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Action Principle and Nonlocal Field Theories

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Lagrangian densities are introduced for nonlocal field theories, which make the application of the action principle possible. The action principle is then applied to classical and quantum nonlocal field theories. General formulas for conserved densities are derived by use of a generalized variation method, under the assumption of c-number variations. These formulas are also applied to a particular model due to Kristensen and Møller. The charge and energy-momentum vector are, thereby, shown to be equal to the expressions given by Pauli, which he derived by other means. This model is also quantized by use of the Yang-Feldman method. But as this method leads to a noncanonical quantization, the above derived quantities are in general no longer conserved. This is due to the fact that the assumption of c-number variations essentially restricts the quantization to a canonical one. The action principle with q-number variations is therefore considered. Thus new integral conserved quantities are derivable. However, one also gets a general consistency condition for the above model as well as for all other similar models. The fulfilment of this condition is required by (i) stationarity of the total action and (ii) uniqueness of the integral conserved quantities. It is also a necessary condition for the existence of a unitary S operator. The Kristensen-Møller model is shown to violate the above condition, and is therefore not consistent.

INTRODUCTION

Twenty years ago there was considerable interest in nonlocal field theories. Yukawa[1] was led to a nonlocal theory by use of the reciprocity principle, and Kristensen and Møller[2] proposed a nonlocal model for the nucleon-meson interaction. The aim of these theories was to get convergent perturbation expansions. However, this was not easily established in a satisfactory way,[3] which is one of the reasons why the whole idea was abandoned at that time.

There are several reasons why nonlocal field theories may be of interest today. Evidently the most important physical reason is the fact that there has been little success so far in the efforts to apply local field theories to weak and especially strong interactions. Nonlocal theories can here perhaps provide the extra structure necessary in order to fit the experimental data. Furthermore, from a mathematical point of view, local field theories are not quite satisfactory. The field operators are distributions which are difficult to handle, and one has only been able to prove the existence of free fields. Note also that, as products of distributions in general do not exist, one has to, in some way, introduce nonlocality (regularizations) in order to be able to handle the perturbation expansions in a rigorous manner. The hope is that a nonlocal field theory would overcome these difficulties.

In this paper we shall consider nonlocal theories of the Kristensen-Møller type. The action functional for such theories may be written

$$W=\int\cdots\int d^4x_1\cdots d^4x_N w(x_1,\ldots,x_N),$$

where $w(x_1,\ldots,x_N)$ is the action integrand, which may contain terms like $F(x_1,\ldots,x_N)\phi(x_1)\cdots\phi(x_N)$, where $\phi(x_i)$ are the field operators and $F(x_1,\ldots,x_N)$ is an arbitrary function called the form function. Poincaré invariance, macroscopic causality, convergence, etc. will strongly restrict the class of possible form functions. A functional variation of the above action gives equations of motion which are integro-differential equations, a typical feature of this kind of theories. We shall in particular investigate the explicit form and properties of conserved quantities. This will be done by a direct use of the action principle,[4] which here is made applicable to nonlocal field theories. When one first considered this problem,[2,5,6] one thought that there did not exist any conserved quantities in the ordinary sense for nonlocal theories. One thought that one could only have collision invariance. By this, one meant that the conserved quantities were strictly conserved only over infinitely large time intervals, i.e., they kept the same value in the limits $t\to-\infty$ and $t\to+\infty$, but were not constant in between. Pauli,[7] however, pointed out the following: "It has been correctly emphasized by A. S. Wightman that the invariance of a quantized theory with respect to any continuous group must necessarily be connected with the existence of corresponding integrals (constants in time) of the field equations (equations of motion) and that these integrals also determine the variations of the field quantities (observables) of the theory for an infinitesimal transformation of this group." Pauli then actually calculated the conserved energy-momentum vector and charge operator (noncovariant expressions) for the Kristensen-Møller theory. This he did in a straightforward manner by use of the equations of motion. He also showed that there may exist a conserved energy-momentum tensor and current density as well. Such quantities (only energy-momentum tensors) were also derived by Ôno[8] by use of a particular variation technique. But the connection between the action principle and conserved quantities in the case of nonlocal field theories has not been investigated before, and this is the main object of the present paper.

In Sec. I we will show how one may define the Lagrangian density necessary for the application of Schwinger's action principle. It is thereby shown that there exist an infinity of different possible Lagrangian densities to any given set of equations of motion. In Sec. II we will then apply the action principle by use of a generalized variation method presented in detail. The variation method to be generalized is that for higher-order Lagrangians, which is briefly presented in Appendix A. In Sec. II A are general formulas for conserved densities derived under the assumption of c-number variations. In Sec. II B we apply these formulas to the Kristensen-Møller model and take a closer look at the resulting conserved quantities. In Appendix B we apply the generalized variation method to nonlocal terms containing derivatives of the field operators, which essentially completes our derivations. In Sec. III we show that the possible set of Lagrangian densities is even larger than indicated in Sec. I, which has consequences even for local field theories. In Sec. IV we quantize the Kristensen-Møller model by use of the Yang-Feldman method, which leads to a noncanonical quantization of the fields. Because of this, the energy-momentum tensor (and vector) given in Sec. II B is no longer conserved. It is, however, shown that the action principle with q-number variations yields new integral conserved quantities, but they are shown not to be unique. Still it is made plausible and checked up to the third order in the perturbation expansions that they have the right generator properties, though in different senses. Finally, in Sec. V we briefly summarize our results and make some further comments.

