$$A_6 = s_1^{\alpha_1}\left(\frac{s_{12}}{s_1}\right)^{\alpha_2}\left(\frac{s_{31}}{s_1}\right)^{\alpha_3} \times \frac{1}{(s_1 s_2/s_{12})(s_3 s_1/s_{31}) + 1/s_1^{\alpha_v(0)-\alpha_1}} \, .$$

[14]These limits have been discussed and studied in the dual resonance model by C. E. DeTar and J. H. Weis, Phys. Rev. D 4, 3141 (1971).

[15]Since the behavior of the amplitude is less specified, it is generally more difficult to rigorously exclude "nonsense" helicity singularities in limits like (2.4) with few variables asymptotic when they do not also occur in the Regge limits. These would be peculiar helicity singularities whose location differed from that of a given angular momentum singularity by an integer, but do not occur in the part of the amplitude with the angular momentum singularity.

[16]The criterion $\lambda \leq 0$ for an helicity limit being inside the physical region has been emphasized by Abarbanel and Schwimmer (Ref. 1). For a nice presentation of kinematics in standard notation, see M. N. Misheloff, Phys. Rev. 184, 1732 (1969).

[17]Note that $\lambda(t_1,t_2,t_3) \leq 0$, only if all t_i have the same sign. Then, given, say, t_2 and t_3, $\lambda \leq 0$ for

$$|\sqrt{t_2} - \sqrt{t_3}| \leq |t_1| \leq |\sqrt{t_2} + \sqrt{t_3}| \, .$$

[18]From Eq. (2.12) one sees that the triple-Regge limit ($s_2, s_3, s_{23} \to \infty$) is obtained for fixed φ_2 and φ_3, whereas, the Regge-helicity limit (s_2, s_3, s_{23} fixed) is obtained for φ_2 and φ_3 varying with s_{12}/s_1 and s_{31}/s_1. Thus, depending on the relative orientation of the states on either side of $\sqrt{s_1}$, either one or the other limit is obtained inside the physical region.

[19]A. Patrascioiu, MIT report (unpublished).

[20]C cannot be a constant since the coefficient of s_{12}/s_1 in (2.13) can be made arbitrarily large for $\lambda \leq 0$ by scaling all the t_i to large values.

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Chiral Symmetry Limit in the SU(3) σ Model with Bilinear Breaking

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We add to the usual σ model a $(3^*, 3) + (3, 3^*)$ term which is bilinear in the scalar-meson fields and study the behavior of the tree-approximation solutions as the explicit symmetry breaking is turned off. We find that *both* $c \simeq -\sqrt{2}$ and $c \simeq 0$ occur for the same mass spectrum, 0-8 mixing, and F_K/F_π. Solutions with different c values approach the symmetry limit differently, but in this limit one can smoothly reach either a Goldstone pseudoscalar octet, or a symmetric vacuum, or neither for both values of c. No clear indication is found that solutions are near a critical value of the mass parameter.

I. INTRODUCTION

The assumption that expansions in symmetry-breaking parameters exist provides a basis for study of approximate symmetries of the strong-interaction Hamiltonian. For systems whose underlying symmetry is chiral SU(3)×SU(3),[1] the question of the nature of the symmetry limit and the existence of expansions in powers of symmetry-breaking parameters about such a limit is complicated by the Goldstone phenomenon[2] where the solutions of the symmetric theory do not exhibit the full symmetry of the Hamiltonian. Dashen[3] and Dashen and Weinstein[4] developed chiral-symmetry-breaking expansions and systematically exploited the technique in deriving a number of correlations among symmetry-breaking effects. Li and Pagels[5] subsequently showed that Goldstone-boson intermediate states can give rise to singularities in the symmetry limit, thus invalidating many attempts to extrapolate "soft pion" theorems to the pion mass shell. In a study of Lagrangian models solved in the tree-graph approximation, Carruthers and Haymaker[6] noticed a different phenomenon which has the same negative implication for expansions about a Goldstone symmetry solution. They found that vacuum expectation values in the σ model[7] are multivalued functions of the Lagrangian symmetry-breaking parameters and that the power-series expansion of their solution had a radius of convergence much smaller than the value of the SU(3)×SU(3)-breaking parameter needed to produce correct pseudoscalar masses.

Subsequent work on Lagrangian models has clarified the nature of the singularities.[8,9,10] Carruthers and Haymaker[10] have suggested pursuing the idea that the n-point functions are analytic functions of the breaking parameters in order to extract model-independent results. The effects of single-loop diagrams on the tree-approximation parameters have also been studied and found to be small.[11] In this connection, single-loop diagrams in the SU(2)×SU(2) σ model have been shown to be analytic in the symmetry limit,[12] indicating that only higher-order loop corrections contribute to the nonanalytic Li-Pagels[5] behavior near the Goldstone limit. The singularities of the tree approximation itself may still be a problem, of course.

In this work, we extend the study of one- and two-point functions as multivalued functions of the chiral-symmetry-breaking parameters. We stay within the tree approximation and push the σ model further to include breaking terms in the Lagrangian which are bilinear in the basis fields [though still transforming as $P=+$, $I=Y=0$ members of a $(3,3^*)+(3^*,3)$ representation[13,14]]. Since the model with bilinear breaking terms is renormalizable,[11] loop corrections could be calculated in principle. However, the tree-approximation solutions themselves are varied and complex enough to discourage any idea of systematically evaluating loop effects. We look closely at the numerous tree-approximation solutions of this more complicated model to see what features, if any, are common to all solutions of both models.

In the following section, we review and extend the work on the σ model with linear $(3^*,3)+(3,3^*)$ breaking. In Sec. III we introduce the bilinear term and analyze the way the solutions to this model approach the SU(3)×SU(3)-symmetry limit. In Sec. IV, we summarize and discuss the implications of our results. The Appendix covers the procedure for obtaining solutions and includes necessary definitions and formulas.

II. LINEAR SYMMETRY BREAKING

In this section we shall review the tree-approximation solutions to the linear σ model with linear $(3,3^*)+(3^*,3)$ breaking.[7] The Lagrangian density is

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu u_i \partial_\mu u_i + \partial_\mu v_i \partial_\mu v_i) - \alpha I_2 - \alpha'(I_2)^2 - \beta I_3^+ - \gamma I_4 + \delta a(u_0 + c u_8), \quad (1)$$

where u_i and v_i are, respectively, scalar and pseudoscalar fields which belong to a $(3^*,3)+(3,3^*)$ representation of SU(3)×SU(3). The invariants I_2, I_3^+, and I_4 are defined in the Appendix. Predictions of this model[7] and of its generalizations[15] have been extensively studied. In addition, the analytic structure of the solutions as functions of δ and c has been examined, and those symmetry limits have been determined which can be reached smoothly from solutions fitted to physical parameters, such as pseudoscalar masses and one of the $I=Y=0$ members of the scalar nonet.[16] The indications are[6,10] that the linear model can connect smoothly only to the "normal" symmetry limit, $\delta=0$, $\langle u_0\rangle=0$, $\langle u_8\rangle=0$, and all masses nonzero degenerate, *or* to the Goldstone pseudoscalar octet symmetry limit $\langle u_0\rangle\neq 0$, $\delta=0$, $\langle u_8\rangle=0$, pseudoscalar octet masses $=0$, $m_{\eta'}\neq 0$, and the scalar nonet masses degenerate. We review this situation now and comment on the observation[10] that one solution is near a "critical point" in the parameter α.

The most reliable experimental meson information is, of course, the pseudoscalar masses. The pseudoscalar mass-squared matrix $m^2_{ij}=\partial^2\mathcal{L}/\partial v_i\partial v_j$, after diagonalization in the 0-8 sector, provides four equations (see Appendix) for the four quantities

$$b = \frac{\langle 0|u_8|0\rangle}{\sqrt{2}\,\langle 0|u_0|0\rangle},$$

$$x_1 = 2\alpha + 4\alpha'\lambda_p^2(1+2b^2), \quad (2)$$

$$x_2 = 4\sqrt{6}\,\beta\lambda_p,$$

$$x_3 = 8\gamma\lambda_p^2, \quad \lambda_p \equiv \langle u_0\rangle_{\text{physical}}.$$

In particular, m_π^2, m_K^2, and $m_\eta^2+m_{\eta'}^2$ enable one to express the latter three parameters, x_1, x_2, and x_3, in terms of b_p. The quantity $m_{\eta'}^2-m_\eta^2$ then gives essentially[17] a quadratic equation for b. We label the solutions to this equation b(physical) $\equiv b_p$. A set of values of x_1, x_2, and x_3 are then found for *each* b_p, but the value of $\lambda_p=\langle u_0\rangle_{\text{physical}}$ for the physical masses cannot be determined separately from the coefficients of the invariants without additional input, such as F_π, to set the scale of the vacuum-breaking effects.[18] The necessary conditions for stability, $\partial\mathcal{L}/\partial u_0=0$ and $\partial\mathcal{L}/\partial u_8=0$,[19] are

$$\frac{\delta a}{\lambda_p} = x_1 + x_2(1-b^2) + \tfrac{2}{3}x_3(1+6b^2-2b^3), \quad (3)$$

$$\frac{\delta a c}{\lambda_p} = \sqrt{2}\,b[x_1 - x_2(1+b) + 2x_3(1-b+b^2)], \quad (4)$$

respectively. Equations (3) and (4) yield values for $\delta a/\lambda_p$ and c for *each* b_p value. For the scalar mass matrix $\partial^2\mathcal{L}/\partial u_i\partial u_j = M^2_{ij}$, the values of $M_{\pi_N}^2$

TABLE I. Solutions to linear-breaking model. Input masses are $m_\pi^2=0.019$, $m_K^2=0.246$, $m_\eta^2=3.01$, and $m_{\eta'}^2=0.917$. All masses are in GeV.

b_p	c_p	M_{π_N}	M_K	$\tan\theta_P$	$M_\epsilon^2+M_{\epsilon'}^2$ [a]	$M_{\epsilon'}$	M_ϵ	$\tan\theta_S$
−0.218	−1.30	0.906	0.888	0.052	1.70	1.140	0.630	0.955
					1.40	1.050	0.550	0.682
−0.352	−1.33	0.902	0.725	0.228	1.70	1.225	0.456	1.290
					1.40	1.095	0.440	1.146

[a] These are input values. Both c_p values go with each $M_\epsilon^2+M_{\epsilon'}^2$ value.

