^2)^l\,dt}{(\cosh a + t\sinh a)^{l+1-i\rho}} . \tag{115}$$

In the limit (114) we can replace $(\cosh a + t\sinh a)^{-l-1+i\rho}$ with e^{ikrt} (for $c\to\infty$) and we obtain an integral representation for a Bessel function. Finally,

$$\frac{\Gamma(i\rho)}{\Gamma(i\rho-l)}\frac{1}{(\sinh a)^{1/2}}P^{-l-1/2}_{-1/2+i\rho}(\cosh a) \to i^l\left(\frac{2}{\pi}\right)^{1/2} j_l(kr) . \tag{116}$$

Since the angular part of the basis function $P_l(\cos\theta)$ [or more generally $Y_{lm}(\theta,\phi)$] does not change under contraction, we find that the $O(3,1)\supset O(3)\supset O(2)$ expansion (3) does indeed go over into the $E(3)\supset O(3)\supset O(2)$ one (87).

2. *The Hyperbolic System*

The relativistic momentum is parametrized as

$$p = c^2(\cosh\alpha\cosh\beta, \cosh\alpha\sinh\beta\cos\phi, \cosh\alpha\sinh\beta\sin\phi, \sinh\alpha) , \tag{117}$$

and we can write the nonrelativistic one as

$$(E,\vec{p}) = (E, \kappa_\parallel\cos\phi, \kappa_\parallel\sin\phi, \kappa_\perp) , \tag{118}$$

with

$$E = \tfrac{1}{2}(\kappa_\parallel^2 + \kappa_\perp^2)$$

(we set $m=1$). The limit is obtained by setting $\alpha = \kappa_\perp/c$ and $\beta = \kappa_\parallel/c$, and then taking $c\to\infty$.

The Casimir operator of the O(2, 1) subgroup is

$$L_3^2 - K_1^2 - K_2^2 = l(l+1), \quad l = -\tfrac{1}{2} + iq . \tag{119}$$

Setting $l+\frac{1}{2} = iq = icb$, we find that for $c\to\infty$ (119) goes into the Casimir operator for E(2)

$$P_1^2 + P_2^2 = b^2 . \tag{120}$$

Thus, the $O(3,1)\supset O(2,1)\supset O(2)$ basis functions should go into the $E(3)\supset E(2)\supset O(2)$ ones if we set

$$\rho = cr, \quad \alpha = \kappa_\perp/c, \quad \beta = \kappa_\parallel/c, \quad q = bc, \quad c\to\infty . \tag{121}$$

Indeed, the hyperbolic-system basis functions for unitary representations of O(3, 1) are

$$\Phi^{\pm}_{\rho q}(\alpha,\beta) = \frac{\Gamma(\frac{1}{2}+i(\rho+q))\,\Gamma(\frac{1}{2}+i(\rho-q))}{2\pi\Gamma(1+i\rho)} \times \frac{1}{\cosh\alpha}P^{-i\rho}_{-1/2+iq}(\mp\tanh\alpha)P_{-1/2+iq}(\cosh\beta) . \tag{122}$$

Similarly as in (115) we find that

$$P_{-1/2+iq}(\cosh\beta) \xrightarrow[q\beta=\kappa_\parallel b]{c\to\infty} J_0(b\kappa_\parallel) .$$

Writing the α-dependent part of (122) in terms of a combination of two hypergeometric functions and taking the limit $c\to\infty$ term by term, we find

$$(\tfrac{1}{2}i\rho)^{1/2}\frac{\Gamma(\frac{1}{2}+i(\rho+q))\,\Gamma(\frac{1}{2}+i(\rho-q))}{2\pi\Gamma(1+i\rho)}\frac{1}{\cosh\alpha}P^{-i\rho}_{-1/2+iq}(\mp\tanh\alpha) \xrightarrow[c\to\infty]{} \exp[\mp i(r^2-b^2)^{1/2}\kappa_\perp] \tag{123}$$

Thus the hyperbolic expansion (5) does indeed go over into the cylindrical one (92).