I. FORMULATION OF THE ACTION PRINCIPLE AND DEFINITION OF A LAGRANGIAN DENSITY

An action of the type

$$W=\int\cdots\int d^4x_1\cdots d^4x_N w(x_1,\ldots,x_N),\quad N\geq 2 \tag{1.1}$$

is generally only a part of the total action. Usually one lets the interaction part be of the above type and the free part an integral of the ordinary free local Lagrangian. Such theories are called field theories with nonlocal interaction. The great advantage with these theories is that one can use the Yang-Feldman[9] formalism for quantization and for calculation of the S matrix (explicitly up to second order so far[10]).

Consider now the following rather general expression of the action integrand in (1.1) [for simplicity we exclude derivatives of field operators; note, however, that the introduction of derivatives of the fields in the action integrand offers no new problems (see Appendix B)]:

$$w(x_1,\dots,x_N)=E_{\alpha_1\cdots\alpha_M}(x_1,\dots,x_N)\phi^{\alpha_1}(x_1)\cdots\phi^{\alpha_M}(x_p), \tag{1.2}$$

where $E_{\alpha_1\cdots\alpha_M}(x_1,\dots,x_N)$ is the form function, which is assumed among other things to provide convergence. α_i, $i=1,\dots,N$, denote spinor, vector indices, etc. $N\geqslant p$, which means that the form function may contain more points than the polynomial field expression. M is the number of fields involved (not necessarily different). Furthermore, $M\geqslant p$, i.e., some of the fields may be taken at one and the same point.

For simplicity we shall restrict the form function to the following form:

$$E_{\alpha_1\cdots\alpha_M}(x_1,\dots,x_N)=F(x_1,\dots,x_N)\Lambda_{\alpha_1\cdots\alpha_M}, \tag{1.3}$$

where $\Lambda_{\alpha_1\cdots\alpha_M}$ is a constant matrix. (The following results and methods can, however, be extended to more general form functions.)

Poincaré invariance implies that $w(x_1,\dots,x_N)$ shall transform like a scalar. This in turn implies that the form function $F(x_1,\dots,x_N)$ also must transform like a scalar under Poincaré transformations. Thus $F(x_1,\dots,x_N)$ may be represented by the following Fourier expression:

$$F(x_1,\dots,x_N)=\frac{1}{(2\pi)^{4(N-1)}}\times\int\cdots\int d^4l_1\cdots d^4l_N\,\delta^4(l_1+\cdots+l_N)\times G(l_1,\dots,l_N)e^{i(l_1x_1+\cdots+l_Nx_N)}, \tag{1.4}$$

where $G(l_1,\dots,l_N)$ is a Lorentz-invariant function of the four-vectors l_i. The δ function provides translation invariance.

Let us consider the interesting case of a nonlocal interaction. There the free part of the total action is written as $W_0=\int d^4x\,\mathcal{L}_0(x)$, where $\mathcal{L}_0(x)$ is the usual free Lagrangian density. The total action is then $W_{\text{tot}}=W_0+W_I$, where W_I is given by (1.1). The equations of motion are derived by putting the functional derivatives of W_{tot} equal to zero. Let, e.g., the field ϕ^{α_R} in (1.2) be associated with the point x_s, and let it furthermore be a boson field. $\delta W_{\text{tot}}/\delta\phi^{\alpha}(x)=0$ then gives explicitly [for simplicity the action integrand (1.2) is assumed to be linear in ϕ^{α_R} in this example]

$$(\Box+m^2)\phi_{\alpha_R}(x)=\int\cdots\int d^4x_1\cdots d^4x_{s-1}d^4x_{s+1}\cdots d^4x_N\times F(x_1,\dots,x_{s-1},x,x_{s+1},\dots,x_N)\times\Lambda_{\alpha_1\cdots\alpha_M}\phi^{\alpha_1}\cdots\phi^{\alpha_{R-1}}\phi^{\alpha_{R+1}}\cdots\phi^{\alpha_M}(x_p), \tag{1.5}$$

which is a typical equation of motion for theories with nonlocal interaction. Note that (1.5) is an integro-differential equation.