$=\partial^2\mathcal{L}/\partial u_{1,2,3}{}^2$ and $M_K{}^2=\partial^2\mathcal{L}/\partial u_{4,5,6,7}{}^2$ are predicted for each b_p solution. The isoscalar- and scalar-nonet masses squared, $M_\epsilon{}^2$ and $M_{\epsilon'}{}^2$, are determined, and the value of one or of a combination of the two assumed in order to separate α and α'. The latter information is necessary to study the behavior of the solutions as functions of δ and/or c. For each of several reasonable values of $M_\epsilon{}^2+M_{\epsilon'}{}^2$, the two solutions of the model which correspond to the pseudoscalar mass values (in units of GeV^2 always unless specified otherwise) $m_\pi{}^2=0.019$, $m_K{}^2=0.246$, $m_\eta{}^2=0.301$, and $m_{\eta'}{}^2=0.917$ are shown in Table I. The solutions differ in some details, but the features that $c\approx-\sqrt{2}$ (near the SU(2)×SU(2)-invariant Hamiltonian) and that b_p is small and negative (small vacuum breaking relative to chiral breaking) are common to both. We are primarily interested in following the solutions as functions of δ in the region around δ(physical) $\equiv\delta_p$ down to $\delta=0$, and we shall keep $c=c_{\text{physical}}$ for this study.[8] We ask whether or not the physical solution connects smoothly to a symmetric limit as $\delta\to0$, and, if so, what the character of the symmetric solution is. A reasonable criterion of "smoothness" was presented by Carruthers and Haymaker,[6] who require that all masses squared remain positive (a no-ghost or stability criterion)

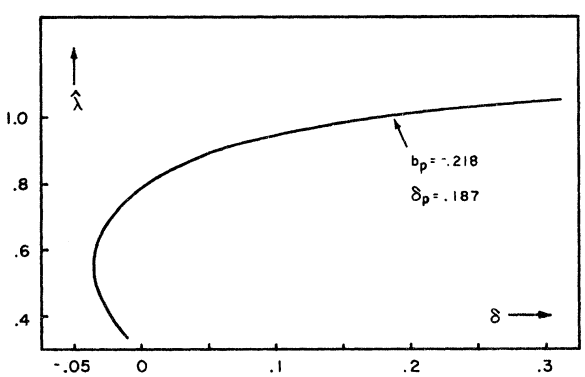

FIG. 1. Linear-breaking case $b_p=-0.218$, $M_\epsilon{}^2+M_{\epsilon'}{}^2=1.7$ GeV^2 for $\hat\lambda$ vs δ region including physical point, symmetry point, and singularity nearest symmetry point. $\hat\lambda$ is dimensionless and δ is actually $\delta a/\lambda_p$ in units of GeV^2, so $\delta_p\approx10m_\pi{}^2$.

and that no singularities of $\lambda(\delta)$, $b(\delta)$, or masses are encountered in the path $\delta\to0$. We note that the mass positivity requirement usually implies that no singularities occur, since singularities are generally accompanied by a zero of M_ϵ or $M_{\epsilon'}$.[9]

The actual procedure we use is dictated by calculational convenience. We study λ, δ, and masses squared as functions of $b=b(\delta)\equiv\langle u_8\rangle/\sqrt{2}\langle u_0\rangle$ (with x_1, α, $\alpha'\lambda_p{}^2$, x_2, x_3, and c fixed) in the neighborhood of $b_p=b(\delta_p)$. The results, however, are plotted as $\lambda=\langle u_0\rangle$ vs δ, since the behavior is more clearly shown in this way. In Figs. 1 and 2 we show the dependence of the vacuum expectation values $\lambda\equiv\langle u_0\rangle$ on the Hamiltonian symmetry-breaking parameter, δ, for the solutions $b_p=-0.218$ and $b_p=-0.352$, respectively. The scalar-mass values used are such that $M_\epsilon{}^2+M_{\epsilon'}{}^2=1.7$ GeV^2, a convenient input combination for computational purposes. Both solutions have the same character, that is, they connect smoothly to $\langle u_0\rangle\neq0$, $\langle u_8\rangle=0$ Goldstone pseudoscalar-octet-symmetry limits as $\delta\to0$. The one $b_p=-0.218$ yields $M_\epsilon=630$ MeV and $M_{\epsilon'}=1140$ MeV; the $b_p=-0.352$ yields $M_\epsilon=456$ MeV and $M_{\epsilon'}=1204$ MeV. However, both solutions also exhibit singularities at negative δ values for which $|\delta|_{\text{singularity}}<\delta$(physical), thus prohibiting a power-series expansion about $\delta=0$ from reaching δ(physical). The singularity occurs when $M_{\epsilon'}=0$.[9] These solutions have identical features to the solution studied by Carruthers and Haymaker[6] for the model with $\alpha=0$.

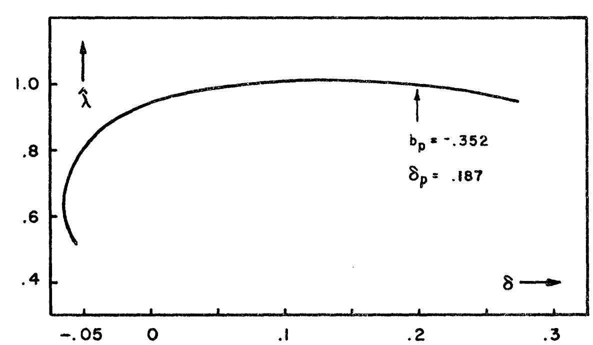

FIG. 2. Case $b_p=-0.352$, $M_\epsilon{}^2+M_{\epsilon'}{}^2=1.7$ GeV^2.

We have chosen to examine the details of both solutions because the only argument against the $b_p = -0.352$ solution at this stage is its somewhat low-κ-mass prediction of 725 MeV and, with the input combination $M_\epsilon^2 + M_{\epsilon'}^2 = 1.7$ GeV^2, the low value of M_ϵ. It is, however, not clear whether or not one should try to compare to better than 100–200 MeV with the illusive scalar-mass spectrum. An additional reason for treating this solution is the sensitivity of the b_p values to the choice of average kaon mass, as pointed out by Chan and Haymaker.[11]

Let us now choose several different $M_\epsilon^2 + M_{\epsilon'}^2$ values to study the sensitivity of the conclusions about the nature of the symmetry limit to the values of the scalar masses. The value $M_\epsilon^2 + M_{\epsilon'}^2 = 2.0$ GeV^2 yields the *same* essential features as shown in Figs. 1 and 2, and no conclusions are altered. However, the choice $M_\epsilon^2 + M_{\epsilon'}^2 = 1.4$ GeV^2 leads to new behavior for the $b_p = -0.218$ solution for which $M_\epsilon = 552$ MeV and $M_{\epsilon'} = 1046$ MeV, but the *same* behavior as before for the solution $b_p = -0.352$, for which $M_\epsilon = 440$ MeV and $M_{\epsilon'} = 1095$ MeV. The curves are shown in Figs. 3 and 4 for $b_p = -0.218$ and $b_p = -0.352$, respectively. The former solution ($b_p = -0.218$, Fig. 3) has changed character, now connecting to a *normal* ($\langle u_0 \rangle = 0$, $\langle u_8 \rangle = 0$ as $\delta \to 0$) solution with degenerate pseudoscalar and scalar nonets. A continuation from $\delta = 0$ to $\delta = \delta_p$ is blocked by a singularity, at which $M_{\epsilon'} = 0$,[9] at $\delta \approx 0.007$. The details of this region are shown in the inset in Fig. 3. The effect exhibited is essentially that pointed out by Olshansky.[8] There is a slight difference here, since we reevaluate all of the coefficients for the new values M_ϵ and $M_{\epsilon'}$ rather than view the effect as being dependent on α alone. This phenomenon has been likened to a "critical point" in the solution to the theory.[10] The suggestion was made that model-independent results might be obtained by hypothesizing that one is near a critical point in the meson mass parameter α and then applying Wilson's renormalization-group techniques.[20]

However, we wish to point out that the other solution (see Fig. 4) for which $b_p = -0.352$ and $c = -1.32$, does *not* exhibit this critical-point effect, and retains the Goldstone-limit character shown in Figs. 2 and 4, for $M_\epsilon^2 + M_{\epsilon'}^2$ values in the range $1.2 \leq M_\epsilon^2 + M_{\epsilon'}^2 \leq 2.0$ GeV^2, with no accompanying transition from Goldstone to normal symmetry limit. The transition will occur for this solution, just as for the solution $b_p = -0.218$, $c = -1.30$, but it occurs for a *very* low value of the scalar mass combination $M_\epsilon^2 + M_{\epsilon'}^2$. Only the $b_p = -0.218$ solution, therefore, provides support for the notion that the meson system is *close* to a transition from a Goldstone solution to a normal

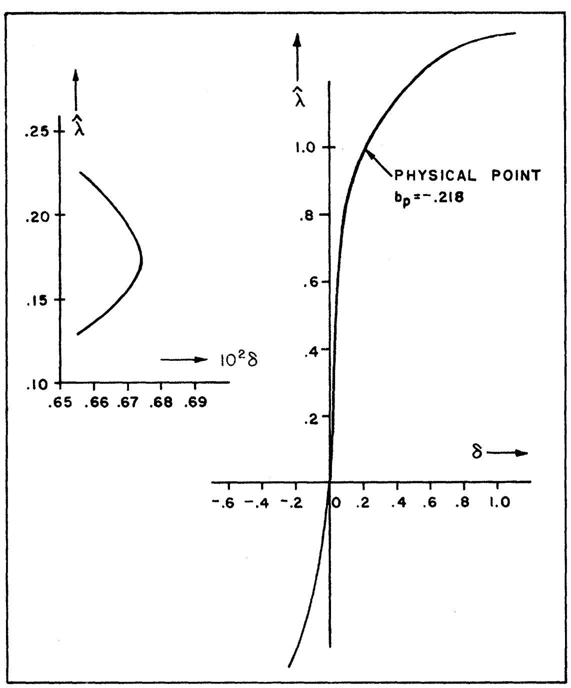

FIG. 3. Case $b_p = -0.218$, $M_\epsilon^2 + M_{\epsilon'}^2 = 1.4$ GeV^2.