E. Comparison Between Relativistic and Nonrelativistic Expansions

As is to be expected, the variety of expansions provided by the group E(3) for nonrelativistic scattering amplitudes is much less than provided by O(3, 1) for the relativistic case. We have noticed that the seven subgroup-type expansions of O(3, 1) contract to three E(3) ones. Of particular physical significance is the fact that both the $O(3,1)\supset O(2,1)\supset O(2)$ expansion, in which the O(2, 1) subgroup provides a Regge-pole expansion, and the $O(3,1)\supset E(2)\supset O(2)$ expansion, in which the E(2) subgroup provides a relativistic eikonal expansion, contract to the nonrelativistic eikonal expansion corresponding to $E(3)\supset E(2)\times T_\perp\supset O(2)\times T_\perp$. Thus complex angular momentum theory seems to have a fundamental group-theoretical foundation in the relativistic case, but not in the nonrelativistic case,[29] where similar group-theoretical arguments lead to the eikonal expansion. It is paradoxical that complex angular momentum was first introduced for potential scattering,[30,31] but it should be remembered that its most fruitful applications and further developments lie completely within the relativistic field of elementary-particle interactions.[7]

IV. GENERAL PROPERTIES OF SCATTERING AMPLITUDES, INCORPORATED IN THE TWO-VARIABLE EXPANSIONS

Let us now consider the obtained two-variable expansions of interest, i.e., the spherical expansion (104) and the cylindrical expansion (108), and investigate their behavior in various physically interesting cases. We again consider the two cases separately.

A. The Spherical Expansion

1. *Threshold Behavior*

We write the partial-wave amplitude as

$$a_l(k)=\int_{\beta-i\infty}^{\beta+i\infty} r^2 dr\, B_l(r)\zeta_l(kr)\,, \tag{124}$$

with

$$B_l(r)=\frac{2}{\pi^2 i}\int_0^\infty k^2 dk\, a_l(k)\kappa_l(kr)\,, \tag{125}$$

where the spherical Bessel functions $\zeta_l(kr)$ and $\kappa_l(kr)$ are given by (106).

Let us now consider the limit $k\to 0$. We have

$$\zeta_l(kr)\underset{kr\to 0}{\sim}\frac{\sqrt{\pi}}{2^{l+1}\Gamma(l+\frac{3}{2})}(kr)^l\,. \tag{126}$$

This immediately shows that for $k\to 0$ all partial-wave amplitudes $a_l(k)$ with $l\neq 0$ vanish correctly and only the s wave ($l=0$) survives (as it should).

For "reasonable" potentials the partial-wave amplitude behaves as

$$a_l(k)\underset{k\to 0}{\sim} ic_l k^{2l}\,, \tag{127}$$

where c_l is a real constant.[31] We have[18]

$$\zeta_l(kr)=\frac{\sqrt{\pi}}{2^{l+1}\Gamma(l+\frac{3}{2})}(kr)^l \times\left[1+\sum_{n=1}^\infty\frac{\Gamma(l+\frac{3}{2})}{n!\,\Gamma(l+\frac{3}{2}+n)}(\tfrac{1}{2}kr)^{2n}\right]. \tag{128}$$

Substituting (128) into (124) we find that we obtain the correct threshold behavior (127) if the Galilei amplitudes $B_l(r)$ satisfy the constraints

$$\int_{\beta-i\infty}^{\beta+i\infty} r^2 dr\, B_l(r) r^{l+2n}=0\,, \qquad 2n=0,2,\ldots,l-1 \text{ (or } l). \tag{129}$$

This will be satisfied if, e.g., $B_l(r)r^{l+2n+2}\to 0$ for $|r|\to\infty$, $\operatorname{Re}r\geqslant\beta$, and $B_l(r)$ has no singularities in the half-plane $\operatorname{Re}r\geqslant\beta$.

Let us note here that formula (124) cannot represent the partial-wave amplitude in the entire complex k plane, since the property

$$\zeta_l(e^{im\pi}kr)=e^{iml\pi}\zeta_l(kr) \tag{130}$$

would imply a too-simplistic symmetry property,

$$a_l(-k)=e^{\pm il\pi}a_l(k)\,.$$

2. *Asymptotic Behavior*

Let us investigate the expansion (124) for $k\to\infty$. So far the Galilei amplitude $B_l(r)$ is an unknown function and we can make various assumptions about it. We have

$$\zeta_l(kr)\underset{kr\to\infty}{\sim}\frac{1}{2kr}\left[e^{kr}+(-1)^{l+1}e^{-kr}\right]. \tag{131}$$