In order to formulate the action principle in the usual way,[4] one needs a Lagrangian density. The problem is now to define it. Obviously, in the above example the Lagrangian $\mathcal{L}(x)=\mathcal{L}_0(x)+\phi^{\alpha_R}(x)f_{\alpha_R}(x)$ gives the equation of motion (1.5), if $f_{\alpha_R}(x)$ is equal to the right-hand side of (1.5). The derivation of this Lagrangian density may be put in words as follows: In the action integrand $w(x_1,\dots,x_N)$, one fixes the point with the field ϕ^{α_R}, and integrates over all the other points. Thereby one extracts the contribution from all the other fields to the point x_s. Then one identifies the point x_s with the point x in the free Lagrangian $\mathcal{L}_0(x)$.

We may now generalize the above procedure to derive Lagrangian densities to the general case. To get a Lagrangian $\mathcal{L}(x)$ one has to fix a point x in some way related to the points $x_1,\dots,x_N$ in the action integrand (1.2). The most arbitrary choice, if one requires that a translation $x_i^\mu\to x_i^\mu+a^\mu$ imply $x^\mu\to x^\mu+a^\mu$, is $x^\mu=\sum_{i=1}^N a_ix_i^\mu$, where a_i are real constants such that $\sum_{i=1}^N a_i=1$.

Thus we see that out of a given action like (1.1), one may define an infinity of different Lagrangian densities, all of which give the same equations of motion; this because of the simple fact that they give one and the same action functional. However, this nonuniqueness implies a corresponding nonuniqueness of the conserved quantities and is therefore unsatisfactory. Of course, if nonlocal models are formulated in terms of Lagrangian densities, one will no longer have any uniqueness problems.

We shall now give explicit expressions for the Lagrangian densities. It will thereby and later be shown convenient to introduce relative coordinates defined by

$$\xi_{ij}\equiv x_j-x_i\,. \tag{1.6}$$

These coordinates have the following properties:

$$\xi_{kk}=0,\qquad \xi_{jk}=-\xi_{kj},\qquad \xi_{ik}+\xi_{kj}=\xi_{ij}\,. \tag{1.7}$$

As we will only consider Poincaré-invariant actions, the form functions will be translation-invariant. This implies that $F(x_1, \ldots, x_N)$ only depends on the differences between coordinates, or equivalently it depends only on the relative coordinates ξ_{ij}. We may therefore define N new functions F_k, $k=1, \ldots, N$ with the following relation to $F(x_1, \ldots, x_N)$ [see (1.4)]:

$$F(x_1, \ldots, x_N) = F_k(\xi_{k1}, \ldots, \xi_{kN}), \quad k=1, \ldots, N. \tag{1.8}$$

By use of (1.6) and (1.7) one gets

$$x_k^\mu = x^\mu + \sum_{i=1} a_i \xi_{ik}^\mu, \quad k=1, \ldots, N \tag{1.9}$$

where $x^\mu = \sum_{i=1}^N a_i x_i^\mu$.

The action integrand (1.2) may now be written

$$w(x_1, \ldots, x_N) = F_k(\xi_{k1}, \ldots, \xi_{kN}) \Lambda_{\alpha_1 \cdots \alpha_M} \times \phi^{\alpha_1}\left(x + \sum_{i=1}^N a_i \xi_{i1}\right) \cdots \phi^{\alpha_M}\left(x + \sum_{i=1}^N a_i \xi_{ip}\right), \tag{1.10}$$

any $k \in \{1, \ldots, N\}$. Out of this action integrand we may define a Lagrangian density by simply integrating over the relative coordinates ξ_{kj}:

$$\mathcal{L}(x) = \int \cdots \int d^4\xi_{k1} \cdots d^4\xi_{k\,k-1} d^4\xi_{k\,k+1} \cdots d^4\xi_{kN} F_k(\xi_{k1}, \ldots, \xi_{kN}) \Lambda_{\alpha_1 \cdots \alpha_M} \phi^{\alpha_1}(x + \xi_{k1} + \rho_k) \cdots \phi^{\alpha_M}(x + \xi_{kp} + \rho_k), \tag{1.11}$$

where $-\rho_k \equiv \sum_{i=1}^N a_i \xi_{ki}$. Use has been made of (1.7) and $\sum_{i=1}^N a_i = 1$. Note that we get the same $\mathcal{L}(x)$ whatever set of relative coordinates $\xi_{k1}, \xi_{k2}, \ldots, \xi_{kj}$ $(k=1, \ldots, N)$ we are integrating over.