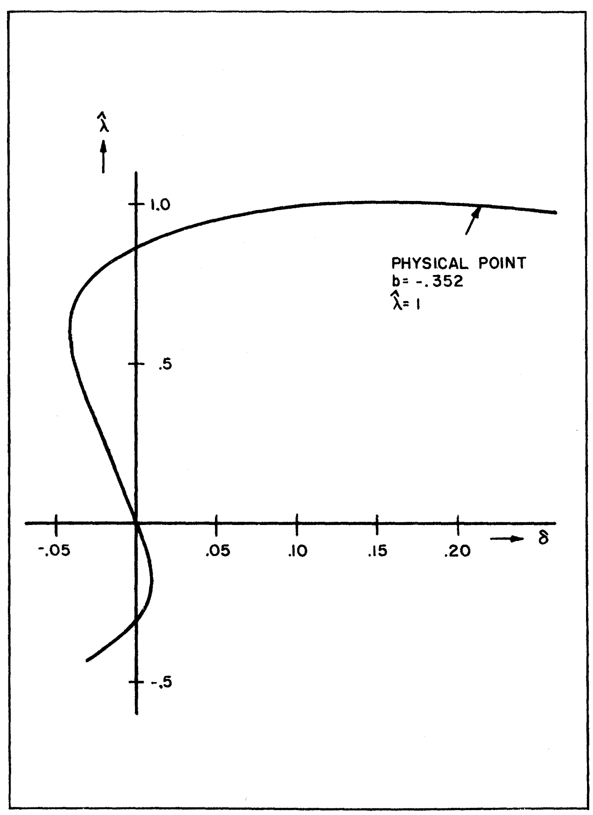

FIG. 4. Case $b_p = -0.352$, $M_\epsilon^2 + M_{\epsilon'}^2 = 1.4$ GeV^2.

solution, the latter separated by singularities from the physical values of masses and couplings. We shall return to this point in the next section, where we discuss the effects which arise when bilinear chiral-symmetry-breaking terms are added to the Lagrangian, Eq. (1).

III. BILINEAR $(3, 3^*)+(3^*, 3)$-BREAKING EFFECTS

A Lagrangian with exactly the same SU(3)×SU(3) structure as the original σ model, Eq. (1), is obtained by adding a term $\delta d(U_0+cU_8)$, bilinear in the fields u_i and v_i, which is parallel in SU(3) space to the bilinear breaking term of Eq. (1). The scalar quantities U_0 and U_8 are members of the $(3, 3^*)+(3^*, 3)$ component in the reduction of the product of the basic multiplet with itself. Explicitly we have

$$U_0=4(\tfrac{2}{3})^{1/2}(u_0{}^2-v_0{}^2)-2(\tfrac{2}{3})^{1/2}\sum_{i=1}^{8}(u_i{}^2-v_i{}^2),$$
$$U_8=-4(\tfrac{2}{3})^{1/2}(u_0u_8-v_0v_8)+4\sum_{i=1}^{8}d_{8ii}(u_i{}^2-v_i{}^2). \qquad (5)$$

The mass formulas, extremum equations, and divergence equations are all modified in the presence of the new terms. We relegate to the Appendix the details of the equations and their solutions. The essential point is that we have one new parameter, d, and must therefore provide one more piece of information. We choose to work with the mass of the scalar, $I=1$ particle π_N, which may be tentatively identified with the $I=1$, $J^P=(0^+?)$ effect which has been observed at about 980 MeV. The standard σ model predicts this mass to be about 900 MeV, somewhat lower than one would like. While our main purpose is not to "fix up" the scalar mass spectrum, it is of interest to determine the behavior of the tree-approximation solutions when the π_N mass is lifted to a possibly more realistic value and to compare the behavior with the results of the linear-breaking case. Qualitatively, the most striking difference between the solutions of the modified model and those of the original one is the added complexity of the former. We find that there are eight solutions for each set of mass $m_\pi{}^2$, $m_K{}^2$, $m_\eta{}^2$, $m_{\eta'}{}^2$, and $M_{\pi_N}{}^2$. The quantity b_p $(b=\langle u_8\rangle/\sqrt{2}\langle u_0\rangle)$, the ratio of SU(3) octet to singlet vacuum breaking, satisfies a quartic equation. For each of the four b_p values which solves the quartic equation that determines the b_p's, there are now two corresponding values of c_p, the ratio of SU(3) octet to singlet breaking in the Lagrangian.

Before entering into the detailed results of our solutions to the model with the bilinear breaking terms added, we note several new possibilities that this expanded model allows which are evident from the general tree-approximation equations themselves. First of all, as is clear from the bilinear terms in the axial-vector divergence equations

$$\partial_\mu A_\mu^i=\delta(\tfrac{2}{3})^{1/2}[a+4(\tfrac{2}{3})^{1/2}d\langle u_0\rangle]v_i+4(\tfrac{2}{3})^{1/2}\delta d(\tilde{u}_0v_i+v_0\tilde{u}_i+\sum_{j,k=1}^{8}d_{ijk}\tilde{u}_jv_k)+\text{octet breaking}, \qquad (6)$$

where $u_0=\langle u_0\rangle+\tilde{u}_0$, $u_8=\langle u_8\rangle+\tilde{u}_8$, one can discuss broken-chiral-symmetry relations from the point of view of Dashen's power-counting arguments[3] even within the tree approximation. Even in σ models, as mentioned by Carruthers and Haymaker,[6] this is the essential question in regard to the validity of low-energy theorems. More technical points are (i) that the $\langle u_8\rangle=0$, $c\neq 0$ solution, impossible in the linear-breaking case, is allowed in the bilinear-breaking version; and (ii) that δdU_0 contributes to splitting the π^0 and η' masses and, in contrast with the linear-breaking case, the I_3^+ is no longer essential to keeping π and η' nondegenerate in the SU(3) limit, $c=0$.[21]

To begin the discussion of the solutions of the model with the added term, we choose the input value $M_{\pi_N}{}^2=0.8229\ \text{GeV}^2$ (equal to the "preferred" one[6] predicted by the model when only linear-breaking terms are present; see Sec. II). In Table II we list all of the eight solutions found for the mass inputs $m_\pi{}^2=0.019$, $m_K{}^2=0.246$, $m_\eta{}^2=0.301$, $m_{\eta'}{}^2=0.917$, and $M_{\pi_N}{}^2=0.8229$. Of the eight solutions, one is identical to the solution of the linear-breaking model with the same mass spectrum, having $b_p=-0.218$, $c=-1.30$, and $d=0$. Another solution is close to the other linear-model solution which had $M_{\pi_N}{}^2=0.816$, $b_p=-0.354$, and $c=-1.32$. As we let $M_{\pi_N}{}^2$ decrease to this latter value, 0.816 GeV^2, we recapture this other linear-breaking solution, with $d=0$, as one of the eight solutions to the modified model.

The remaining six solutions, however, are completely different from either of the solutions discussed in Sec. II. The cases with $b=-20.65$ are clearly outside of currently popular ideas about chiral-symmetry breaking, as it is impossible to make them even roughly compatible with estimates of b and c from $F_K/F_\pi f_+(0)$ or from a quark-model description of baryon mass splittings. In addition, if we let the value of $M_{\pi_N}{}^2$ increase, we find that the real branches $b=-0.3945$ and -20.65 meet and split off again into complex conjugate branches as $M_{\pi_N}{}^2\approx 0.90\ \text{GeV}^2$ is reached. For these reasons, we choose not to consider further the last four solutions of Table II. The behavior just described

TABLE II. Solutions to bilinear-breaking model. The mass value $M_{\pi_N}=0.906$ is assumed, identical to one *predicted* for the $b_p=-0.218$ solution of linear case ($M_{\pi_N}{}^2=0.823$). All masses are in GeV.

b_p	c_p	M_{π_N}	$\tan\theta_P$	$M_\epsilon{}^2+M_{\epsilon'}{}^2$ [a]	$M_{\epsilon'}$	M_ϵ	$\tan\theta_S$
−0.218	−0.308	0.888	0.052	1.70	1.140	0.630	0.955
	−1.30			1.20	0.993	0.456	0.508
+0.120	−0.010	0.766	0.184	1.70	1.050	0.736	−6.07
	−1.17			1.20	0.810	0.715	−6.67
−0.390	−0.60	0.715	0.244	1.70	1.23	0.413	1.204
	−1.32			1.20	1.02	0.391	0.982
−20.55	−1.48	0.550	0.947	1.70	1.30	0.120	0.054
	4.28			1.20	1.09	0.123	0.063

[a] These are input values. Both c_p values go with each $M_\epsilon{}^2+M_{\epsilon'}{}^2$ value.

gives some added justification for discarding the $b_p=-0.352$ case of the linear model, since it disappears as a viable candidate if one tries to "lift" the scalar spectrum by adding a bilinear breaking term. At the same time, of course, new candidates for "physical" solutions are introduced, and it is these that we wish to discuss in the remainder of this section.

The most striking feature of Table II is that for each of the b_p values, $b_p=-0.218$ and $b_p=0.120$, one has *both* of the currently competing pictures of chiral-symmetry breaking as possible solutions. Namely, for *each* b_p *either* $c\approx-\sqrt{2}$ (Ref. 14) [near SU(2)×SU(2)] *or* c small[22] [near SU(3) Lagrangian symmetry]. This effect becomes even more pronounced as $M_{\pi_N}{}^2$ is allowed to rise. Thus the restriction to $c\approx-\sqrt{2}$ of the usual σ model, a result characteristic of strict pole dominance of all nonvanishing divergences,[23] disappears when deviations from operator partial conservation of axial-vector current (PCAC) are allowed by the introduction of bilinear-breaking terms.

