derivations to follow, if one understands time-ordered products to really be covariant T^* products these terms can be ignored.

[8]S. Weinberg, Phys. Rev. Lett. 29, 388 (1972); H. Georgi and S. L. Glashow, Phys. Rev. D 6, 2977 (1972).

[9]To establish the relation of this terminology to that of Weinberg, Georgi, and Glashow we note that *a controlled but not computable mass difference* would be a nonvanishing quantity whose value in the zero-loop approximation is fixed by Goldstone-boson coupling constants in a Lagrangian containing all renormalizable terms consistent with the symmetry principles adopted; a computable mass difference is one whose value in the zero-loop approximation vanishes.

[10]For example see T. Hagiwara and B. W. Lee, Phys. Rev. D 7, 459 (1973), for an example of the controlled but noncalculable case. For a discussion of the second type of mass formula see S. Weinberg, Phys. Rev. Lett. 29, 388 (1972).

[11]If we had assumed that $|A\rangle$ and $|B\rangle$ were fermion states having opposite intrinsic parity we would have obtained

$$\langle A|j^\mu(0)|B\rangle = \bar{u}_A(p')\{[\gamma^\mu g^5_{AB}(q^2) + q^\mu h^5_{AB}(q^2) + q_\nu \sigma^{\mu\nu} s^5_{AB}(q^2)]\gamma^5\}u_B(p).$$

Taking the divergence of this expression yields $(m_A + m_B)g^5_{AB}(0) = 0$; hence, the conclusions for the vector and axial-vector current are simply interchanged.

[12]Actually, we could assume a more complicated structure for the f's (i.e., a general matrix $f_{\alpha\beta}$); however, for the purposes of this brief argument this would add nothing to the point we wish to make.

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Complex Helicity Plane*

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We argue that the locations of poles in complex helicity are determined completely by the Regge poles in complex angular momentum. They lie at "sense" values of the helicity, $m = \alpha_i, \alpha_i - 1, \alpha_i - 2, \ldots$, relative to the angular momentum singularities at $j = \alpha_i$. Thus, through the determination of helicity singularities, singularities in angular momentum determine asymptotic limits in addition to the conventional multi-Regge limits.

I. INTRODUCTION

Increased theoretical and experimental interest in multiparticle scattering amplitudes has recently sparked interest in the complex-helicity plane.[1] This is because, in multiparticle amplitudes, singularities in complex helicity play a role very similar to singularities in complex angular momentum—both control specific, distinct asymptotic limits.[2] Despite their similar manifestations, however, there are fundamental differences between complex angular momentum and complex helicity. Whereas angular momentum is a Poincaré-invariant quantity and singularities in complex angular momentum are manifestations of dynamical objects like bound states and resonances, helicity is not a Poincaré invariant, and thus singularities in complex helicity are not expected to be manifestations of independent dynamical objects. Indeed, it is usually assumed that the helicity singularities are completely determined by the angular momentum singularities—a Regge pole in angular momentum at $j=\alpha_i$ yielding singularities at "sense" values of the helicity,

$$m=\alpha_i - p, \tag{1.1}$$

where p is a positive integer or zero. Here we give arguments for this rule.

In order to illustrate the importance of the rule (1.1), let us briefly mention two of its interesting consequences. It has recently been shown that the vanishing of the Pomeranchon-particle-Reggeon vertex for a Pomeranchon with unit intercept[3] implies the vanishing of the Pomeranchon-particle-particle elastic vertex.[4] The Pomeranchon-particle-Reggeon vertex contains a term [see Eqs. (1.15) to (1.17) below]

$$\eta^{\alpha_R}\frac{\beta}{\alpha_R}, \tag{1.2}$$

where β is the elastic coupling and in Fig. 1 $\alpha(t_1)=\alpha_P$, $\alpha(t_2)=\alpha_R$, $\eta=s_{12}/s_1s_2$ $(=1/\kappa)$. This is the only term $\approx 1/\alpha_R$, so the vanishing of (1.2) implies the vanishing of β. Since the decoupling is proven only for $\eta=(m^2-t_2)\approx-\alpha_R$, an additional term in the vertex of the form

$$\eta^{\alpha_R+1}\beta$$

would cancel (1.2) and cause the proof of the vanishing of the elastic coupling to fail[5]—thereby saving the simple picture of the Pomeranchon as a Regge pole with exactly unit intercept. However, such a term corresponds to a "nonsense" helicity singularity at $m=\alpha_R+1$ and is excluded by (1.1).

As a second example, we note that (1.1) implies the uniformity of the interchange of Regge and scaling limits for inclusive cross sections. The inclusive cross section for $1+2\to 2'+X$ (see Fig. 2) is

$$E\frac{d^3\sigma}{dp^3}\sim\frac{1}{s}\,\mathrm{Disc}_{M^2}A_6(s,t,M^2),$$

where $s=(p_1+p_2)^2$, $t=(p_2+p_2')^2$, and M is the mass of X. In the Regge limit ($s\to\infty$, M^2, and t fixed), Regge behavior of the exclusive processes gives

$$\mathrm{Disc}_{M^2}A_6\sim s^{2\alpha(t)}f_R(M^2,t), \tag{1.3}$$

whereas in the scaling limit ($M^2\to\infty$, s/M^2, and t fixed) we have

$$\mathrm{Disc}_{M^2}A_6\sim(M^2)^{\alpha(0)}f_S(s/M^2,t). \tag{1.4}$$

Uniformity[6] of these limits would require

$$f_S(s/M^2,t)\sim(s/M^2)^{2\alpha(t)}\gamma(t)$$

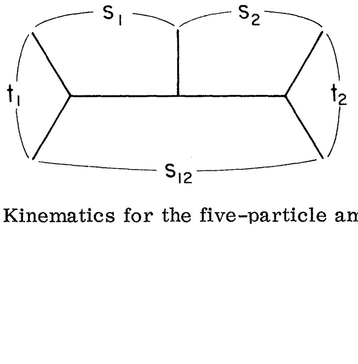


FIG. 1. Kinematics for the five-particle amplitude.