The action functional is $W = \int d^4x\, \mathcal{L}(x)$, which may be checked by simply changing the integration variables in (1.1). Now let the field ϕ^{α_R} have the argument x_s as before. Then a functional derivation of W with respect to ϕ^{α_R} gives $[\delta\phi^{\alpha_i}(y)/\delta\phi^{\alpha_j}(x) = \delta_{ij}\delta^4(x-y)]$

$$\frac{\delta W}{\delta \phi^{\alpha_R}(x)} = \int \cdots \int d^4\xi_{k1} \cdots d^4\xi_{k\,k-1} d^4\xi_{k\,k+1} \cdots d^4\xi_{kN} F_k(\xi_{k1}, \ldots, \xi_{kN}) \Lambda_{\alpha_1 \cdots \alpha_M} \times \phi^{\alpha_1}(x + \xi_{k1} - \xi_{ks}) \cdots \phi^{\alpha_{R-1}} \phi^{\alpha_{R+1}} \cdots \phi^{\alpha_M}(x + \xi_{kp} - \xi_{ks}), \tag{1.12}$$

which is independent of ρ_k, and as $\xi_{kl} - \xi_{ks} = \xi_{sl}$, a change of integration variables shows that (1.12) is equal to the right-hand side of (1.5), as it should be. Notice that the assumption that (1.2) is linear in ϕ^{α_R} in this example is no restriction. A functional derivation in the nonlinear case yields simply a sum of terms like the one in (1.12).

We may summarize our results so far as follows: To every action integrand of the type (1.2) (containing derivatives of fields or not) we may define a Lagrangian $\mathcal{L}(x)$. This $\mathcal{L}(x)$, is, however, not uniquely defined, but all the different Lagrangians associated with a given action integrand give the same equations of motion. $\mathcal{L}(x)$ is explicitly given by the convenient expression (1.11), where the nonuniqueness is expressed by the real parameters a_i, $i=1, \ldots, N$. In contradistinction to local field theory, the fields are no longer associated just with the point x in $\mathcal{L}(x)$, but a space-time domain around x. It is then the form function that determines how large this domain is.

The question is now how a particular Lagrangian should be chosen, for even if the condition that $\mathcal{L}(x)$ has to be Hermitian will restrict the possible choices somewhat, one requires much stronger conditions to get a unique $\mathcal{L}(x)$ out c given action integrand. Of course, one can let oneself be guided by simplicity here, but a particular choice $\mathcal{L}(x)$ seems also to be a dynamical assumption as it will be reflected into a corresponding choice of, e.g., currents, which in turn might have a physical meaning. The last remark then suggests that if different current densities have different physical interpretations, then we have a real means of selecting the Lagrangian.

As soon as we have chosen a particular Lagrangian density $\mathcal{L}(x)$, the formulation of the action principle offers no problems. The action with boundaries will look as usual:

$$W_{21} = \int_{\sigma_1}^{\sigma_2} d^4x\, \mathcal{L}(x),$$

as well as the action principle

$$\delta W_{21} = F[\sigma_1] - F[\sigma_2], \tag{1.13}$$

where δ is an infinitesimal variation and $F[\sigma]$ the corresponding infinitesimal generator. (If and in what way $F[\sigma]$ is a generator in the nonlocal case is not *a priori* clear. However, in Sec. IV we shall make it plausible that $F[\sigma]$ has the expected generator property in consistency with the Yang-Feldman quantization.)

Looking on the action functional (1.1), one sees that the introduction of boundaries in one of the integrations, e.g.,

$$W_{21}=\int_{-\infty}^{\infty}d^4x_1\cdots\int_{\sigma_1}^{\sigma_2}d^4x_i\cdots\int_{-\infty}^{\infty}d^4x_N \times w(x_1,\ldots,x_N),$$

is equivalent to defining a Lagrangian where $x=x_i$.

This is in sharp contrast to an earlier approach[11] to the formulation of the action principle for nonlocal field theories. There one used an action with boundaries introduced in every integration, i.e.,

$$W_{21}=\int_{\sigma_1}^{\sigma_2}\cdots\int_{\sigma_1}^{\sigma_2}d^4x_1\cdots d^4x_N w(x_1,\ldots,x_N).$$

Such an action does not have the usual additivity property $W_{31}=W_{32}+W_{21}$, and a variation of the action functional does not give any infinitesimal generator $F[\sigma]$ (without doing any unjustifiable intermediate redefinition). Consequently, one was not able to derive any conserved quantities in a straightforward way.

II. APPLICATION OF THE ACTION PRINCIPLE

A. General Formulas

Even if we are now able to define a Lagrangian density, the application of the action principle is not a straightforward matter. The technique to be used here is to expand the different fields in the chosen Lagrangian successively around the point x in a Taylor expansion. The resulting Lagrangian will be of infinite order, which means that it contains field operators and derivatives thereof up to infinite order. We then make use of the results from the application of the action principle to higher-order Lagrangians, generalized to infinite order.

Even if this method is mathematically justifiable, it is here only considered as a formal technique, as one is always able to transform the expression into a closed form.

The action principle for higher-order Lagrangians is reviewed in Appendix A. The useful formulas, generalized to infinite-order Lagrangians, are given below (c-number variations are assumed here).