Our primary interest is in the behavior of solutions as symmetry breaking is turned off. We see in Fig. 5, where $\hat\lambda$ ($\hat\lambda=\langle u_0\rangle/\langle u_0\rangle_{\text{physical}}$) is plotted against $\hat\delta=\delta/\delta_{\text{physical}}$, that the $b=-0.219$ and $c=-0.3$ solution chooses the Goldstone pseudoscalar octet in the symmetry limit, $\hat\delta\to0$. Note that one can subsequently reach the normal limit by circling two branch points. This behavior is found for all of the values 1.2 GeV$^2\leqslant M_\epsilon{}^2+M_{\epsilon'}{}^2\leqslant1.7$ GeV^2. There is no singularity of λ between $\hat\delta=1$ and $\hat\delta=0$. Moreover, there are no real δ singularities in the whole range $-1\leqslant|\hat\delta|\leqslant1$, and $\hat\lambda$ may be expandable about $\hat\delta=0$ in this entire region, which includes the physical solution. Clearly, from Fig.

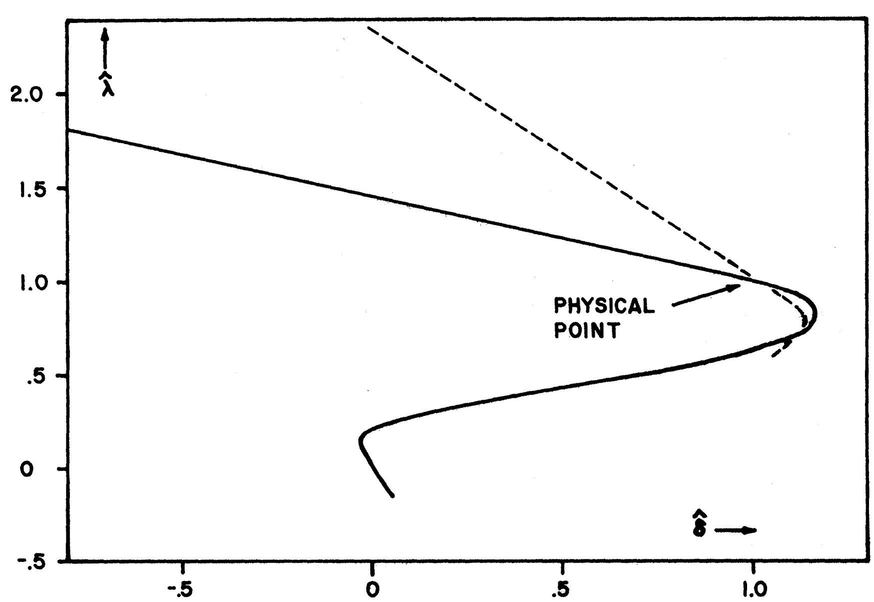

FIG. 5. Bilinear-breaking cases $M_{\pi_N}{}^2=0.823$ GeV^2, $b_p=-0.218$, $c=-0.308$, $M_\epsilon{}^2+M_{\epsilon'}{}^2=1.7$ GeV^2, (solid line) and $M_\epsilon{}^2+M_{\epsilon'}{}^2=1.2$ GeV^2 (dashed line). $\hat\lambda=\lambda/\lambda_p$ and $\hat\delta=\delta/\delta_p$ are dimensionless.

5, a linear fit to $\hat{\lambda}$ in this region is a good approximation.

When we examine the pseudoscalar masses, however, we find that they cannot be described by a linear expansion in $\hat{\delta}$ from $\hat{\delta}=0$ to $\hat{\delta}=1$ (physical $\hat{\delta}=1$ with our normalization). This is shown in Fig. 6, where the pion mass is plotted against $\hat{\delta}$. The pion mass squared rises from zero at $\hat{\delta}=0$, reaches a local maximum before $\hat{\delta}=1$, and then falls sharply through the physical value $m_\pi^2=0.019$ GeV^2. In addition, as the value of $M_\epsilon^2+M_{\epsilon'}^2$ is lowered from 1.7 GeV^2, the position of the local minimum, which occurs at small negative $\hat{\delta}$ values when $M_\epsilon^2+M_{\epsilon'}^2$ is above 1.7 GeV^2, moves toward more positive $\hat{\delta}$ values and occurs at about $\hat{\delta}=0.5$ when $M_\epsilon^2+M_{\epsilon'}^2=1.2$ GeV^2. This behavior is shown in Fig. 7, and the negative mass-squared values that occur between the physical point at $\delta=1$ and the symmetry limit at $\delta=0$ clearly violate the criterion that the symmetry limit be reached via physically admissable values of the observable quantities. Thus the $b=-0.218$ and $c=0.3$ solution is unable to connect naturally to an SU(3)×SU(3) symmetry for small scalar mass values. This effect persists as $M_{\pi_N}^2$ is raised to 0.95.

The behavior of the solution having $b_p=0.12$ (see Ref. 24) and $c=-1.17$ is shown in Fig. 8. With $M_\epsilon^2+M_{\epsilon'}^2=1.7$ GeV^2, the Goldstone pseudoscalar octet is reached smoothly as $\hat{\delta}\to 0$. A branch point in $\hat{\lambda}(\delta)$ occurs at $\hat{\delta}=-0.42$, so a power-series expansion about $\hat{\delta}=0$ cannot represent the physical solution. As $M_\epsilon^2+M_{\epsilon'}^2$ is lowered to 1.2 GeV^2, the symmetry limit changes from Goldstone to normal, and the latter case is shown by the dashed line in Fig. 8. This critical behavior is nearly the same as that found in the case $b=-0.218$, $c=-1.30$, but with the interesting difference that the branch points that occur at small positive $\hat{\delta}$ in the latter case are absent. In summary, the description $b_p=+0.12$, $c_p=-1.17$ connects smoothly to the limit $\delta=0$, $b=0$, $\lambda\neq 0$ (Goldstone pseudoscalar octet) when $M_\epsilon^2+M_{\epsilon'}^2=1.7$ GeV^2, while it connects smoothly to the limit $\delta=0$, $b=0$, $\lambda=0$ (normal limit) when $M_\epsilon^2+M_{\epsilon'}^2=1.2$ GeV^2.

The $b_p=+0.12$ and $c=-0.09$ solution (not shown in a figure) is completely isolated from the symmetry limit for all values of $M_\epsilon^2+M_{\epsilon'}^2$ which we have tried (1.2 to 2.0 GeV^2). Negative pion masses appear at about $\hat{\delta}=0.9$ as $\hat{\delta}$ is reduced from $\hat{\delta}=1$, the physical point. We have not investigated the details of the approach to the symmetry limit for this case.

A summary of the interesting solutions is presented in Table III for the value $M_{\pi_N}^2=0.822$ GeV^2. At this value of $M_{\pi_N}^2$ the $b=-0.218$, $c=-1.30$, $d=0$ linear-model solution is reproduced, as expected, while the remaining solutions are peculiar to the bilinear-breaking model.

Details of Solutions for Choice $M_{\pi_N}^2=0.950$ GeV^2

As mentioned earlier, the solutions $b_p=-0.395$ and $b_p=-20.6$ become physically uninteresting (meet and become complex) as $M_{\pi_N}^2$ is allowed to rise. Taking the round figure $M_{\pi_N}^2=0.950$ GEV^2 ($M_{\pi_N}=0.974$ GeV), we examine the real solutions which evolve from the $b_p=-0.218$ and $b_p=+0.12$ cases discussed above. These roots move closer together, remaining real, to values -0.142 and $+0.013$, respectively, as $M_{\pi_N}^2$ is lifted to 0.950 GeV^2. For each b(physical), the two c roots again come in a "small" [near SU(3) Hamiltonian] and a "large"[14] [$c\approx-\sqrt{2}$, near SU(2)×SU(2) Hamiltonian] size. A summary of the solutions is presented in Table IV for the scalar mass inputs $M_\epsilon^2+M_{\epsilon'}^2=1.7$ GeV^2 and $M_\epsilon^2+M_{\epsilon'}^2=2.0$ GeV^2. We note that both the $b_p=-0.142$ and $b_p=0.013$ solutions produce acceptable values for the scalar nonet masses. The mixing between ϵ and ϵ' is, however, considerably different in these two cases.

The transition from normal to Goldstone limit again occurs for the case $b=-0.142$, $c=-1.29$,

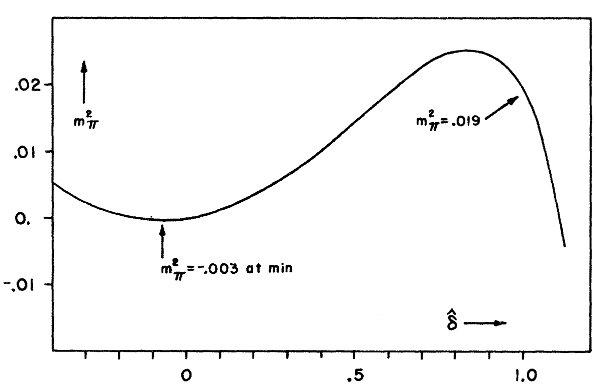

FIG. 6. m_π^2 vs $\hat{\delta}$ for solid-line solution of Fig. 5 ($M_\epsilon^2+M_{\epsilon'}^2=1.7$ GeV^2).

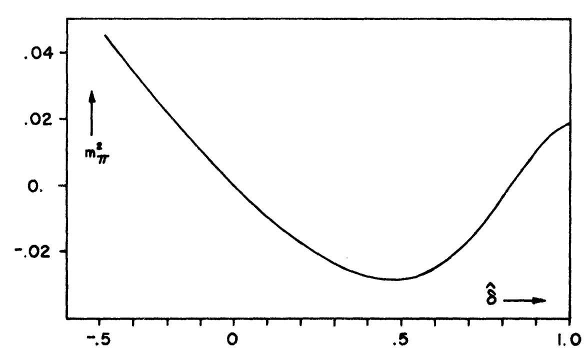

FIG. 7. m_π^2 vs $\hat{\delta}$ for dashed-line solution of Fig. 5. ($M_\epsilon^2+M_{\epsilon'}^2=1.2$ GeV^2).