Let us be inspired by Regge-pole theory and assume that $B_l(r)$ is an analytic function in the complex r plane with a finite number of simple poles in a strip $0<\beta'<\operatorname{Re}r<\beta$. We can then shift the integration path in (124) to the left and write $a_l(k)$ as a sum over the pole contributions plus a "background" integral over the unitary representations:

$$a_l(k)=2\pi i\sum_n r_n{}^2\zeta_l(kr_n)\beta_l(r_n) + \int_{-i\infty}^{+i\infty} r^2 dr\, B_l(r)\zeta_l(kr)\,, \tag{132}$$

where r_n is the position of the nth pole and $\beta_l(r_n)$ is the residue of $B_l(r)$ at r_n. It follows from (131) that the pole r_0 furthest to the right will dominate for $k\to\infty$ and we find

$$a_l(k)\underset{k\to\infty}{\sim}\pi i r_0\beta_l(r_0)\frac{1}{k}e^{kr_0}\,,$$

i.e., the partial-wave amplitude increases asymptotically as $k \to \infty$ (we have $0 < \mathrm{Re}\, r_0 \leq \beta$).

Alternatively, the high-energy behavior of $a_l(k)$ may be generated by the specific form of $B_l(r)$. Indeed, if we make the simple ansatz

$$B_l(r) = A r^{-B-3} \tag{133}$$

(A and B are constants), we find[23] that $a_l(k)$ has a simple power behavior

$$a_l(k) \underset{k\to\infty}{\sim} A \frac{\pi i}{\Gamma(B+2)} k^B . \tag{134}$$

3. *Breit-Wigner Resonances and Other Dynamical Singularities*

Let us assume that the formula (124) can be extended to at least part of the complex k plane and consider mechanisms producing singularities in the k plane. Clearly a singularity of $a_l(k)$ will correspond to the diverging of the integral representation (124). This can occur for one of two reasons. The first is that a singularity, e.g., a second-order pole of the integrand, can lie on the integration path for $|r|$ finite and may cause a divergence. The second is that the asymptotic behavior of the integrand for $\mathrm{Im}\, r \to \pm\infty$, $\mathrm{Re}\, r = \beta$ fixed, may be such that the integral diverges. It should be noted that $\zeta_l(kr)$ is an entire function of r. Hence, the only part of the integrand that can have a singularity for $|r|$ finite is the Galilei amplitude $B_l(r)$. Since this does not depend on the energy k, it cannot give rise to physically meaningful singularities.

Now let us consider the integrand for $r = \beta + i\sigma$, $\sigma \to \pm\infty$. We split the integral (124) into three parts, setting

$$a_l(k) = \left(\int_{\beta - i\infty}^{\beta - iM} + \int_{\beta - iM}^{\beta + iN} + \int_{\beta + iN}^{\beta + i\infty} \right) r^2 dr\, B_l(r) \zeta_l(kr) . \tag{135}$$

We take M and N large but finite, disregard the integral over the finite region (since it cannot give rise to dynamical singularities), and replace the integrands in the other two integrals by the corresponding asymptotics. For definiteness, let us consider

$$a_l^N(k) = \int_{\beta + iN}^{\beta + i\infty} r^2 dr\, B_l(r) \zeta_l(kr) . \tag{136}$$

We replace $\zeta_l(kr)$ according to (131) and make an ansatz for the Galilei amplitude:

$$B_l(r) \underset{\substack{\mathrm{Im}\, r \to \infty \\ \mathrm{Re}\, r = \beta}}{\sim} \frac{1}{r} e^{k_0 r}, \quad \mathrm{Im}\, k_0 > 0 . \tag{137}$$

We have

$$\begin{aligned} a_L^N(k) &= \frac{1}{2k} \int_{\beta + iN}^{\beta + i\infty} dr \left[e^{(k+k_0) r} + (-1)^{l+1} e^{(-k+k_0) r} \right] \\ &= \frac{-1}{2k} \left[\frac{e^{(k+k_0)(\beta + iN)}}{k + k_0} + (-1)^l \frac{e^{(-k+k_0)(\beta + iN)}}{k - k_0} \right] . \end{aligned} \tag{138}$$

Similarly we define

$$a_l^M(k) = \int_{\beta - i\infty}^{\beta - iM} r^2 dr\, B_l(r) \zeta_l(k) . \tag{139}$$

The ansatz

$$B_l(r) \underset{\substack{-\mathrm{Im}\, r \to \infty \\ \mathrm{Re}\, r = \beta}}{\sim} \frac{1}{r} e^{k_0 r} \quad (\mathrm{Im}\, k_0 < 0) \tag{140}$$

will lead to a term similar to (138).