The action functional is

$$W_{21}=\int_{\sigma_1}^{\sigma_2}d^4x\,\mathcal{L}(x), \tag{2.1}$$

where $\mathcal{L}(x)$ partly may look like (1.11). Consider first functional variations δ_0 of the action (2.1). The action principle [see (1.13) or (A2)]

$$\delta_0 W_{21}=\int_{\sigma_1}^{\sigma_2}d^4x\,\delta_0\mathcal{L}(x) = F[\sigma_1]-F[\sigma_2]$$

gives (for simplicity, the index on α, which identifies the fields in $\mathcal{L}(x)$, is suppressed in this subsection as well as in the appendixes; notice that repeated α means also a summation over this index, except when the index is explicitly written out)

(1) the equations of motion [see (A7)]

$$\sum_{i=0}^{\infty}(-\partial_\mu)^i\frac{\partial\mathcal{L}}{\partial(\partial_\mu)^i\phi^\alpha}=0; \tag{2.2}$$

(2) the infinitesimal generator [see (A4) and (A5)]

$$\begin{aligned}F[\sigma]&=-\int_\sigma d\sigma_\mu\sum_{s=0}^{\infty}\pi_\alpha^{\mu\nu_1\cdots\nu_s}\delta_0\partial_{\nu_1}\cdots\partial_{\nu_s}\phi^\alpha\\&=-\int_\sigma d\sigma_\mu\sum_{s=0}^{\infty}\sum_{i=0}^{\infty}(-\partial_\lambda)^i\\&\quad\times\frac{\partial\mathcal{L}}{\partial\partial_\mu(\partial_\nu)^s(\partial_\lambda)^i\phi^\alpha}\delta_0(\partial_\nu)^s\phi^\alpha,\end{aligned} \tag{2.3}$$

where we have used the notation $(\partial_\nu)^s\equiv\partial_{\nu_1}\cdots\partial_{\nu_s}$. (Such a formula has in fact been given by Ôno.[8] However, he did not give any derivation of it.) Note that we could have used the formula (2.2) equally well as the functional derivatives in Sec. I.

The coordinate variation induced by the infinitesimal Poincaré transformation $x_\mu\to x_\mu+\delta x_\mu$, where $\delta x_\mu=\epsilon_\mu+\epsilon_{\mu\nu}x^\nu$, gives via the action principle the infinitesimal generator [see (A13)]

$$F[\sigma]=-\int_\sigma d\sigma_\mu\left(\mathcal{L}\delta x^\mu+\sum_{s=0}^{\infty}\pi_\alpha^{\mu\nu_1\cdots\nu_s}\delta_{\mathcal{P}}\partial_{\nu_1}\cdots\partial_{\nu_s}\phi^\alpha\right) \tag{2.4}$$

where

$$\delta_{\mathcal{P}}\partial_{\nu_1}\cdots\partial_{\nu_s}\phi^\alpha(x)=-\delta x_\mu\partial^\mu\partial_{\nu_1}\cdots\partial_{\nu_s}\phi^\alpha(x)+\tfrac{1}{2}\epsilon_{\mu\nu}\Sigma_{\alpha\beta\nu_1\cdots\nu_s}^{\mu\nu\mu_1\cdots\mu_s}\partial_{\mu_1}\cdots\partial_{\mu_s}\phi^\beta(x). \tag{2.5}$$

$\Sigma_{\alpha\beta\nu_1\cdots\nu_s}^{\mu\nu\mu_1\cdots\mu_s}$ is the same matrix that occurs in (A16). The explicit expression for this matrix may easily be derived by making use of (A11) and the fact that $\delta_{\mathcal{P}}$ commutes with the derivatives ∂_{ν_i}. One finds

$$\begin{aligned}\delta_{\mathcal{P}}\partial_{\nu_1}\cdots\partial_{\nu_s}\phi^\alpha(x)&=\partial_{\nu_1}\cdots\partial_{\nu_s}[-\delta x_\nu\partial^\nu\phi^\alpha(x)+\tfrac{1}{2}\epsilon_{\nu\rho}\Sigma_{\alpha\beta}^{\nu\rho}\phi^\beta(x)]\\&=-\delta x_\nu\partial^\nu\partial_{\nu_1}\cdots\partial_{\nu_s}\phi^\alpha(x)+\tfrac{1}{2}\epsilon_{\nu\rho}\Sigma_{\alpha\beta}^{\nu\rho}\partial_{\nu_1}\cdots\partial_{\nu_s}\phi^\beta(x)-s\epsilon_{\nu\nu_s}\partial^\nu\partial_{\nu_1}\cdots\partial_{\nu_{s-1}}\phi^\alpha(x).\end{aligned} \tag{2.6}$$