as $s/M^2\to\infty$. Thus we have

$$\mathrm{Disc}_{M^2}A_6\sim(M^2)^{\alpha(0)}(s/M^2)^{2\alpha(t)}\gamma(t), \tag{1.5}$$

which gives the behavior usually assumed for inclusive cross sections near the phase-space boundary. Although this uniformity and the consequent equality of powers of s in (1.3) and (1.5) may seem trivial, it is not. For example, the function

$$\mathrm{Disc}_{M^2}A_6=(s/M^2)^{2\alpha(t)+2}\frac{(M^2)^{\alpha(0)+3}}{a(M^2)^3+bs^2} \tag{1.6}$$

satisfies (1.3) and (1.4) but not (1.5). The limit in (1.5) is a combined Regge ($M^2\to\infty$)-helicity ($s/M^2\to\infty$) limit. Thus the power of $(s/M^2)^2$ is given by the leading singularity in complex helicity, whereas the power of s^2 in the Regge limit (1.3) is given by the leading singularity in complex angular momentum. The rule (1.1) says these are equal and thus (1.5) holds [the example (1.6) would require a nonsense helicity pole at $m=\alpha(t)+1$].

In the remainder of this section we establish our terminology and notation and review the basic elements of complex-helicity analysis by discussing the five-particle amplitude since it is the simplest example of the use of complex helicity. In Sec. II we discuss in detail the arguments for (1.1). Consider a double partial-wave analysis in the t_1 and t_2 channels as shown in Fig. 1:

$$A_5=\sum_{m=0}^{\infty}\sum_{j_1=m}^{\infty}\sum_{j_2=m}^{\infty}(2j_1+1)(2j_2+1)d^{j_1}_{0m}(\cos\theta_1)\times d^{j_2}_{m0}(\cos\theta_2)(\cos m\omega)a(j_1,j_2,m;t_1,t_2), \tag{1.7}$$

where parity invariance has been used to remove $\sin m\omega$ terms. We expect that the behavior for large $\cos\theta_1$, $\cos\theta_2$, and $\cos\omega$ can be obtained by performing Sommerfeld-Watson transforms in j_1, j_2, and m, respectively. Since

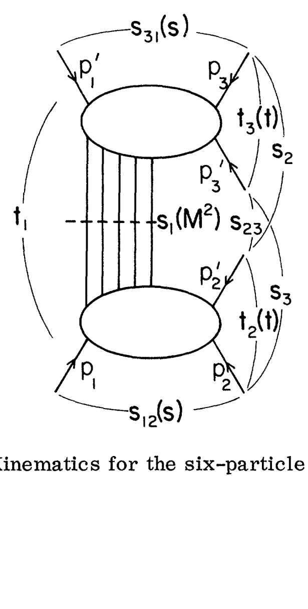


FIG. 2. Kinematics for the six-particle amplitude.

$$s_1 \propto \cos\theta_1,$$
$$s_2 \propto \cos\theta_2, \tag{1.8}$$
$$s_{12} \propto (m^2 - t_1 - t_2)\cos\theta_1\cos\theta_2 - 2(t_1 t_2)^{1/2}\sin\theta_1\sin\theta_2\cos\omega,$$

these correspond to large s_1, s_2, and s_{12}, respectively. Thus singularities in complex angular momenta j_1 and j_2 determine the behavior for

Single-Regge limits.

$$s_1 \to \infty; \quad s_{12}/s_1, \; s_2, \; t_1, \; t_2 \text{ fixed}, \tag{1.9a}$$

or

$$s_2 \to \infty; \quad s_{12}/s_2, \; s_1, \; t_1, \; t_2 \text{ fixed}, \tag{1.9b}$$

and

Double-Regge limit.

$$s_1, s_2 \to \infty; \quad \eta \equiv s_{12}/s_1 s_2, \; t_1, \; t_2 \text{ fixed}. \tag{1.10}$$

Singularities in the complex helicity m determine the behavior for

Complex-helicity limit.

$$s_{12} \to \infty; \quad s_1, s_2, t_1, t_2 \text{ fixed}. \tag{1.11}$$

Singularities in both angular momentum and helicity determine the mixed limits:

Regge-helicity limits.

$$s_1, s_{12}/s_1 \to \infty; \quad s_2, t_1, t_2 \text{ fixed}, \tag{1.12a}$$

$$s_2, s_{12}/s_2 \to \infty; \quad s_1, t_1, t_2 \text{ fixed}, \tag{1.12b}$$

and

Double-Regge-helicity limit.

$$s_1, s_2, s_{12}/s_1 s_2 \to \infty; \quad t_1, t_2 \text{ fixed}. \quad \text{(See Ref. 7.)} \tag{1.13}$$

In order to relate the behavior in these limits to singularities in complex angular momentum and helicity, we perform a multiple Sommerfeld-Watson transform of (1.7). The essential features of this transform are exhibited by the following representation for $A_5{}^1$:

$$A_5 = \left(\frac{1}{2\pi i}\right)^3 \int dm \int dj_1 \int dj_2 \Gamma(-m) \times \Gamma(-j_1 + m)\Gamma(-j_2 + m) \times (-s_1)^{j_1 - m}(-s_2)^{j_2 - m} \times (-s_{12})^m a(j_1, j_2, m; t_1, t_2). \tag{1.14}$$

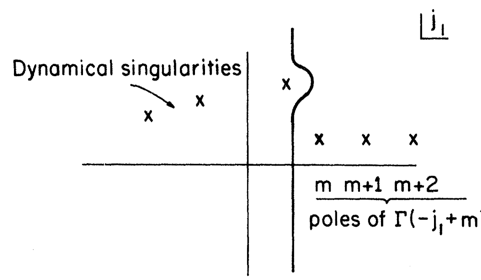
The nontrivial dependence on j_1, j_2, and m of the group representation functions is exhibited by the three Γ functions. The remaining dependence, as well as kinematic factors from converting to invariants using (1.8), have been absorbed into the partial-wave amplitude $a(j_1, j_2, m; \; t_1, t_2)$. Equation (1.14) has the advantage of exhibiting the dependence of the amplitude on the invariants in terms of which its analyticity properties are most simply stated.