Thus a quite simple ansatz on the behavior of $B_l(r)$ for $\mathrm{Im}\, r \to \pm\infty$ leads to the appearance of simple poles in the partial-wave amplitude $a_l(k)$. Clearly the conditions on the imaginary part of k_0 allow for bound states, resonances, and antibound states.[31]

Several comments are in order here. The first is that the exponent k_0 in (137) and (140) determines the position of the pole in the energy plane, but not its residue. The second is that the exponential behavior in our ansatz does not have to be the leading asymptotic term for $\mathrm{Im}\, r \to \pm\infty$, and that more singular terms may be present. Examples of such behavior will be presented in a forthcoming paper.[32] The mechanism for generating dynamic singularities, proposed above, is completely analogous to that generating Regge poles[33] in the relativistic O(3, 1)$\supset$O(2, 1)$\supset$ O(2) expansions (5).

B. The Cylindrical Expansion

Let us now investigate the eikonal-type expansion (108).

1. *Low-Energy Behavior*

We have

$$\begin{aligned} I_0(\kappa_\parallel b) &\underset{\kappa_\parallel b \to 0}{\sim} 1 + (\tfrac{1}{2}\kappa_\parallel b)^2 , \\ e^{\kappa_\perp z} &\underset{\kappa_\perp z \to 0}{\sim} 1 + \kappa_\perp z . \end{aligned} \tag{141}$$

Thus, nothing of particular interest happens to the expansion (108) in the low-energy limit. Anyhow, the expansion is suitable for treating scattering at high energies.

2. *Asymptotic Behavior*

For $\kappa_\parallel \to \infty$ we have

$$I_0(\kappa_\parallel b) \underset{\kappa_\parallel \to \infty}{\sim} \frac{1}{(2\pi\kappa_\parallel b)^{1/2}} (e^{\kappa_\parallel b} \pm i e^{-\kappa_\parallel b}) . \tag{142}$$

Let us consider the consequences of various assumptions concerning the Galilei amplitude $B(b, z)$. First, assume that $B(b, z)$ is an analytic function of b in the strip $0 \leqslant \beta' \leqslant \mathrm{Re}\, b \leqslant \beta$ with a simple pole at $b = b_0$. We can then shift the b-integration path in (108) to the left and find that for $\kappa_\parallel \to \infty$ the pole term will dominate, so that

$$F(\kappa_\perp, \kappa_\parallel) \underset{\substack{\kappa_\parallel \to \infty \\ \kappa_\perp = \text{const}}}{\sim} i(2\pi b_0)^{1/2} \frac{e^{b_0 \kappa_\parallel}}{(\kappa_\parallel)^{1/2}} \int_{\gamma - i\infty}^{\gamma + i\infty} \mathrm{Res}[B(b_0, z)]\, e^{-\kappa_\perp z}. \tag{143}$$

Similarly, if $B(b, z)$ is an analytic function of z for $\mathrm{Re}\, z \leqslant \gamma$ with a simple pole at $0 < z_0 < \gamma$,

$$F(\kappa_\perp, \kappa_\parallel) \underset{\substack{\kappa_\perp \to \infty \\ \kappa_\parallel = \text{const}}}{\sim} 2\pi i\, e^{\kappa_\perp z_0} \int_{\beta - i\infty}^{\beta + i\infty} b\, db\, \mathrm{Res}[B(b, z_0)]\, I_0(b\kappa_\parallel). \tag{144}$$

Thus, the simplest analyticity assumptions again lead to exponentially increasing amplitudes. If the residue of $B(b, z)$ at the pole in one variable is an analytic function of the other variable, with a pole somewhere in the analyticity strip, we obtain

$$F(\kappa_\perp, \kappa_\parallel) \underset{\substack{\kappa_\perp \to \infty \\ \kappa_\parallel \to \infty}}{\sim} -2\pi(2\pi b_0)^{1/2} \frac{e^{b_0 \kappa_\parallel} e^{z_0 \kappa_\perp}}{(\kappa_\parallel)^{1/2}} \mathrm{Res}[B(b, z)] \Big|_{b = b_0,\, z = z_0}. \tag{145}$$

The variables $\kappa_\perp$ and $\kappa_\parallel$ are given by (78) for the "backward" brick-wall system and by (82) for the "forward" one.