Equation (2.4) may now be written

$$F[\sigma]=\int_\sigma d\sigma_\mu\left[\left(-\mathcal{L}g^{\mu\nu}+\sum_{s=0}^{\infty}\pi_\alpha^{\mu\nu_1\cdots\nu_s}\partial^\nu\partial_{\nu_1}\cdots\partial_{\nu_s}\phi^\alpha\right)\delta x_\nu\right.$$
$$\left.-\tfrac{1}{2}\epsilon_{\nu\rho}\sum_{s=0}^{\infty}\left[\pi_\alpha^{\mu\nu_1\cdots\nu_s}\Sigma_{\alpha\beta}^{\nu\rho}\partial_{\nu_1}\cdots\partial_{\nu_s}\phi^\beta+s(\pi_\alpha^{\mu\nu\nu_1\cdots\nu_{s-1}}\partial^\rho-\pi_\alpha^{\mu\rho\nu_1\cdots\nu_{s-1}}\partial^\nu)\partial_{\nu_1}\cdots\partial_{\nu_{s-1}}\phi^\alpha\right]\right]. \tag{2.7}$$

Identification of (2.7) with $F[\sigma]=\epsilon_\mu P^\mu[\sigma]+\frac{1}{2}\epsilon_{\mu\nu}J^{\mu\nu}[\sigma]$, $P^\nu[\sigma]=\int_\sigma d\sigma_\mu T_c^{\mu\nu}$, and $J^{\nu\rho}[\sigma]=\int_\sigma d\sigma_\mu M_c^{\mu\nu\rho}$ gives the canonical energy-momentum tensor [cf. (A15)]

$$T_c^{\mu\nu}=-\mathcal{L}g^{\mu\nu}+\sum_{s=0}^{\infty}\pi_\alpha^{\mu\nu_1\cdots\nu_s}\partial^\nu\partial_{\nu_1}\cdots\partial_{\nu_s}\phi^\alpha$$
$$=-\mathcal{L}g^{\mu\nu}+\sum_{s=0}^{\infty}\sum_{i=0}^{\infty}(-\partial_\lambda)^i\frac{\partial\mathcal{L}}{\partial\partial_\mu(\partial_\rho)^s(\partial_\lambda)^i\phi^\alpha}\partial^\nu(\partial_\rho)^s\phi^\alpha \tag{2.8}$$

and the corresponding angular momentum tensor [cf. (A16)]

$$M_c^{\mu\nu\rho}=x^\rho T_c^{\mu\nu}-x^\nu T_c^{\mu\rho}-\sum_{s=0}^{\infty}\left[\pi_\alpha^{\mu\nu_1\cdots\nu_s}\Sigma_{\alpha\beta}^{\nu\rho}\partial_{\nu_1}\cdots\partial_{\nu_s}\phi^\beta+s(\pi_\alpha^{\mu\nu\nu_1\cdots\nu_{s-1}}\partial^\rho-\pi_\alpha^{\mu\rho\nu_1\cdots\nu_{s-1}}\partial^\nu)\partial_{\nu_1}\cdots\partial_{\nu_{s-1}}\phi^\alpha\right]. \tag{2.9}$$

One may also define a symmetric energy-momentum tensor as follows: Call the last term in (2.9) $-A^{\mu\nu\rho}(x)$, then define the tensor [see (A17)]

$$f^{\mu\nu\rho}\equiv\tfrac{1}{2}(A^{\mu\nu\rho}+A^{\rho\nu\mu}+A^{\nu\rho\mu}), \tag{2.10}$$

which has the properties

$$f^{\mu\nu\rho}=-f^{\nu\mu\rho},\quad \tfrac{1}{2}(f^{\mu\nu\rho}-f^{\mu\rho\nu})=\tfrac{1}{2}A^{\mu\nu\rho}. \tag{2.11}$$

Using then the fact that $\epsilon_{\mu\nu}=\partial_\nu\delta x\mu$ and assuming that $\delta x_\rho f^{\mu\nu\rho}$ approaches zero sufficiently rapidly at infinitely remote points on σ so that $\int_\sigma d\sigma_\mu\partial_\nu(\delta x_\rho f^{\mu\nu\rho})=0$, one finds the symmetric or Belinfante energy-momentum tensor

$$T_B^{\mu\nu}=T_c^{\mu\nu}+\partial_\rho f^{\rho\mu\nu}. \tag{2.12}$$

The corresponding angular momentum tensor in turn is given by

$$M_B^{\rho\mu\nu}=T_B^{\rho\mu}x^\nu-T_B^{\rho\nu}x^\mu. \tag{2.13}$$

We shall now apply these formulas to a Lagrangian whose interaction part looks like

$$\mathcal{L}_I(x)=\int d^4\xi\, f_{\alpha_k}(x,\xi)\phi^{\alpha_k}(x+\xi)$$
$$=\int d^4\xi\, f_{\alpha_k}(x,\xi)\sum_{l=0}^{\infty}\frac{(\xi^\mu\partial_\mu)^l}{l!}\phi^{\alpha_k}(x), \tag{2.14}$$

where $f^\alpha(x,\xi)$ contains the form function and the other fields involved. This Lagrangian comprises all nonlocal field theories with nonlocal interaction and nonderivative coupling.