The integration contour in (1.14) is such that the partial-wave sum is recovered by closing it to the right—see Fig. 3. Thus the singularities in $\Gamma(-m)$ lie to the right of the m contour and the singularities in $\Gamma(-j_i + m)$ lie to the right of the j_i contour. The latter correspond to sense values of the helicity, $m = j_i - p$ (p non-negative integer).[8] The singularities due to the $\Gamma(-j_i + m)$ lie to the left of the m contour, however, and thus will give contributions to the behavior as $s_{12}/s_1 s_2 \to \infty$. As the m contour is swept to the left the poles in $\Gamma(-j_i + m)$ will pinch the j_i contour against any dynamical singularities in j_i in the partial-wave amplitude (e.g., Regge poles) and produce singularities in m. Thus a singularity at $j_i = \alpha_i(t_i) \equiv \alpha_i$ will lead to helicity singularities at

$$m = \alpha_i - p \; (p \text{ non-negative integer}). \tag{1.1}$$

Hence, if we assume the partial-wave amplitude has no singularities in m, the complex-helicity singularities are determined completely by the "dynamical" complex angular momentum singularities (e.g., those in the partial-wave amplitude).

Let us now discuss what arguments can be made



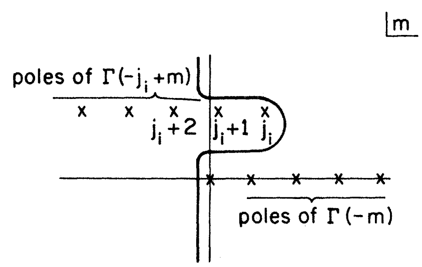


FIG. 3. Integration contour for Eq. (1.14). (a) Complex j_1 plane when j_1 integration is performed first. (b) Complex m plane when m integration is performed first.

for this assumption. Suppose we consider the double-Regge limit of (1.14). Then singularities at $j_i = \alpha_i$ lead to

$$A_5 \sim \frac{1}{2\pi i}\int dm\,\Gamma(-m)\Gamma(-\alpha_1+m)\Gamma(-\alpha_2+m) \times(-s_1)^{\alpha_1-m}(-s_2)^{\alpha_2-m}(-s_{12})^m \times\beta(m;t_1,t_2)\,. \tag{1.15}$$

In general, the behavior $(-s_1)^{\alpha_1-m}(-s_2)^{\alpha_2-m}$ represents a simultaneous discontinuity in s_1 and s_2 in the double-Regge region of phase space. Such physical-region simultaneous discontinuities in overlapping variables are prohibited by the Steinmann relation. Therefore, assuming the m contour can be closed to the left,[9] we see that singularities in m are allowed only for m differing from either α_1 or α_2 by an integer.[1] We then obtain

$$A_5 \sim (-s_{12})^{\alpha_1}(-s_2)^{\alpha_2-\alpha_1}V_1(\eta;t_1,t_2) + (-s_{12})^{\alpha_2}(-s_1)^{\alpha_1-\alpha_2}V_2(\eta;t_1,t_2)\,, \tag{1.16}$$

where $\eta = s_{12}/s_1 s_2$. The helicity integral (1.15) naturally provides Laurent expansions for V_i about $\eta=\infty$:

$$V_i(\eta;t_1,t_2) = \sum_{k=-\infty}^{\infty}\eta^{-k}\,V_i^k(t_1,t_2)\,. \tag{1.17}$$

The "kinematic" singularities in helicity in the $\Gamma(-\alpha_i+m)$ provide the terms with non-negative k, i.e., sense values of the helicity $m=\alpha_i-p$. Nonsense terms for negative k would correspond to dynamical singularities in helicity, i.e., singularities in the residue $\beta(m;t_i,t_2)$ of the *partial-wave amplitude*.

The nonsense terms in (1.17) clearly cannot give any contributions to the residues of the poles for α_1 or α_2 integral, since they correspond to nonsense helicities (nonpolynomial residues). Thus they would be terms which contribute to none of the resonances but which do contribute to the Regge trajectories. Such a situation is quite at odds with our usual notions of the trajectory interpolating the resonances but not *a priori* inconceivable.

We close this section with a technical remark in order to eliminate a possible source of confusion. In general one expects fixed-pole dynamical singularities as well as moving poles and cuts.[1] These are located at $j_i - m = n$, where n is a *negative* integer. They lie to the *right* of the m contour, however, and thus do not produce the nonsense terms η^{-n}. [Alternatively, if the j_i integrals are done first, one sees that fixed poles do not produce singularities in m since they cannot pinch the contour against the other dynamical singularities on the left of the j_i contour *or* the singularities in $\Gamma(-j_i+m)$.] The nonsense terms would be produced by singularities at $\alpha_i - m = n$ to the *left* of the m contour.

II. ARGUMENTS AGAINST "NONSENSE" HELICITY SINGULARITIES

In this section we give four rather different arguments against nonsense helicity singularities, each proceeding from different technical assumptions. The first two arguments (Secs. II A and II B) rely on the existence of the Sommerfeld-Watson transform or, equivalently, the uniformity of interchange of Regge and helicity limits. Most readers will find this assumption convincing. However, in Sec. II C we present a more fundamental argument which proceeds directly from analyticity by use of the Steinmann relation and does not require this assumption. Since the Steinmann relation only applies to the physical region, this argument needs to be supplemented by the assertion of Sec. II D that other singularities do not contribute to the Regge or helicity limits.