More realistic asymptotic behavior is the Regge asymptotics:

$$F(\kappa_\perp, \kappa_\parallel) = -\frac{\pi\beta(\kappa_\perp)\kappa_\parallel^{\,2\alpha(\kappa_\perp)}}{\sin\pi\alpha(\kappa_\perp)}, \tag{146}$$

where $\alpha(\kappa_\perp)$ and $\beta(\kappa_\perp)$ are arbitrary functions (the Regge trajectory and residue). In order to obtain this behavior, let us define the eikonal partial amplitude as

$$a(b, \kappa_\perp) = \int_{\gamma - i\infty}^{\gamma + i\infty} dz\, B(b, z) e^{\kappa_\perp z}. \tag{147}$$

Now if we set[21]

$$a(b, \kappa_\perp) = i \frac{\beta(\kappa_\perp) 2^{2\alpha(\kappa_\perp)}}{\sin\pi\alpha(\kappa_\perp) b^{2\alpha(\kappa_\perp)+2}} [\Gamma(\alpha(\kappa_\perp)+1)]^2, \tag{148}$$

we obtain precisely the expression (146).

3. *Dynamic Singularities*

It is again very easy to point out a mechanism leading to dynamic singularities in the variables $\kappa_\perp$ or $\kappa_\parallel$. Consider for instance the integral (147). Singularities of $B(b, z)$ for $|z|$ finite could at most generate nonphysical fixed singularities of $a(b, \kappa_\perp)$. Let us however consider the asymptotic behavior of the Galilei amplitudes. Let us make the ansatz

$$B(b, z) \underset{\substack{\mathrm{Im}\, z \to \infty \\ \mathrm{Re}\, z = \gamma}}{\sim} e^{-z f(b)}, \quad \mathrm{Im} f(b) > 0. \tag{149}$$

We have

$$a^N(b, \kappa_\perp) \equiv \int_{\gamma + iN}^{\gamma + i\infty} dz\, B(b, z) e^{\kappa_\perp z} \to \frac{e^{[\kappa_\perp - f(b)][\gamma + iN]}}{f(b) - \kappa_\perp}. \tag{150}$$

Thus the term (149) in the Galilei amplitude contributes a simple pole (150) to $a(b, \kappa_\perp)$ and the position of this pole is determined by the function $f(b)$. In turn, according to (143), this pole would give rise to the asymptotic behavior

$$F(\kappa_\perp, \kappa_\parallel) \underset{\substack{\kappa_\parallel \to \infty \\ \kappa_\perp = \text{const}}}{\sim} \frac{1}{(\kappa_\parallel)^{1/2}} e^{f^{-1}(\kappa_\perp)\kappa_\parallel}, \tag{151}$$

where $b = f^{-1}(\kappa_\perp)$ is the inverse function to $\kappa_\perp = f(b)$.

The contribution to $a(b, \kappa_\perp)$ from the other end of the integration path can be considered similarly.

We can also rewrite (108) in a different form, setting

$$a(z, \kappa_\parallel) = \int_{\beta - i\infty}^{\beta + i\infty} b\, db\, B(b, z) I_0(b\kappa_\parallel). \tag{152}$$

Using the asymptotic formula (142) we define

$$a_N(z, \kappa_\parallel) = \frac{1}{(2\pi\kappa_\parallel)^{1/2}} \int_{B + iN}^{\beta + i\infty} \sqrt{b}\, db\, B(b, z)(e^{\kappa_\parallel b} + i e^{\kappa_\parallel b}). \tag{153}$$