Equation (2.2) now gives

$$\sum_{i=0}^{\infty}(-\partial_\mu)^i\frac{\partial\mathcal{L}_I}{\partial(\partial_\mu)^i\phi^\alpha}=\int d^4\xi\, f_\alpha(x-\xi,\xi). \tag{2.15}$$

A calculation of the interaction part of the infinitesimal current in (2.3) yields

$$j_I^\mu(x)=-\sum_{s=0}^{\infty}\sum_{i=0}^{\infty}(-\partial_\lambda)^i\frac{\partial\mathcal{L}_I}{\partial\partial_\mu(\partial_\nu)^s(\partial_\lambda)^i\phi^\alpha(x)}\delta_0(\partial_\nu)^s\phi^\alpha(x)$$
$$=-\sum_{s=0}^{\infty}\sum_{i=0}^{\infty}\int d^4\xi\frac{\xi^\mu}{(s+i+1)!}(-\xi\partial)^i f_\alpha(x,\xi)\delta_0(\xi\partial)^s\phi^\alpha(x), \tag{2.16}$$

where we use the notation $(\xi\partial)\equiv\xi^\mu\partial_\mu$. j_I^μ has the property that

$$\partial_\mu j_I^\mu=\int d^4\xi[f_\alpha(x-\xi,\xi)\delta_0\phi^\alpha(x)-f_\alpha(x,\xi)\delta_0\phi^\alpha(x+\xi)]. \tag{2.17}$$

One may show in a straightforward manner that (2.16) may be written as

$$j_I^\mu = -\int d^4\xi\, \xi^\mu \frac{\sin\frac{1}{2}i(\xi\partial)}{\frac{1}{2}i(\xi\partial)} f_\alpha(x-\tfrac{1}{2}\xi,\,\xi)\delta_0\phi^\alpha(x+\tfrac{1}{2}\xi). \tag{2.18}$$

This is done by writing

$$\frac{\sin\frac{1}{2}i(\xi\partial)}{\frac{1}{2}i(\xi\partial)} f_\alpha(x-\tfrac{1}{2}\xi,\,\xi)\delta_0\phi^\alpha(x+\tfrac{1}{2}\xi) = \frac{1}{(\xi\partial)}(1-e^{-\xi\partial})[f_\alpha(x,\xi)e^{\xi\partial}\delta_0\phi^\alpha(x)],$$

then expanding $e^{\xi\partial}$ as an infinite power series and changing summation variables, and then making use of

$$\sum_{s=0}^{\infty}(-1)^s\frac{1}{s!(i-s)!(s+k+1)} = \frac{k!}{(i+k+1)!}.$$

The integrand on the right-hand side of (2.18) may be expressed by a convolution integral:

$$j_I^\mu(x) = -\int d^4\xi\,\xi^\mu\int d^4x' D(x-x',\tfrac{1}{2}\xi)f_\alpha(x'-\tfrac{1}{2}\xi,\,\xi)\delta_0\phi^\alpha(x'+\tfrac{1}{2}\xi), \tag{2.19}$$

where

$$\begin{aligned} D(x,a) &= \frac{1}{(2\pi)^4}\int d^4p\, e^{ixp}\frac{\sin(ap)}{(ap)} \\ &= \frac{1}{2a_0}[\theta(t+a_0)-\theta(t-a_0)]\delta^3\left(\vec{x}-\frac{t}{a_0}\vec{a}\right) \end{aligned} \tag{2.20}$$

if $a_0 \neq 0$, and similar expressions if $a_0 = 0$ and $\vec{a} \neq \vec{0}$.

$D(x,a)$ may be said to be a smeared δ function as

$$\lim_{a_\mu\to 0} D(x,a) = \delta^4(x). \tag{2.21}$$

Note also that $D(x,a)$ has support along a line, i.e., $D(x,a)\neq 0$ if and only if $x^\mu = ca^\mu$, where c is any real constant fulfilling $|c| \leq 1$.

Furthermore, $D(x,a)$ has the important property

$$a^\mu\partial_\mu D(x,a) = \tfrac{1}{2}[\delta^4(x+a)-\delta^4(x-a)]. \tag{2.22}$$

As a simple check, one may take the divergence of (2.19) and use the property (2.22) to see that one gets (2.17).