A. Argument from Regge Behavior

The Sommerfeld-Watson representation (1.14) suggests that the asymptotic limits $s_1\to\infty$, $s_2\to\infty$, $\eta\to\infty$ are simply determined by the singularities in j_1, j_2, and m in the partial-wave amplitude, and give the same result when taken in any order. Indeed, the complex angular momentum and helicity language really only makes sense if this is the case. If the limits can be uniformly interchanged in this way, consistency with Regge behavior can be used to exclude nonsense helicity singularities in certain cases.

Consider the discontinuity of A_5 in s_1 in the physical region for the $2\to 3$ process. The discontinuity is then given by a sum over intermediate states as shown in Fig. 4:

$$\mathrm{Disc}_{s_1}A_5 = \sum_{k=0}^{\infty} s_2^{\,k}\,\gamma_k(s_{12},s_1;t_1,t_2)\,. \tag{2.1}$$

The sum over powers of s_2 is equivalent to a partial-wave expansion in the angular momentum of

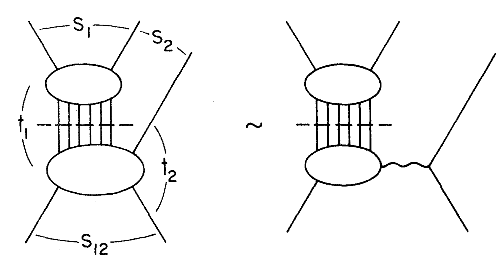


FIG. 4. Discontinuity of A_5.

the intermediate states. In the Regge limit, $s_{12} \to \infty$ and s_{12}/s_2 fixed, we can use Regge behavior of the individual exclusive processes to obtain

$$\mathrm{Disc}_{s_1} A_5 \underset{\substack{s_{12}\to\infty \\ s_{12}/s_2 \text{ fixed}}}{\sim} s_{12}{}^{\alpha_2} \sum_{k=0}^{\infty} \left(\frac{s_2}{s_{12}}\right)^k \beta_k(s_1; t_1, t_2)\,. \tag{2.2}$$

The behavior

$$\gamma_k \sim s_{12}{}^{\alpha_2 - k} \beta_k \tag{2.3}$$

arises because s_2 corresponds to helicity k of the intermediate state [see Eq. (2.12)]. Comparing with (1.14) we see that (2.2) apparently corresponds to only sense values of helicity $m = \alpha_2 - p$. Unfortunately we cannot conclude that nonsense terms are absent since $s_{12}/s_2 \to \infty$ is not inside the physical region for the $2 \to 3$ process and it is possible that (2.2) diverges when continued to that point. [In this case Eq. (2.2) should be rearranged into an expansion about a point in the physical region where the series converges.] Such divergences of partial-wave expansions are common phenomena.

The simple Regge argument can be used in the corresponding case of $\mathrm{Disc}_{s_1} A_6$ shown in Fig. 2,[10] however, since then s_{12}/s_2, $s_{31}/s_3 \to \infty$ is inside the physical region. The kinematics for this case will be reviewed in Sec. IIC below. We also discuss below the possibility of continuing this result to $\alpha_3 = 0$ to obtain the absence of nonsense helicity singularities for A_5.

B. Argument from Unitarity

If the interchange of limits is uniform, we can also argue against nonsense helicity singularities directly from unitarity. Again we must consider A_6 so we can work inside the physical region. The discontinuity of A_6 in s_1 (see Fig. 2) in the Regge-helicity limit analogous to (1.12), s_1, s_{12}/s_1, $s_{31}/s_1 \to \infty$ with the other invariants fixed, is

$$\mathrm{Disc}_{s_1} A_6 \sim s_1{}^{\alpha_1} \left(\frac{s_{12}}{s_1}\right)^{\alpha_2 - n_2} \left(\frac{s_{31}}{s_1}\right)^{\alpha_3 - n_3} \times V^{n_2 n_3}(t_1, t_2, t_3; s_2, s_3, s_{23})\,. \tag{2.4}$$

We have exhibited the contributions of possible nonsense helicity singularities at $\alpha_2 - n_2$ and $\alpha_3 - n_3$ (n_2 and n_3 are negative).

The general limit (2.4) can be related to the forward limit, where $t_1 = 0$, $t_2 = t_3$, $s_{23} = 0$, $s_2 = s_3 = m_2{}^2 = m_3{}^2$ by the Schwarz inequality, where the inner product is taken as a sum over intermediate states.[11] This is illustrated graphically in Fig. 5 and gives

$$\left| s_1{}^{\alpha_1} \left(\frac{s_{12}}{s_1}\right)^{\alpha_2 - n_2} \left(\frac{s_{31}}{s_1}\right)^{\alpha_3 - n_3} V^{n_2 n_3} \right|^2 \leq \left| s_1{}^{\alpha_v(0)} \left(\frac{s_{12}}{s_1}\right)^{2(\alpha_2 - n_2')} V^{n_2 n_2} \right| \left| s_1{}^{\alpha_v(0)} \left(\frac{s_{31}}{s_1}\right)^{2(\alpha_3 - n_3')} V^{n_3 n_3} \right|, \tag{2.5}$$

where α_v is the leading trajectory with vacuum quantum numbers. Thus when α_1 lies above $\bar{\alpha}_v(0)$, the next singularity below $\alpha_v(0)$, the existence of nonsense helicity singularities in the general limit requires corresponding singularities at $n_2 = n_2$ and $n_3 = n_3$ in the forward limit. However, for negative t_2 (or t_3) the forward amplitude is forbidden by unitarity to grow by a power larger than $s_{12}{}^2$ (or $s_{31}{}^2$) (see Ref. 12). Thus $V^{n_2 n_3}$ vanishes for all $\alpha(t_1) \geq \bar{\alpha}_v(0)$ and $t_2, t_3 \leq 0$ when $\alpha_2 - n_2 > 1$ or $\alpha_3 - n_3 > 1$. [The condition $\lambda(t_1, t_2, t_3) \leq 0$ must also be satisfied in order to assure that one is in the physical region—see Sec. IIC below.] For most trajectories and negative integers n there will be some region of negative t_2, t_3, where $V^{n_2 n_3}$ must vanish. Since $V^{n_2 n_3}$ is expected to be an analytic function of t_2 and t_3, it will then vanish for all t_2 and t_3. However, for trajectories like the pion with $\alpha(0) < 0$ we cannot exclude $n = -1$ terms by this argument.