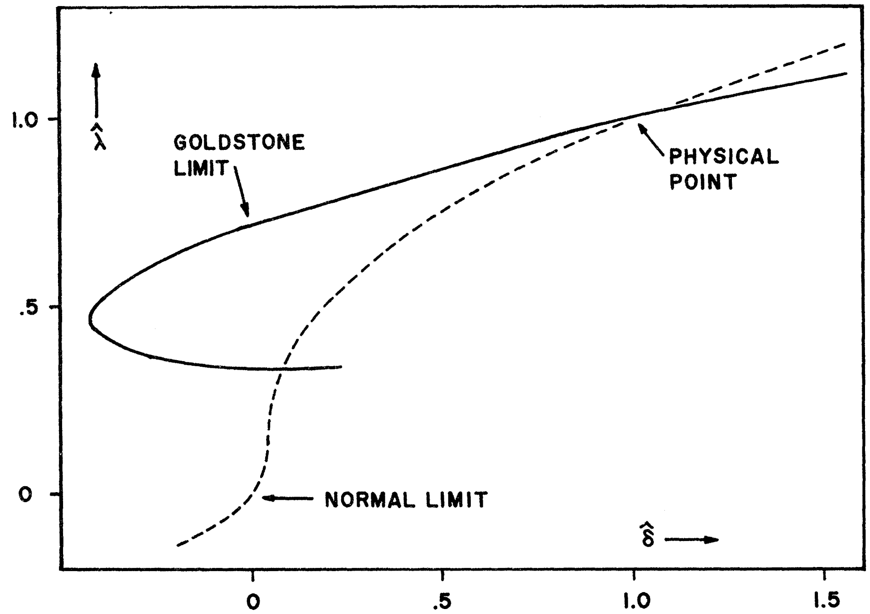

FIG. 8. $\hat{\lambda}$ vs $\hat{\delta}$ for the bilinear-breaking cases $M_{\pi_N}{}^2 = 0.823$ GeV^2, $b_p = 0.12$, $c = -1.17$. $M_\epsilon{}^2 + M_{\epsilon'}{}^2 = 1.7$ GeV^2 (solid line) and $M_\epsilon{}^2 + M_{\epsilon'}{}^2 = 1.2$ GeV^2 (dashed line). There is no singularity in dashed path.

but at higher scalar mass values than before. This is shown in Fig. 9, where the solid line ($M_\epsilon{}^2 + M_{\epsilon'}{}^2 = 2.0$ GeV^2) is the Goldstone limit and the dashed line ($M_\epsilon{}^2 + M_{\epsilon'}{}^2 = 1.7$ GeV^2) is the normal limit. The basic character of this solution, which is the one that tends in the limit $d = 0$ (no bilinear breaking) to the linear-breaking-model case studied in detail by several authors,[6,8] remains unchanged as $M_{\pi_N}{}^2$ is increased.

For the alternative c value (−0.178), the behavior of the solution near $\hat{\delta} = 0$ is shown in Fig. 10. The Goldstone nature of the limit is unchanged by varying $M_\epsilon{}^2 + M_{\epsilon'}{}^2$ over a wide range, but the limit is screened from the physical point by a region of instability (pion mass squared negative) for all $M_\epsilon{}^2 + M_{\epsilon'}{}^2$ values less than about 2.6 GeV^2. The pion mass squared is plotted vs $\hat{\delta}$ for the cases $M_\epsilon{}^2 + M_{\epsilon'}{}^2 = 1.7$ GeV^2 and $M_\epsilon{}^2 + M_{\epsilon'}{}^2 = 2.0$ GeV^2 in Fig. 11.

Next, we see in Fig. 12 that the $b_p = 0.013$, $c = -1.25$ case connects smoothly to the Goldstone limit for both values of $M_\epsilon{}^2 + M_{\epsilon'}{}^2$ tried. The transition to a normal limit occurs when $M_\epsilon{}^2 + M_{\epsilon'}{}^2$ is lowered to 0.8 GeV^2. The small-c case, $c = -0.001$, has highly erratic behavior, and the pion mass squared becomes negative for $\hat{\delta} \approx 0.99$,

TABLE III. Summary of symmetry limits of the interesting solutions to the bilinear-breaking model with $M_{\pi_N}{}^2 = 0.823$ GeV^2. Smooth limit means that no instabilities and/or singularities are encountered as $\delta \to 0$. Goldstone limit is $\langle u_0 \rangle \neq 0$, $b = 0$ as $\delta \to 0$; normal limit means $\langle u_0 \rangle = 0$, $\langle u_8 \rangle = 0$ as $\delta \to 0$. $\langle u_0 \rangle_{\text{physical}} \equiv \lambda(\delta_p)$ expansion exists if there are no $\lambda(\delta)$ singularities in $|\delta| \leq |\delta_p|$.

$M_{\pi_N}{}^2 = 0.823$ GeV^2 b_p	c_p	$M_\epsilon{}^2 + M_{\epsilon'}{}^2$ [a] (GeV^2)	Smooth limit?	Character of limit	$\langle u_0 \rangle$ expansion?
−0.218	−1.30	1.70	yes	Goldstone	no
		1.20	no	normal	no
	−0.30	1.70	yes	Goldstone	maybe
		1.20	no	Goldstone	maybe
+0.120	−1.17	1.70	yes	Goldstone	no
		1.20	yes	normal	maybe
	−0.09	1.70	no	···	no
		1.20	no	···	no

[a] Input

TABLE IV. Real solutions to bilinear-breaking model with $M_{\pi_N}{}^2 = 0.950$ GeV^2.

$M_{\pi_N}{}^2 = 0.950$ GeV^2 b_p	c_p	M_K	$\tan\theta_P$	$M_\epsilon{}^2 + M_{\epsilon'}{}^2$ [a] (GeV^2)	$M_{\epsilon'}$ (GeV)	M_ϵ (GeV)	$\tan\theta_S$
−0.143	−0.178	1.15	−0.015	1.70	1.130	0.650	0.525
	−1.29			2.00	1.192	0.760	0.776
0.013	−0.001	0.845	0.186	1.70	1.040	0.783	5.44
	−1.25			2.00	1.173	0.792	9.64

[a] Input

very near the physical point. We have not followed this highly unusual case in detail, since there is clearly no possibility that one can talk about a smooth symmetry limit for any reasonable choice of input parameters.

A recapitulation of the solutions is presented in Table V. Comparing with Table III, we see that the nature of each solution is essentially unchanged by the increase in $M_{\pi_N}{}^2$ from 0.822 GeV^2 to 0.95 GeV^2. The only effect is that scalar-mass-dependent changes in the character of the $\delta \to 0$ limit occur at different values of $M_\epsilon{}^2 + M_{\epsilon'}{}^2$ as $M_{\pi_N}{}^2$ is increased. For example, the first solution in Table III, where $M_{\pi_N}{}^2 = 0.822$ GeV^2, changes from a "normal" limit, but one that can only be reached by circling two branch points, to a smooth "Goldstone" limit at about $M_\epsilon{}^2 + M_{\epsilon'}{}^2 \approx 1.5$ GeV^2, while, referring to Table V, this solution makes the change at $M_\epsilon{}^2 + M_{\epsilon'}{}^2 \approx 1.9$ GeV^2 when $M_{\pi_N}{}^2 = 0.95$ GeV^2.

IV. DISCUSSION

In this section we shall summarize the essential features of the solutions to the linear- and bilinear-breaking σ models, elaborating on several features which are possible only in bilinear breaking, and comment on the model dependence of the approach to symmetry limits.

(i) Linear-breaking model. There are two solutions to this model, one of which has been discussed by Carruthers and Haymaker[6] for the special case in which the mass term $u_i{}^2 + v_i{}^2$ is dropped ($\alpha = 0$). (Following an observation by Olshansky,[8] Carruthers and Haymaker subsequently discussed the influence of the mass term, but in the context of a model with no octet breaking.[10]) We looked at both solutions in detail in Sec. II. In agreement with the analysis of Refs. 8 and 10, both solutions failed to have a smooth Goldstone limit when the originally assumed value of $M_\epsilon{}^2 + M_{\epsilon'}{}^2 = 1.7$ GeV^2 was decreased to a value small enough that the "critical" value of α was exceeded. However, the solution with $b_p = -0.218$ (see Ref. 25) became unstable when M_ϵ and $M_{\epsilon'}$ were in the neighborhood of 0.500 GeV and 1.0 GeV, respectively, while the $b_p = -0.352$ case lost the Goldstone limit at much lower values of the scalar masses. Whether or not the model is near a critical[10] value of the mass parameter depends upon the solution one chooses. The fact that the $b_p = -0.218$ solution, which alternates between the Goldstone pseudoscalar octet limit and the "pseudonormal"[26] limit as M_ϵ and $M_{\epsilon'}$ are varied around 0.500 GeV and 1.0 GeV, respectively, gives a more plausible picture of ϵ and ϵ' masses, and mixing might be suggestive. However, the solution into which it develops

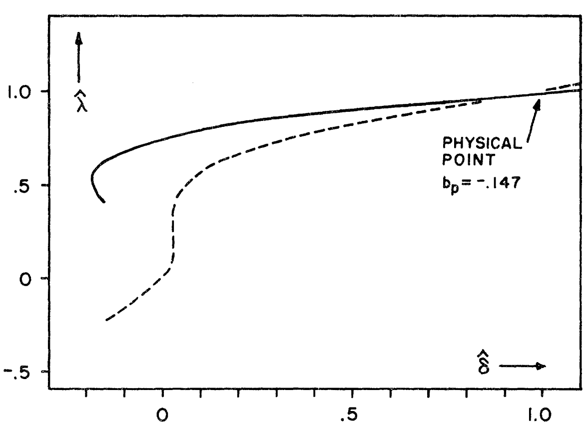

FIG. 9. Bilinear-breaking cases $M_{\pi_N}{}^2 = 0.95$ GeV^2, $b_p = -0.142$, $c = -1.29$, $M_\epsilon{}^2 + M_{\epsilon'}{}^2 = 1.7$ GeV^2 (dashed line) and $M_\epsilon{}^2 + M_{\epsilon'}{}^2 = 2.0$ GeV^2 (solid line). The s-shaped curve for the former case involves two branch points.