Clearly the ansatz

$$B(b, z) \underset{\substack{\mathrm{Im}\, b \to \infty \\ \mathrm{Re}\, b = \beta}}{\sim} \frac{1}{\sqrt{b}} e^{f(z) b}, \quad \mathrm{Im} f(z) > 0 \tag{154}$$

leads to dynamical poles in $a(z, \kappa_{\|})$, since

$$a_N(z, \kappa_{\|}) = \frac{1}{(2\pi\kappa_{\|})^{1/2}} \left\{ \frac{-e^{[\kappa_{\|}+f(z)](\beta+iN)}}{f(z)+\kappa_{\|}} + i\,\frac{e^{[f(z)-\kappa_{\|}](\beta+iN)}}{\kappa_{\|}-f(z)} \right\}. \quad (155)$$

All considerations that we have performed for the nonunitary expansions can be performed just as well for the unitary ones ($\beta=\gamma=0$) but we shall not go into that here.

V. CONCLUSIONS

The main content of this article is the presentation of two-variable expansions of nonrelativistic scattering amplitudes in terms of basis functions of representations of the group E(3). We have presented all three expansions, corresponding to subgroup reductions of E(3), and consider two of them to be of particular physical interest. These are the spherical expansions corresponding to the $E(3)\supset O(3)\supset O(2)$ reduction, that is related to the usual partial-wave expansion, and the cylindrical expansions corresponding to the $E(3)\supset E(2)\times T_{\perp} \supset O(2)\times T_{\perp}$ reduction, related to the usual eikonal expansion. We present expansions in terms of unitary representations for square-integrable amplitudes [see (87), (88) and (92), (93)] and also in terms of nonunitary representations for asymptotically growing amplitudes [see (104–(109)]. In the present paper we concentrate mainly on formal aspects of the theory, in particular various mathematical properties of the expansions, the Clebsch-Gordan coefficients of E(3), etc. We do, however, consider some physical properties, in particular the appearance of dynamical singularities of the amplitudes—like Breit-Wigner resonances, bound states, etc. We also discuss the threshold behavior and the mechanisms for generating high-energy behavior of various types—exponentially growing or decreasing, power-bounded, Regge asymptotics, etc.

We are fully aware of the fact that the usual dynamic theory of nonrelativistic scattering, based on the use of a Schrödinger equation (or some other equation, e.g., the Lippmann-Schwinger one) with a definite potential, is simpler and more straightforward than a scattering theory based on Galilei amplitudes. However, the concept of a potential does not generalize in a simple manner to the relativistic case, whereas Galilei amplitudes not only do this, but are actually obtained as $c\to\infty$ limits of Lorentz amplitudes.

Besides providing insight into the case of relativistic scattering of elementary particles, the Galilei expansions presented in this article have their own intrinsic mathematical and physical interest. From the mathematical point of view they constitute a further development of the representation theory of E(3) in various bases, in particular harmonic analysis on this group (or rather on the corresponding homogeneous manifold) both for square-integrable and for considerably more general functions (use is made of nonunitary infinite-dimensional representations). From the physical point of view the Galilei expansions can be used to help separate variables in the Schrödinger equation and to provide representations of amplitudes in the case when the Schrödinger equation is not solvable. What is more, it is well known that very few solvable potentials exist in quantum mechanics (even for the S wave, still more so for all waves) and also that it is very difficult to construct potentials, providing scattering amplitudes with definite required properties.[31] In a subsequent article[32] we demonstrate simple Galilei amplitudes that generate partial-wave amplitudes $a_l(k)$ having singularities of required types at required positions (e.g., Breit-Wigner resonances), correct threshold behavior, reasonable asymptotic behavior, etc.

Finally, let us mention that one could attempt to derive equations directly for the Galilei amplitudes, e.g., $A_l(r)$ in (87) and (88). Such an approach would be similar to the variable-phase method[34] that provides equations for the partial-wave amplitudes $a_l(k)$. These equations are nonlinear, but are still very useful in many applications. We have so far made no progress in this direction.