C. Argument from Steinmann Relations

We have noted above that uniformity of interchange of limits and Regge behavior (or unitarity) exclude nonsense helicity singularities in certain cases. However, the existence of Sommerfeld-

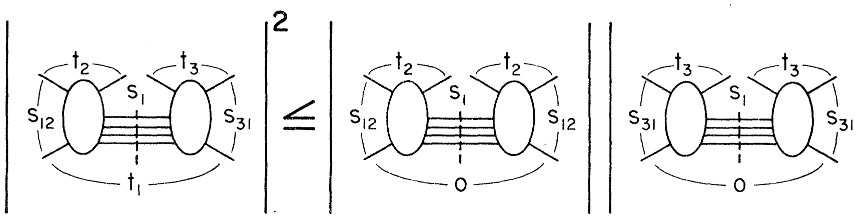


FIG. 5. Schwarz inequality as applied in Sec. IIA.

Watson representations like (1.14) is not rigorously established,[2] so we would like to try to do without the uniformity assumption. There are several types of nonuniformities to be concerned about:

(i) Nonuniformity of interchange of Regge and helicity limits for the same Reggeon. When the Regge argument of Sec. II A can be applied, this is excluded. For example, suppose $s_{12}/s_2 \to \infty$ is inside the physical region for A_5. Consider the two orders of limits $s_2 \to \infty$ [Regge limit (1.9b)] then $s_{12}/s_2 \to \infty$ and $s_{12} \to \infty$ [helicity limit (1.11)] then $s_2 \to \infty$ which both lead to the Regge-helicity limit (1.12b). In the first order Eq. (2.2) gives

$$\mathrm{Disc}_{s_1} A_5 \sim s_{12}{}^{\alpha_2} \beta_0(s_1; t_1, t_2). \tag{2.6}$$

The second order can also be obtained from Regge behavior of the individual intermediate states if $s_{12}/s_2 \to \infty$ is inside the physical region merely by changing the relative orientation of the blobs in Fig. 4 [the corresponding case for A_6 is studied in detail below—see Eq. (2.12) and Ref. 18]. Taking this limit on (2.1) and using (2.3) we obtain (2.6) again.

(ii) Nonuniformity of interchange of Regge-helicity limit for one Reggeon with Regge limit for another Reggeon. The example (1.6) is an instance of such a nonuniformity. An analogous example for A_5 is

$$A_5 = s_{12}{}^{\alpha_2} s_1{}^{\alpha_1 - \alpha_2} \frac{1}{s_1 s_2/s_{12} + 1/s_1}. \tag{2.7}$$

First taking the Regge-helicity limit (1.12b) then $s_1 \to \infty$ (1.9a) yields $s_{12}{}^{\alpha_2} s_1{}^{\alpha_1 - \alpha_2 + 1}$. The reverse order of limits yields $s_{12}{}^{\alpha_2} s_1{}^{\alpha_1 - \alpha_2}(s_{12}/s_1 s_2)$. This type of behavior cannot be excluded using the Regge arguments of Sec. II A since as $s_1 \to \infty$, the number of intermediate states increases and the sum could diverge to produce such behavior.[13]

In order to exclude this type of behavior and provide another type of argument against nonsense helicity singularities, we extend the Steinmann-relation argument[1] of Sec. I. In Sec. I we recalled that the Steinmann relation requires $m = \alpha_i - k$ (k is any integer). Now we show that, in fact, k must be non-negative in certain cases. The argument in many respects is similar to the Regge argument of Sec. II A but it is valid for s_1 large.

Again we consider the six-particle amplitude of Fig. 2 in the single Regge-helicity limit:

$$\mathrm{Disc}_{s_1} A_6 \sim s_1{}^{\alpha_1} \left(\frac{s_{12}}{s_1}\right)^{\alpha_2 - n_2 + n_1} \left(\frac{s_{31}}{s_1}\right)^{\alpha_3 - n_3 + n_1} \times V^{n_2 n_3}(t_1, t_2, t_3; s_2, s_3, s_{23}). \tag{2.4}$$

If we take the further limit $s_2, s_3, s_{23} \to \infty$ with $s_{23}/s_2 s_3$ fixed we expect to obtain the same behavior as in the triple-Regge–helicity limit which is analogous to (1.13):

$$V^{n_2 n_3} \sim s_2{}^{n_2} s_3{}^{n_3} s_{23}{}^{n_1} V^{n_1 n_2 n_3}(t_1, t_2, t_3), \tag{2.8}$$

where we have also picked out a single term $s_{23}{}^{n_1}$. The triple-Regge–helicity limit can also be reached by taking the helicity limit $\eta_{ij} \to \infty$ on the triple-Regge form

$$\mathrm{Disc}_{s_1} A_6 \sim s_1{}^{\alpha_1} \left(\frac{s_{12}}{s_1}\right)^{\alpha_2} \left(\frac{s_{31}}{s_1}\right)^{\alpha_3} \times V_{23}(t_1, t_2, t_3; \eta_{12}, \eta_{23}, \eta_{31}), \tag{2.9}$$

where $\eta_{ij} = s_{ij}/s_i s_j$. Thus the form (2.9) corresponds to a nonsense term

$$V_{23} \approx \eta_{12}{}^{-n_2} \eta_{31}{}^{-n_3} \left(\frac{\eta_{31} \eta_{12}}{\eta_{23}}\right)^{-n_1} V^{n_1 n_2 n_3}(t_1, t_2, t_3) \tag{2.10}$$

in the expansion analogous to (1.17). The particular ratios of invariants occurring in (2.8) or (2.10) may appear peculiar to the reader unfamiliar with these limits, but we will see below that they have a very natural meaning.[14]