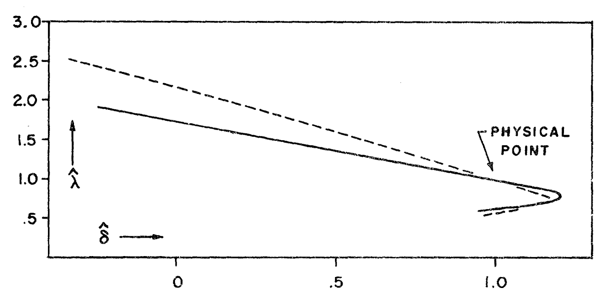

FIG. 10. Bilinear-breaking cases $M_{\pi_N}{}^2 = 0.95$ GeV^2, $b_p = -0.142$, $c = -0.178$, $M_\epsilon{}^2 + M_{\epsilon'}{}^2 = 1.7$ GeV^2 (dashed line) and $M_\epsilon{}^2 + M_{\epsilon'}{}^2 = 2.0$ GeV^2 (solid line).

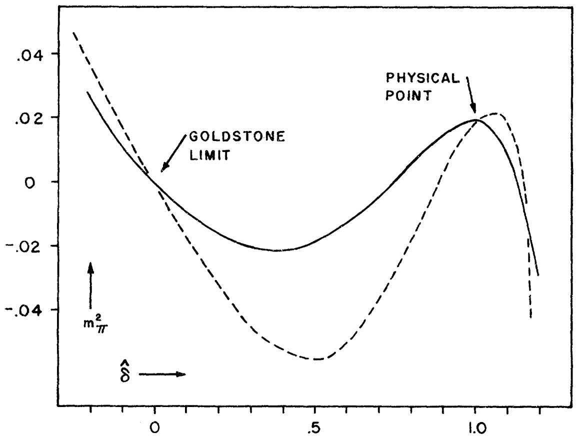

FIG. 11. Plot of m_π^2 in GeV^2 vs $\hat\delta$ for solutions of Fig. 10. Dashed line is $M_\epsilon^2+M_{\epsilon'}^2=1.7$ GeV^2 case and solid line is $M_\epsilon^2+M_{\epsilon'}^2=2.0$ GeV^2 case.

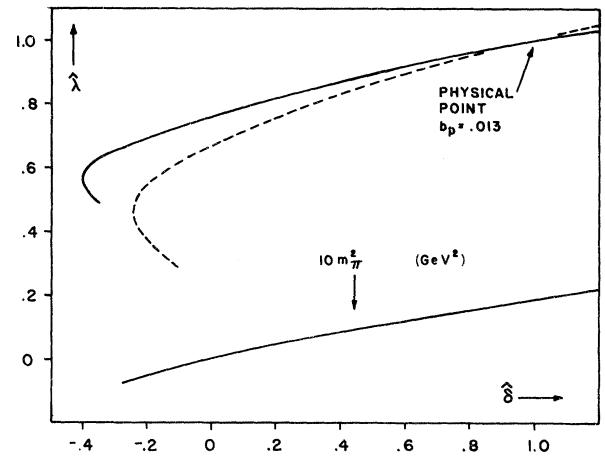

FIG. 12. Plot of $\hat\lambda$ vs $\hat\delta$ for $M_{\pi_N}^2=0.95$ GeV^2, $b_p=+0.013$, $c=-1.25$, $M_\epsilon^2+M_{\epsilon'}^2=1.7$ GeV^2 (dashed) and $M_\epsilon^2+M_{\epsilon'}^2=2.0$ GeV^2 (solid). m_π^2 dependence on $\hat\delta$ for latter case is shown at bottom.

as the bilinear term is introduced is only one of several reasonable candidates in this more complicated model.

(ii) Bilinear-breaking model. There are eight solutions possible in this model, though four are complex and of no interest in the case $M_{\pi_N}\approx 0.980$ GeV, as discussed in the latter part of Sec. III. In fact, the solution $b_p=-0.352$ to the linear model, which had been discounted in previous work,[6] evolves into one of the complex solutions as M_{π_N} is raised from the (predicted) value $M_{\pi_N}=0.904$ GeV of the linear model to (assumed) values above 0.940 GeV in the bilinear model. This is perhaps another justification for preferring the $b_p=-0.218$ solution[6] as a guide to general results in the linear-breaking studies. However, the number of viable solutions certainly increases when the extra term is added to $\mathcal{L}$. The most interesting new twist is the appearance of two values of c for each value of b_p; a "small" c ($c\approx-0.2$, supporting the Brandt and Preparata[19] point of view) and a "large" c ($c\approx-\sqrt{2}$, supporting the Gell-Mann–Oakes–Renner and Glashow-Weinberg point of view). All masses, the mixing angles, and F_K/F_π are the same for these two widely differing values of c. Depending upon the assumed values for M_{π_N} and $M_\epsilon{}^2+M_{\epsilon'}{}^2$, both solutions, one or the other, or neither may connect smoothly to a Goldstone SU(3) ×SU(3) limit. The value of c found from the tree approximation does not depend upon the way in which the symmetry limit is achieved, and one can have both, either, or neither of the Brandt-Preparata and Gell-Mann–Oakes–Renner pictures of SU(3)×SU(3) Hamiltonian breaking compatible with the pseudoscalar octet as Goldstone particles as $\hat\delta\to 0$.

In principle, bilinear breaking affords two interesting possibilities which are not present in the linear-breaking model. Solutions are actually found among those studied which closely corre-

TABLE V. Summary of symmetry limits of solutions to bilinear model with $M_{\pi_N}{}^2=0.950$ GeV^2.

$M_{\pi_N}{}^2=0.950$ GeV^2 b_p	c_p	$M_\epsilon{}^2+M_{\epsilon'}{}^2$ [a] (GeV^2)	Smooth limit?	Character of limit	$\langle u_0\rangle$ expansion?
−1.42	−1.29	1.70	no	normal	no
		2.00	yes	Goldstone	no
	−0.178	1.70	no	Goldstone	maybe
		2.00	no	Goldstone	maybe
0.013	−1.25	1.70	yes	Goldstone	no
		2.00	yes	Goldstone	no
	−0.001	1.70	no	···	no
		2.00	no	···	no

[a] Inputs.

spond to these *two* cases. We can have (1) $b=0$, $c\approx-\sqrt{2}$ and we find a case $b_p=0.01$, $c_p=-1.25$ when $M_{\pi_N}=0.975$ GeV. Note that $m^2_{\pi,K,\eta}\sim\delta$ as $\delta\to 0$ in this solution. This is almost a literal reproduction of the picture in the original Gell-Mann–Oakes–Renner paper.[14] We can also have (2) $m^2_{\pi,K,\eta}\sim\delta^2$ as $\delta\to 0$, and this is the case for the solution $b\approx-0.2$, $c\approx-0.2$ [approximate SU(3) symmetry of the Lagrangian]. Let us briefly discuss the origin of these effects. (a) In the σ model which has breaking linear in the fields, the extremum equation $\partial\mathcal{L}/\partial u_8=0$ requires the condition $\lambda'=0\Rightarrow c=0$ (or $b=\lambda'/\sqrt{2}\,\lambda=0\Rightarrow c=0$). However, if $a/d=4\lambda_p(\frac{2}{3})^{1/2}$ (see Ref. 27) this requirement is lifted in the model with breaking which is bilinear in the basis fields. (b) If we write

$$\mathcal{H}=\mathcal{H}_{\text{symmetric}}-(u_0'+u_8'c)\,, \tag{7}$$

then we can express the pseudoscalar masses

$$m_i^2=\frac{1}{F_i^2}[(\tfrac{2}{3})^{1/2}+cd_{8ii}]\times[(\tfrac{2}{3})^{1/2}\langle 0|u_0'|0\rangle+d_{8ii}\langle 0|u_8'|0\rangle]\,. \tag{8}$$

In the usual σ model, $u_0'=\delta u_0$ and $u_8'=\delta u_8$, where δ is the explicit SU(3)×SU(3)-breaking parameter. In the Goldstone limit where $\langle u_0\rangle$ and $F_i\to$ constant and $\langle u_8\rangle\to 0$, then $m_i^2\sim$ constant $\times\,\delta$. This is the only way in which the Goldstone limit can be achieved in the model with linear breaking. With breaking which is bilinear in the basis fields u_i and v_i, we can write

$$u_0'=\delta(au_0+dU_0)\,,\quad u_8'=\delta(au_8+dU_8)\,. \tag{9}$$

Since

$$U_0=4(\tfrac{2}{3})^{1/2}u_0^{\,2}+\cdots\,, \tag{10}$$

it is possible to have

$$\langle 0|u_0'|0\rangle=\delta[a\langle u_0\rangle+4(\tfrac{2}{3})^{1/2}d\langle u_0\rangle^2+\cdots] \tag{11}$$

and

$$\langle 0|u_0'|0\rangle\sim\delta^2$$

if

$$\langle 0|u_0|0\rangle|_{\delta=0}=-\frac{a}{4(\frac{2}{3})^{1/2}d}\,.$$

Referring to Eq. (4.2), this means $m_i^2\sim\delta^2$ as $\delta\to 0$. This quadratic behavior is observed to occur in the Brandt-Preparata solution $b\approx-0.2$, $c\approx-0.2$, as we have mentioned. It is interesting that, of all the solutions studied, this is one of several which are free of singularities on the real line $|\hat{\delta}|\leq 1$ (see Table III). However, though a straight line $\lambda=\lambda_0+(d\lambda/d\delta)\delta$ describes the vacuum dependence on δ very well, the pseudoscalar masses require at least cubic terms in δ for a description of their behavior in the interval $|\hat{\delta}|\leq 1$. Implications of this curious behavior may be worth pursuing.