An important feature is that, contrary to the relativistic case, the E(3) group does not provide a framework for complex angular momentum theory, since E(3) does not have an O(2, 1) subgroup. In particular, the nonrelativistic limit of the Regge expansion is the eikonal expansion, rather than the Sommerfeld-Watson transform of the partial-wave expansion. The absence of a group-theoretic motivation for Regge-pole theory in nonrelativistic scattering, while demonstrated clearly in our approach, actually has deeper roots. Indeed for relativistic scattering the Regge-pole expansion can be obtained by performing crossed-channel partial-wave analysis,[7] in which O(2, 1) makes its appearance since "negative mass squared" states [with $t=(p_1-p_3)^2<0$] are present in the direct product of two positive mass squared single-particle states transforming according to irreducible unitary representations of the Poincaré group (one in the initial state, one in the final one and hence corresponding to negative energy). A similar direct product of representations of the Galilei group will contain zero-mass representations, corresponding to an E(2) little group, but not to an O(2, 1) group.[29]

An application of the E(3) expansions to potential scattering will be published separately.[32] There we consider further properties of the Galilei amplitudes $B_l(r)$ and $B(b,z)$; in particular we study their behavior for various specific potentials. We also present formulas for the Galilei amplitude in the first Born approximation for arbitrary potentials. Interestingly, for the spherical expansions these involve the Clebsch-Gordan coefficients for E(3).

We also plan to give a more comprehensive treatment of the representation theory of the group E(3) in different bases and particularly to give a complete treatment of the Clebsch-Gordan problem. In this connection we would like to mention that the E(3) Clebsch-Gordan coefficients have been obtained as a limit of the O(4) coefficients[35] and have also been considered using different techniques.[36]

Finally let us mention that our main aim is to gain a better understanding of the connection between the Galilei amplitudes and the potential and of further general properties of these amplitudes. We will then return to the relativistic case of O(3, 1) expansions, where no potential exists, but where scattering can be directly treated and data directly analyzed in terms of the corresponding Lorentz amplitudes. In this connection it will also be of some interest to consider the non-subgroup-type E(3) expansions, which were not discussed at all in this article.[5]

ACKNOWLEDGMENTS

We would like to thank Professor W. Miller Jr. for a discussion of the Clebsch-Gordan coefficients of E(3) and in particular for calling Refs. 35 and 36 to our attention.

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PHYSICAL REVIEW D VOLUME 8, NUMBER 8 15 OCTOBER 1973

Singular Cores in the Three-Body Problem. II. Numerical Solution*

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The formalism for singular cores previously introduced is investigated numerically for a simple model. Calculations of three-particle binding energies and the analog of *N-d* scattering for this model demonstrate the practicality of the earlier theoretical development.

I. INTRODUCTION

The first paper of this series discussed a new generalization of the three-particle formalism to include two-body interactions characterized by a hard core or by a boundary condition on the wave function (BCM).[1] Specific questions such as uniqueness and three-body unitarity were investigated in some detail at that time. The present article is concerned with a numerical investigation of the formalism for a simple model. Calculations of three-particle binding energies and the analog of *N-d* elastic scattering within the context of this model demonstrate the practicality of the earlier theoretical development for the treatment of singular cores.

The principal motivation for this development is the versatility afforded by being able to utilize this additional class of interactions in the three-body problem. For example, calculations to date in the three-nucleon system with realistic interactions have been almost exclusively restricted to soft-core models, the single exception being the long and difficult variational calculation on the Hamada-Johnston hard core by Delves, *et al.*[2] The results of these computations have generated some doubt as to the ability of such models to fit the experimental data. For example, it appears that any soft-core model which fits the two-nucleon phase shifts reasonably well will underbind the triton by about 2 MeV. A significant discrepancy also appears to exist in the case of the ^{3}He charge form factor[3] (for a more complete discussion see SCI). More recently, the present author has generalized the boundary-condition approach to provide a complete phenomenology of three-particle final states.[4] The model discussed below may also be regarded as a first approximation to this general scheme, and hence has a significance quite apart from singular cores per se.

We begin in Sec. II with a description of our model and the relevant integral equation to be solved. We also take this opportunity to present simplified equations for the general case considered in SCI; these formulas are germane to applications of the formalism with realistic interactions. In Sec. III we introduce a numerical technique which is particularly advantageous for solving integral equations of a certain class; the method is illustrated by means of an exactly soluble example. Numerical results for our model are presented in Sec. IV, which concludes with a discussion of these results and implications for future calculations with our formalism.

II. A SIMPLE MODEL

We shall consider a model in which three identical spinless particles of mass M interact via the BCM alone (no potentials external to the core) and only in relative s waves. If a denotes the core