The behavior (2.8) which allows the nonsense term (2.4) to survive in the triple-Regge–helicity limit implies nonpolynomial dependence in $\mathrm{Disc}_{s_1} A_6$ in s_2, s_3, and s_{23} since n_i are negative. Such behavior generally would be expected to arise from singularities in $\mathrm{Disc}_{s_1} A_6$. For example,

$$\int ds_2' \frac{\rho(s_2')}{s_2 - s_2'} \underset{s_2 \to \infty}{\sim} \sum_{i=1}^{\infty} \left(\frac{1}{s_2}\right)^i \int ds_2' \rho(s_2')(s_2')^{i-1}, \tag{2.11}$$

as long as the integrals converge. A cut of finite length thus naturally produces such inverse powers. We shall now argue that these singularities would be in the physical region in (2.4). Such singularities are forbidden by the Steinmann relation since the discontinuity in s_1 cannot have simultaneous discontinuities in the overlapping variables s_2, s_3, and s_{23}. Thus nonsense terms are excluded in (2.9) and probably are then also absent in (2.4).[15]

We now wish to show directly that the physical region for (2.4) covers essentially all real s_2, s_3, and s_{23}. To do this we parametrize the momenta as follows. Let frame 2 be such that the intermediate state of mass $\sqrt{s_1}$ is at rest and p_1 and $q_2 = p_2 + p_2'$ are along the z axis. This differs from the t_2-channel center-of-mass frame by a boost along the z axis. Similarly frame 3 has $\sqrt{s_1}$ at rest and p_1' and $q_3 = p_3 + p_3'$ along the z axis. Since $\sqrt{s_1}$ is at rest in both frames, they must differ by a rotation $R_z(\varphi_3) R_y(\theta_1) R_z(\varphi_2)$. In the physical regions for the reactions $p_1 + p_2 \to -p_2' + p_{s_1}$ and $-p_{s_1} - p_3' \to p_1' + p_3$ we have $t_2, t_3 < 0$ and $\varphi_2, \theta_1, \varphi_3$ are physical angles of rotation. We then find in the limit $s_1, s_{12}/s_1, s_{31}/s_1 \to \infty$

$$\begin{aligned}
\cos\theta_1 &\sim 1+\frac{2t_1}{s_1}\,,\\
s_2 &\sim (s_{12}/s_1)[t_1+t_2-t_3-2(t_1t_2)^{1/2}\cos\varphi_2]\,,\\
s_3 &\sim (s_{31}/s_1)[t_3+t_1-t_2-2(t_3t_1)^{1/2}\cos\varphi_3]\,,\\
s_{23} &\sim \frac{s_{12}}{s_1}\frac{s_{31}}{s_1}[t_1+t_2+t_3-2(t_1t_2)^{1/2}\cos\varphi_2\\
&\qquad -2(t_3t_1)^{1/2}\cos\varphi_3\\
&\qquad +2(t_2t_3)^{1/2}\cos(\varphi_2-\varphi_3)]\,.
\end{aligned} \tag{2.12}$$

From (2.12) we see that a contribution to the discontinuity in s_1 of spin J must be a polynomial in s_2, s_3, and s_{23} of total order at most J. Such contributions then naturally give the terms in (2.8) or (2.10) for *positive* n_i. However, although any given contribution contributes only polynomials in s_2, s_3, and s_{23}, as s_1 increases more and more terms enter and the series could start to diverge to produce terms with negative n_i. Thus, whereas the series must be convergent inside the physical region, it could be the representation of a function with a singularity outside the physical region. This is a typical phenomenon with partial-wave expansions.

However, the physical region actually encompasses essentially all real s_2, s_3, and s_{23}. This is because the coefficients of the large variables s_{12}/s_1 and s_{31}/s_1 in (2.12) have linear zeros inside the physical region, and thus slight variations in them around zero can give essentially any value of s_2, s_3, or s_{23}. Consider, for example,

$$t_1+t_2-t_3-2(t_1t_2)^{1/2}\cos\varphi_2=0\,. \tag{2.13}$$

This can be solved for physical φ_2, if

$$(t_1+t_2-t_3)^2\le[2(t_1t_2)^{1/2}]^2\,,$$

or

$$\begin{aligned}
\lambda(t_1,t_2,t_3) &= t_1^{\,2}+t_2^{\,2}+t_3^{\,2}-2t_1t_2\\
&\quad -2t_2t_3-2t_3t_1\\
&\le 0\,.
\end{aligned} \tag{2.14}$$

Similarly, the other brackets in (2.12) can vanish if $\lambda(t_1,t_2,t_3)\le 0$.[16] Thus nonsense helicity singularities are forbidden for t_1 such that $\lambda(t_1,t_2,t_3)\le 0$. Assuming $V^{n_1n_2n_3}$ is an analytic function of the t_i, they are also forbidden for all t_i by analytic continuation, since $\lambda\le 0$ is a finite region in the t_i.[17,18]

D. Singularities Other Than Normal Thresholds

Throughout this paper we have made the plausible physical assumption that singularities in the asymptotic behaviors of amplitudes are asymptotic representations of the true singularity structure of the amplitude. Thus the term $A_5\sim(-s_{12})^{\alpha_2}(-s_1)^{\alpha_1-\alpha_2}$ in (1.16) is interpreted as representing a simultaneous discontinuity in s_1 and s_{12} for large s_1, s_2, η. Nonsense helicity singularities would modify this to $A_5\sim(-s_{12})^{\alpha_2-n}(-s_1)^{\alpha_1-\alpha_2+n}(-s_2)^n$, where $n<0$. Since $\mathrm{Disc}_{s_1}A_5\propto(-s_2)^n$, the discontinuity in s_1 apparently has a singularity in s_2. However, since n is a (negative) integer, this need not represent a singularity for large s_2 inside the physical region, and thus is not necessarily in contradiction with the Steinmann relations. Complex singularities at finite values of the invariant must also be taken into account. Inverse powers are usually asymptotic representations of singularities of finite extent located at finite values of the invariant [see Eq. (2.11)].