In summary, we refer to Table VI, where we see that it is as likely as not that the tree-approximation solutions simply cannot be connected smoothly to a meaningful SU(3)×(3) limit. In those cases where the limit *can* be obtained smoothly, the radius of convergence of a power-series expansion of "one-point" functions, and consequently masses, couplings, etc., is generally too small to provide a description of the physical parameters. In one of the possible exceptions to this observation, the masses, though probably analytic in δ, require at least cubic terms in δ to reproduce their behavior. Thus, our study does not lend support to the hope that a model with a smooth SU(3)×SU(3) limit can be unambiguously identified, and then provide a useful expansion point for successfully describing low-energy hadron phenomena.

On the positive side, it is certainly true that if any smooth limit exists in these models it is either "normal" or "pseudoscalar octet of Goldstones." A κ-Goldstone role is not possible,[28] for example. It is also worth pointing out that of the three leading candidates for reasonable, physical solutions to the bilinear-breaking model, one with reasonable scalar mass spectra shows the "critical" switch between Goldstone and "pseudonormal" or normal limits as the relative size of the mass parameter, α, is varied over a "physically reasonable" range such that $M_\epsilon\approx 0.5$ GeV, $M_{\epsilon'}\approx 1$ GeV.

Regarding the question of whether or not the addition to the usual σ model of terms which are bi-

TABLE VI. Condensation of Tables III and V for "most reasonable" solutions with $M_\epsilon^2+M_{\epsilon'}^2=1.2$, 1.7, and 2.0 GeV^2.

$M_\epsilon^2+M_{\epsilon'}^2$ (GeV^2)	$M_{\pi_N}^2=0.822$ GeV^2			$M_{\pi_N}^2=0.950$ GeV^2		
	$b=-0.218$ $c=-0.300$	$b=-0.218$ $c=-1.30$	$b=-0.120$ $c=-1.17$	$b=-0.143$ $c=-0.178$	$b=-0.143$ $c=-1.29$	$b=-0.01$ $c=-1.25$
1.20	no	no	normal	no	no	normal
1.70	Goldstone	Goldstone	Goldstone	no	no	Goldstone
2.00	Goldstone	Goldstone	Goldstone	no	Goldstone	Goldstone

linear in the fields and transform as $(3^*, 3)+(3, 3^*)$ members is "harmless" (by which we mean that flexibility is provided without affecting general results) we find that substantial changes occur in the nature of the new solutions compared to the nature of the linear-model solutions. Several of these solutions may provide as reasonable descriptions of (poorly known) masses and mixings as the "preferred" $b_p=-0.218$ linear-breaking-model solution.[6] In view of the poorly determined K_{l3} parameters[29] and of π-N σ terms,[30] it is not clear where one may turn to find additional evidence which may discriminate among the possibilities. Consequently, we feel that inferences about possible chiral-symmetry limits and about possible "critical point" effects which are drawn from the usual σ model (linear breaking) may not be supported by variants of the model and should be viewed with caution.

APPENDIX

The Lagrangian we work with in Sec. III reads

$$\mathcal{L}=-\tfrac{1}{2}(\partial_\mu u_i\partial_\mu u_i+\partial_\mu v_i\partial_\mu v_i)-\alpha I_2-\alpha'(I_2)^2-\beta I_3 -\gamma I_4+\delta[a(u_0+cu_8)+d(U_0+cU_8)]. \quad \text{(A1)}$$

The results of the linear model in all subsequent formulas are recovered by setting $d=0$. Written in terms of the basis fields, the invariants and bilinear $(3^*, 3)+(3, 3^*)$ members are

$$\begin{aligned}
I_2&=u_i^2+v_i^2 \quad (\text{"mass" term}),\\
I_3^+&=4d_{ijk}(u_i u_j u_k-3v_i v_j u_k)-6\sqrt{6}\,u_0u_i u_i +6\sqrt{6}\,u_0v_i v_i+12\sqrt{6}\,v_0v_i u_i +6\sqrt{6}\,u_0^3-18\sqrt{6}\,v_0u_0, \qquad \text{(A2)}\\
I_4&=2(d_{ijn}d_{nkl}+if_{ijn}d_{nkl}+id_{ijn}f_{nkl}-f_{ijn}f_{nkl}) \times(u_i u_j u_k u_l+v_i v_j v_k v_l+4u_i u_j v_k v_l-2u_i v_j u_k v_l),
\end{aligned}$$

and

$$\begin{aligned}
U_0&=4(\tfrac{2}{3})^{1/2}(u_0^2-v_0^2)-2(\tfrac{2}{3})^{1/2}\sum_{j=1}^{8}(u_j^2-v_j^2),\\
U_8&=-4(\tfrac{2}{3})^{1/2}(u_0u_8-v_0v_8)+4\sum_{j=1}^{8}d_{8jj}(u_j^2-v_j^2).
\end{aligned} \qquad \text{(A3)}$$

The divergences of the axial-vector currents are given by

$$\begin{aligned}
\partial_\mu A_\mu^i&=\delta(\tfrac{2}{3})^{1/2}[a+4(\tfrac{2}{3})^{1/2}d\langle u_0\rangle]v_i +4(\tfrac{2}{3})^{1/2}\delta d\Big(\tilde{u}_0v_i+v_0\tilde{u}_i+\sum_{j,k=1}^{8}d_{ijk}\tilde{u}_j v_k\Big) + \text{octet breaking}, \qquad \text{(A4)}\\
u_0&=\langle u_0\rangle+\tilde{u}_0.
\end{aligned}$$

The locations of the extrema of the classical potential determine the vacuum expectation values of the fields u_0 and u_8 in the lowest order, and are given by the equations

$$\delta a+8\delta d(\tfrac{2}{3})^{1/2}\lambda+\frac{8\delta dc}{\sqrt{3}}\lambda b =\lambda[2\alpha+4\alpha'\lambda^2(1+2b^2)+4\sqrt{6}\,\beta\lambda(1-b^2) +\tfrac{16}{3}\gamma\lambda^2(1+6b^2-2b^3)] \qquad \text{(A5)}$$

from

$$\frac{\partial\mathcal{L}}{\partial u_0}=0,$$

and

$$\delta ac-4\delta d(\tfrac{2}{3})^{1/2}\lambda(c+\sqrt{2}\,b)-8(\tfrac{2}{3})^{1/2}\delta dc\lambda b = \sqrt{2}\,b\lambda[2\alpha+4\alpha'\lambda^2(1+2b^2)-4\sqrt{6}\,\beta\lambda(1+b) +16\gamma\lambda^2(1-b+b^2)] \qquad \text{(A6)}$$

from

$$\frac{\partial\mathcal{L}}{\partial u_8}=0,$$

where

$$\lambda=\langle 0|u_0|0\rangle$$

and

$$b=\frac{\langle 0|u_8|0\rangle}{\sqrt{2}\,\langle 0|u_0|0\rangle}.$$

In order to fix parameters, the expressions for masses are necessary, and we have

$$\begin{aligned}
m_\pi^2&=x_1+(x_2-x_4)-2(bx_2-x_5)+\tfrac{2}{3}x_3(1+b)^2,\\
m_K^2&=x_1+(x_2-x_4)+(bx_2-x_5)+\tfrac{2}{3}x_3(1-b+7b^2),\\
m_\eta^2+m_{\eta'}^2&=2x_1-(x_2-x_4)+2(bx_2-x_5) +\tfrac{2}{3}x_3(2-2b+5b^2), \qquad \text{(A7)}\\
m_{\eta'}^2-m_\eta^2&=\{[3(x_2-x_4)+2(x_2b-x_5)-\tfrac{2}{3}x_3b(2-b)]^2 +8[(x_2b-x_5)+\tfrac{2}{3}x_3b(2-b)]^2\}^{1/2}
\end{aligned}$$

for the pseudoscalars, and

$$\begin{aligned}
M_{\pi_N}^2&=x_1-(x_2-x_4)+2(bx_2-x_5)+2x_3(1+b)^2,\\
M_K^2&=x_1-(x_2-x_4)-(x_2b-x_5)+2x_3(1-b+b^2),\\
M_\epsilon^2+M_{\epsilon'}^2&=2x_1+4\alpha'\lambda^2(1+2b^2)+(x_2-x_4) -2(x_2b-x_5)+2x_3(2-2b+5b^2), \qquad \text{(A8)}\\
M_{\epsilon'}^2-M_\epsilon^2&=\{[8\alpha'\lambda^2(1-2b^2)+x_2(3+2b)+2x_3(2-b)b]^2 +8[(-x_2b+x_5)+2x_3(2-b)+8\alpha'\lambda^2b]^2\}^{1/2}
\end{aligned}$$

for scalars. Here we have defined the quantities

$$\begin{aligned} x_1 &= 2\alpha + 4\alpha'^2\lambda^2(1+2b^2), \\ x_2 &= 4\sqrt{6}\beta\lambda, \\ x_3 &= 8\gamma\lambda^2, \\ x_4 &= 4(\tfrac{2}{3})^{1/2}\delta d, \\ x_5 &= \frac{\delta dc}{\sqrt{3}}. \end{aligned} \tag{A9}$$

The inputs m_π^2, m_K^2, m_η^2, $m_{\eta'}^2$, and $M_{\pi_N}^2$ determine b, x_3, $bx_2 - x_5$, $x_2 - x_4$, and x_1. M_κ^2 is then predicted. Specifying $M_\epsilon^2 + M_{\epsilon'}^2$ then fixes α and $\lambda^2\alpha'$. Our procedure is to express x_3, $bx_2 - x_5$, $x_2 - x_4$, and x_1 in terms of m_π^2, m_K^2, $m_\eta^2 + m_{\eta'}^2$, $M_{\pi_N}^2$, and b and then to substitute into the $m_{\eta'}^2 - m_\eta^2$ equation. This yields a quartic equation for b. For each b solution, values of x_3, $bx_2 - x_5$, $x_2 - x_4$, and x_1 are computed and the value of M_κ^2 found. The values of M_ϵ^2 and $M_{\epsilon'}^2$ can be computed separately at this point. The extremum equations (A5) and (A6) are used to evaluate c, which satisfies a quadratic equation. Thus *two* c values are found for *each* of the four b values. The individual values of x_2, x_4, and x_5 are determined by the expression

$$x_4 = \frac{\sqrt{2}\,(2-b)}{4(1+b)(2\sqrt{2}-c)} - \frac{\sqrt{2}\,bM_\kappa^2}{4(1+b)c} \tag{A10}$$

and the now known values of c, b, $bx_2 - x_5$, and $x_2 - x_4$. The value of $\delta a/\lambda_p$ can be determined from

$$x_6 \equiv \frac{\delta a}{\lambda_p} = \frac{1}{2}\left[\frac{(2-b)m_K^2}{2\sqrt{2}-c} + \frac{bM_\kappa^2}{c}\right] \tag{A11}$$

for each b, c combination. Equations (A10) and (A11) follow by combining the extremum equations and mass formulas. The dimensionless quantity x_4/x_6 measures the ratio of bilinear to linear breaking.