One might indeed expect that such singularities exist in A_5 or A_6 and thus nonsense helicity singularities are required. For example, the usual box singularity (Fig. 6) is present for small s_1 and s_2. However, because $s_{12}\gg s_1$ and s_2, the nature of this singularity is rather different than in the case of the four-particle amplitude. Indeed it is present in the physical region—but only if approached from $(\mathrm{Im}s_1)(\mathrm{Im}s_2)<0$. Since a Regge-type term like $(-s_1)^{\alpha_1-\alpha_2+n}$ represents a singularity whether approached from $\mathrm{Im}s_1>0$ or $\mathrm{Im}s_1<0$, it cannot represent singularities of this type. Furthermore for this diagram there are also complex anomalous thresholds which are closely tied to the above behavior of the double-spectral singularity, and these also cannot be represented by Regge poles. In general we expect Regge poles to represent only normal threshold singularities. Regge cuts may represent the higher-order Landau diagrams. These points will be discussed further by one of us.[19]

In this connection we would like to make a remark on the relationship between nonsense helicity singularities in the six-particle amplitude and the five-particle amplitude. If particles are Reggeized, A_5 can be obtained by taking the residue of A_6 at $\alpha_3=0$ so the absence of nonsense helicity singularities in A_6 shown in Sec. IIC eliminates one source of them in A_5. However, there is an-

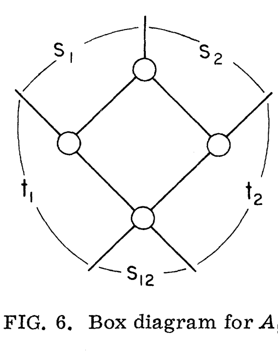


FIG. 6. Box diagram for A_5.

other source. The argument of Sec. IIC should really be made for finite values of the invariants. Thus there we wish to assume that s_1, s_{12}/s_1, and s_{31}/s_1 are very large but finite. Then there are corrections to (2.12) of order $O(1)$, $O(s_1)$, $O(s_{12}/s_1^2)$, etc. Since s_{12}/s_1 is much larger than these, (2.12) essentially still holds. However, we can only exclude singularities from a finite region of $O(s_{12}/s_1)$ or $O(s_{31}/s_1)$. Thus singularities like $s_2' = C(t_1, t_2, t_3)s_{12}/s_1$ cannot generally be excluded for $C \neq 0$.[20] These correspond to a behavior of the triple-Regge vertex in (2.9) like

$$V_{23} \sim [1/\eta_{12} - C(t_1, t_2, t_3)]^{-1}. \tag{2.15}$$

If $C(t_1, t_2, t_3)$ were to vanish for $\alpha_3 = 0$ this would lead to a nonsense helicity singularity in A_5. However, singularities like (2.15) would be asymptotic representations of singularities at $s_1 s_2 = C(t_1, t_2, t_3)s_{12}$. These are not normal threshold-type singularities and we do not expect them to occur in Regge or helicity expansions as we have discussed above.

III. CONCLUSION

We have given arguments for the dependent nature of singularities in complex helicity, i.e., they are related to angular momentum singularities by

$$m = \alpha_i - p, \ (p \text{ non-negative integer}).$$

However, although helicity singularities do not correspond to new dynamical objects, they do determine distinct asymptotic limits. Thus through their determination of helicity singularities, singularities in angular momentum determine asymptotic limits in addition to the conventional multi-Regge limit.

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[1]For a general discussion and references to original papers, see J. H. Weis, Phys. Rev. D 6, 2823 (1972). Complex helicity seems first to have been discussed by V. N. Gribov, I. Ya Pomeranchuk, and K. A. Ter-Martirosyan, Phys. Rev. 139, B184 (1965). Recent developments start with P. Goddard and A. R. White, Nuovo Cimento 1A, 645 (1971); C. E. De Tar *et al.*, Phys. Rev. Lett. 26, 675 (1971), and C. E. Jones, F. E. Low, and J. E. Young, Phys. Rev. D 4, 2358 (1971). A recent general discussion is also given by H. D. I. Abarbanel and A. Schwimmer, Phys. Rev. D 6, 3018 (1972).

[2]Complex helicity is also essential in performing the analytic continuation of multiparticle amplitudes and multiparticle unitarity in complex angular momentum. This has been discussed in an important series of papers by A. R. White, Nucl. Phys. B39, 432 (1972); B39, 461 (1972); and references therein.

[3]C. E. Jones, F. E. Low, S.-H. Tye, G. Veneziano, and J. E. Young, Phys. Rev. D 6, 1033 (1972).

[4]R. C. Brower and J. H. Weis, Phys. Lett. 41B, 631 (1972).

[5]M. B. Einhorn and M. B. Green (unpublished).

[6]The same uniformity assumption is involved in the Regge behavior for electroproduction scaling functions ($M^2 \to q^2$, $s/M^2 \to s/-q^2 = \omega - 1$). In the Regge limit $\nu W_2 \to \beta(q^2)s^\alpha$, whereas in the scaling limit $\nu W_2 \to F_2(\omega)$. Uniformity then gives $F(\omega) \sim \omega^\alpha$ as $\omega \to \infty$.

[7]The names of the limits given here differ from those often used in the past, but we feel these are more descriptive. Note that limits with more than one independent large invariant assume uniformity in the order of taking the limit since otherwise separate limits would have to be defined for different orders. Some of our arguments below will not require this uniformity assumption.

[8]Our use of "sense" and "nonsense" is a generalization of the usual usage. "Sense" is $j - m = 0, 1, 2, \ldots$, and "nonsense" is $j - m = -1, -2, -3, \ldots$. In the Reggeization of four-particle amplitudes m is a fixed integer, so one calls integral values of j "sense" or "nonsense". Here both j and m can be complex.

[9]We need to extract the full double-Regge vertex, not just its leading behavior as $\eta \to \infty$, since $\eta \to \infty$ is not inside the physical region and thus the Steinmann relation need not hold.

[10]C. E. DeTar, C. E. Jones, F. E. Low, C.-I Tan, J. H. Weis, and J. E. Young, Phys. Rev. Lett. 26, 675 (1971); C. E. Jones, F. E. Low, and J. E. Young, Phys. Rev. D 4, 2358 (1971).

[11]For similar uses of the Schwarz inequality, see H. D. I. Abarbanel, S. D. Ellis, M. B. Green, and A. Zee (unpublished) and H. D. I. Abarbanel, V. N. Gribov, and O. V. Kanchelli (unpublished).

[12]A. Patrascioiu, Nuovo Cimento 15A, 249 (1973); G. Tiktopoulos and S. B. Treiman, Phys. Rev. D 6, 2045 (1972). In the present case the reason for the bound in obvious. If the inclusive cross section is integrated over the region of phase space where there is a unique fastest particle, it must be bounded by the total cross section, which in turn is bounded by the Froissart bound. Any small region of $t_2 \leq 0$ and $1 - s_1/s_{12} \approx 1$ would then lead to a violation if $\alpha_2 - k > 1$.

[13]With a slight modification (1.6) or (2.7) circumvents the Schwarz inequality argument as well as the argument of Sec. IIA. For example,

$$A_6 = s_1^{\alpha_1}\left(\frac{s_{12}}{s_1}\right)^{\alpha_2}\left(\frac{s_{31}}{s_1}\right)^{\alpha_3} \times \frac{1}{(s_1 s_2/s_{12})(s_3 s_1/s_{31}) + 1/s_1^{\alpha_v(0)-\alpha_1}} \, .$$

[14]These limits have been discussed and studied in the dual resonance model by C. E. DeTar and J. H. Weis, Phys. Rev. D 4, 3141 (1971).

[15]Since the behavior of the amplitude is less specified, it is generally more difficult to rigorously exclude "nonsense" helicity singularities in limits like (2.4) with few variables asymptotic when they do not also occur in the Regge limits. These would be peculiar helicity singularities whose location differed from that of a given angular momentum singularity by an integer, but do not occur in the part of the amplitude with the angular momentum singularity.

[16]The criterion $\lambda \le 0$ for an helicity limit being inside the physical region has been emphasized by Abarbanel and Schwimmer (Ref. 1). For a nice presentation of kinematics in standard notation, see M. N. Misheloff, Phys. Rev. 184, 1732 (1969).

[17]Note that $\lambda(t_1, t_2, t_3) \le 0$, only if all t_i have the same sign. Then, given, say, t_2 and t_3, $\lambda \le 0$ for

$$|\sqrt{t_2} - \sqrt{t_3}| \le |t_1| \le |\sqrt{t_2} + \sqrt{t_3}| \, .$$

[18]From Eq. (2.12) one sees that the triple-Regge limit ($s_2, s_3, s_{23} \to \infty$) is obtained for fixed φ_2 and φ_3, whereas, the Regge-helicity limit (s_2, s_3, s_{23} fixed) is obtained for φ_2 and φ_3 varying with s_{12}/s_1 and s_{31}/s_1. Thus, depending on the relative orientation of the states on either side of $\sqrt{s_1}$, either one or the other limit is obtained inside the physical region.

[19]A. Patrascioiu, MIT report (unpublished).

[20]C cannot be a constant since the coefficient of s_{12}/s_1 in (2.13) can be made arbitrarily large for $\lambda \le 0$ by scaling all the t_i to large values.

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Chiral Symmetry Limit in the SU(3) σ Model with Bilinear Breaking

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We add to the usual σ model a $(3^*, 3) + (3, 3^*)$ term which is bilinear in the scalar-meson fields and study the behavior of the tree-approximation solutions as the explicit symmetry breaking is turned off. We find that *both* $c \simeq -\sqrt{2}$ and $c \simeq 0$ occur for the same mass spectrum, 0-8 mixing, and F_K/F_π. Solutions with different c values approach the symmetry limit differently, but in this limit one can smoothly reach either a Goldstone pseudoscalar octet, or a symmetric vacuum, or neither for both values of c. No clear indication is found that solutions are near a critical value of the mass parameter.

I. INTRODUCTION

The assumption that expansions in symmetry-breaking parameters exist provides a basis for study of approximate symmetries of the strong-interaction Hamiltonian. For systems whose underlying symmetry is chiral SU(3)×SU(3),[1] the question of the nature of the symmetry limit and the existence of expansions in powers of symmetry-breaking parameters about such a limit is complicated by the Goldstone phenomenon[2] where the solutions of the symmetric theory do not exhibit the full symmetry of the Hamiltonian. Dashen[3] and Dashen and Weinstein[4] developed chiral-symmetry-breaking expansions and systematically exploited the technique in deriving a number of correlations among symmetry-breaking effects. Li and Pagels[5] subsequently showed that Goldstone-boson intermediate states can give rise to singularities in the symmetry limit, thus invalidating many attempts to extrapolate "soft pion" theorems to the pion mass shell. In a study of Lagrangian models solved in the tree-graph approximation, Carruthers and Haymaker[6] noticed a different phenomenon which has the same negative implication for expansions about a Goldstone symmetry solution. They found that vacuum expectation values in the σ model[7] are multivalued func-