To study $\lambda(\delta)$, $b(\delta)$, and masses (δ) we fix x_1, x_2, x_3, x_4, x_5, and x_6 (where $x_5/x_4 = c/\sqrt{2}$) and return to the extremum equations and mass formulas with the substitutions

$$\begin{aligned} &\alpha \to \alpha, \quad x_3 \to x_3\hat{\lambda}^2, \\ &\alpha'\lambda_p^2 \to \alpha'\hat{\lambda}^2, \quad x_4 \to x_4\hat{\delta}, \\ &x_2 \to x_2\hat{\lambda}^2, \quad x_5 \to x_5\hat{\delta}, \end{aligned} \tag{A12}$$

where $\hat{\lambda} \equiv \lambda/\lambda_p$ and $\hat{\delta} \equiv \delta/\delta_p$. That is, since the scales of λ and δ are not fixed, we can only study the dependence of the dimensionless quantity $\hat{\lambda}$ on the dimensionless parameter $\hat{\delta}$. We find a cubic equation for $\hat{\lambda}$ as a function of b, and $\hat{\delta}$ is determined from knowledge of $\hat{\lambda}$. Expressing $\hat{\lambda}$ directly as a function of $\hat{\delta}$ is too complicated, so we choose to find values of $\hat{\lambda}$ and $\hat{\delta}$ by varying b over a range which includes b_p and $b=0$. Working in this way we are able to map out $\hat{\lambda}(\hat{\delta})$ and $b(\hat{\delta})$ in the region including $\hat{\delta}=0$ and $\hat{\delta}=1$. The physical root is defined as the one for which $\hat{\lambda}=1$ and $\hat{\delta}=1$ at $b=b_p$. Our primary interest is the behavior of this root in the region around $\hat{\delta}=0$ which includes $\hat{\delta}=1$.

Mixing angles. We define the $\eta-\eta'$ and $\epsilon-\epsilon'$ mixing angles, θ_P and θ_S, as follows:

$$\begin{aligned} \eta' &= \cos\theta_P v_0 + \sin\theta_P v_8, \\ \eta &= -\sin\theta_P v_0 + \cos\theta_P v_8 \end{aligned} \tag{A13}$$

and

$$\begin{aligned} \epsilon' &= \cos\theta_S u_0 + \sin\theta_S u_8, \\ \epsilon &= -\sin\theta_S u_0 + \cos\theta_S u_8. \end{aligned} \tag{A14}$$

The mixing angle is given in terms of the masses as

$$\tan\theta = \frac{-2m_{08}^2}{m_{88}^2 - m_{00}^2 - [(m_{88}^2 - m_{00}^2)^2 + 4m_{08}^4]^{1/2}}. \tag{A15}$$

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[1]M. Gell-Mann, Physics (N.Y.) 1, 63 (1964).

[2]J. Goldstone, Nuovo Cimento 19, 154 (1961); J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 965 (1962).

[3]R. Dashen, Phys. Rev. 183, 1245 (1969).

[4]R. Dashen and M. Weinstein, Phys. Rev. 183, 1261 (1969); 188, 2330 (1969); Phys. Rev. Lett. 22, 1337 (1969).

[5]L.-F. Li and H. Pagels, Phys. Rev. Lett. 26, 1204 (1971); 27, 1089 (1971).

[6]P. Carruthers and R. W. Haymaker, Phys. Rev. Lett. 27, 455 (1971); Phys. Rev. D 4, 1808 (1971); 4, 1815 (1971).

[7]M. Lévy, Nuovo Cimento 52A, 23 (1967).

[8]R. Olshansky, Phys. Rev. D 4, 2440 (1971).

[9]D. W. McKay and W. F. Palmer, Phys. Rev. D 5, 856 (1972).

[10]P. Carruthers and R. W. Haymaker, Phys. Rev. D 6, 1528 (1972).

[11]L.-H. Chan and R. W. Haymaker, Phys. Rev. D 7, 402 (1973); 7, 415 (1973).

[12]G. S. Guralnik, H. S. Tsao, and T. F. Wong, Phys. Rev. D 7, 2467 (1973).

[13]S. L. Glashow and S. Weinberg, Phys. Rev. Lett. 20, 224 (1968).

[14]M. Gell-Mann, R. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968).

[15]J. Schechter and Y. Ueda, Phys. Rev. D 3, 168 (1971); 3, 2874 (1971).

[16]There is some flexibility in choice of parameters. Olshansky (Ref. 8) fixes b from the Glashow-Weinberg

(Ref. 13) sum rule and predicts an approximately correct value for $m_{\eta'}^2 - m_\eta^2$. He suggests that the value for $m_{\eta'}^2 - m_\eta^2$ can be made to agree with experiment by addition of a bilinear term such as the one we study in Sec. III. Chan and Haymaker (Ref. 11) note that b can be considered a variable parameter over a range -0.2 to -0.35 because of its sensitive dependence on the uncertain value of the kaon average mass.

[17]By essentially we mean that two parameter-independent, complex solutions of an originally quartic equation are discarded and only the two interesting solutions are pursued.

[18]H. Pagels, The Rockefeller University Report No. COO-2232-2B (unpublished) has recently argued that F_π might set the scale for hadron dynamics. He suggests F_π as an order parameter, indicating that hadron dynamics is near a phase-transition point. Carruthers and Haymaker (Ref. 10), suggest α as an order parameter in the σ-model context.

[19]R. Brout, Nuovo Cimento 47A, 932 (1967); P. de Mottoni and E. Fabri, *ibid.* 54A, 42 (1968).

[20]K. G. Wilson, Phys. Rev. B 4, 3174 (1971); 4, 3184 (1971); Phys. Rev. D 3, 1818 (1971).

[21]S. Gasiorowicz and D. Geffen, Rev. Mod. Phys. 41, 531 (1969).

[22]R. A. Brandt and G. Preparata, Ann. Phys. (N.Y.) 61, 119 (1970); Phys. Rev. Lett. 26, 1605 (1971). Other recent works in which small values of c are obtained are: L. Bessler and D. Welling, Phys. Rev. D 6, 1092 (1972) and K. Sebastian, Nucl. Phys. B52, 639 (1973).

[23]R. Olshansky, Phys. Rev. D 6, 2243 (1972).

[24]The small-positive-b solutions imply a large SU(3) breaking in $f_+(0)$, the vector form factor which describes K_{l3} decay. This follows from the tree-approximation identification $F_K/F_\pi = (1-\frac{1}{2}b)/(1+b)$ and the experimentally determined value $F_K/F_\pi f_+(0) = 1.23-1.28$. For example, $b = 0.12$ gives $F_K/F_\pi \approx 0.85$ and $f_+(0) \approx 0.66-0.68$. Most estimates of $f_+(0)$ are within 5–10% of the exact SU(3) value $f_+(0)=1$, in accordance with small SU(3) breaking and the Ademollo-Gatto theorem. Thus, small positive b values force one to accept rather large renormalization effects.

[25]The values of b_p, $\alpha + 4\alpha'\lambda_p^2$, and c do not depend on the scalar masses assumed.

[26]"Pseudonormal" means that singularities, and consequently negative ϵ (or ϵ') mass-squared values, must be encountered before reaching $\hat{\delta}=0$, $\lambda=0$.

[27]a/d is the ratio of linear to bilinear breaking strength in the Hamiltonian.

[28]D. W. McKay and W. F. Palmer, Phys. Rev. D 4, 1018 (1971).

[29]L.-M. Chounet, J.-M. Gaillard, and M. K. Gaillard, Phys. Rep. 4C, 199 (1972).

[30]C. C. Shih and H. K. Shepard, Phys. Lett. 41B, 321 (1972). References to work with less conservative error estimates can be found in this paper.

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Constraints on Higgs Fields in Unified Weak-Electromagnetic Gauge Theories*

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In renormalizable unified gauge theories of the weak and electromagnetic interactions, for a given gauge group we ask the following question: Which representations of the scalar Higgs particles can reproduce a vector-meson mass spectrum in which one vector meson, the photon, remains massless and all other vector mesons acquire mass? Given the Higgs representation, we then examine what restrictions are placed on the charge operator of the theory. The examples of SU(3) and SU(3) × U(1) are worked out in detail. The possible Higgs representations for all the unexceptional classical Lie groups are given. For each such group, there are some low-dimensional representations for which no solution exists.

I. INTRODUCTION AND SUMMARY

The development of renormalizable gauge theories which attempt to unify the weak and electromagnetic interactions[1] has opened a Pandora's box of possible models. Once a gauge group is chosen, the model builder must decide on a representation of scalar Higgs particles, whose nonzero vacuum expectation values (VEV) spontaneously break the gauge symmetry, and assign known and probably unknown fermions to group multiplets in such a way that he satisfies the constraints imposed by the experimental cross sections, masses, moments, decay rates, etc.

In this paper, we present simple criteria for determining, given the underlying group, which representations of the Higgs particles can reproduce a vector-meson mass spectrum in which one vector meson, the photon, remains massless and all other vector mesons acquire mass. Kibble has discussed the choice of Higgs particles in the context of the strong interactions, thus requiring