Equation (2.16) is a frequently occurring expression. It occurs in all currents, in the energy-momentum tensors, and in the angular momentum tensors. The only other kind of expression that occurs is the last one in the canonical angular momentum tensor (2.9), namely

$$\sum_{s=0}^{\infty} s\pi_\alpha^{\mu\nu\nu_1\cdots\nu_{s-1}}\partial^\rho\partial_{\nu_1}\cdots\partial_{\nu_{s-1}}\phi^\alpha - (\nu \leftrightarrow \rho). \tag{2.23}$$

Applying the first term above to the Lagrangian (2.14) one gets

$$\sum_{s=0}^{\infty}\sum_{i=0}^{\infty}\frac{s}{(s+i+1)!}\int d^4\xi\,\xi^\mu\xi^\nu(-\xi\partial)^i f_\alpha(x,\xi)\partial^\rho(\xi\partial)^{s-1}\phi^\alpha(x). \tag{2.24}$$

The divergence is easily calculated to be

$$\partial_\mu B^{\mu\nu\rho} = \int d^4\xi\,\xi^\nu f_\alpha(x,\xi)\partial^\rho\phi^\alpha(x+\xi) - \int d^4\xi\,\xi^\nu\int d^4x' D(x-x',\tfrac{1}{2}\xi)f_\alpha(x'-\tfrac{1}{2}\xi,\,\xi)\partial'^\rho\phi^\alpha(x'+\tfrac{1}{2}\xi), \tag{2.25}$$

from which one may show that the symmetrical energy-momentum tensor contains the same kinds of expressions as the canonical one.

Some calculations show that (2.24) may be written as

$$\sum_{s=0}^{\infty}\sum_{i=0}^{\infty}\int d^4\xi\,\xi^\mu\xi^\nu\tfrac{1}{2}\left[\frac{1}{\frac{1}{2}(\xi\partial)}\left(1-\frac{\sin\frac{1}{2}i(\xi\partial)}{\frac{1}{2}i(\xi\partial)}\right)f_\alpha(x-\tfrac{1}{2}\xi,\,\xi)\partial^\rho\phi^\alpha(x+\tfrac{1}{2}\xi)+\frac{\sin\frac{1}{4}i(\xi\partial)}{\frac{1}{4}i(\xi\partial)}f_\alpha(x-\tfrac{1}{4}\xi)\partial^\rho\phi^\alpha(x+\tfrac{3}{4}\xi)\right], \tag{2.26}$$

which in turn may be transformed into

$$\int d^4\xi\, \xi^\mu \xi^\nu \tfrac{1}{2} \int d^4x' [\overline{D}(x-x', \tfrac{1}{2}\xi) f_\alpha(x'-\tfrac{1}{2}\xi)\partial'^\rho \phi^\alpha(x'+\tfrac{1}{2}\xi) + D(x-x', \tfrac{1}{4}\xi) f_\alpha(x'-\tfrac{1}{4}\xi)\partial'^\rho\phi^\alpha(x'+\tfrac{3}{4}\xi)], \tag{2.27}$$

where we have introduced the function

$$\overline{D}(x,a) = \frac{1}{(2\pi)^4}\int d^4p\, e^{ixp} \frac{1}{i(ap)}\left(1 - \frac{\sin(ap)}{(ap)}\right)$$
$$= \frac{1}{2a_0}\operatorname{sgn}\frac{t}{a_0}\left\{1 - \left|\frac{t}{a_0}\right| + \left(\left|\frac{t}{a_0}\right| - 1\right)\theta\left(\left|\frac{t}{a_0}\right| - 1\right)\right\}\delta^3\left(\vec{x} - \frac{t}{a_0}\vec{a}\right) \tag{2.28}$$

if $a_0 \neq 0$, and similar expressions if $a_0 = 0$ and $\vec{a} \neq \vec{0}$. Evidently, $\overline{D}(x,a)$ has support only for those x values which fulfill $x^\mu \neq 0$ and $x^\mu = ca^\mu$, where c is any real constant such that $|c| \leq 1$.

Note that $\overline{D}(x,a)$ in distinction from $D(x,a)$ is not a kind of smeared δ function as

$$\lim_{a_\mu \to 0} \overline{D}(x,a) = 0\,. \tag{2.29}$$

A useful property is

$$a^\mu \partial_\mu \overline{D}(x,a) = \delta^4(x) - D(x,a)\,. \tag{2.30}$$

By use of (2.22) and (2.30) one may easily check that (2.27) satisfies (2.25).

As the Lagrangian (2.14) is a typical nonlocal Lagrangian, one knows that the application of the action principle to *any* nonlocal field theory with nonlocal interaction and nonderivative coupling gives rise to expressions of the above type. (In Appendix B we have extended the application of the action principle to a nonlocal Lagrangian containing derivatives of the field operators.) An unexpected feature of these expressions is the frequent occurrence of the function $D(x,a)$, (2.20), a feature one would not have guessed in advance.

A comment on the derivation of $T^{\mu\nu}$ and $M^{\mu\nu\rho}$ is necessary here. A fact not mentioned before is that the infinitesimal Poincaré transformation $x^\mu \to x^\mu + \delta x^\mu$, where $\delta x^\mu = \epsilon^\mu + \epsilon^{\mu\nu} x_\nu$, implies an infinitesimal transformation of the relative coordinate ξ^μ as well, namely

$$\xi^\mu \to \xi^\mu + \delta\xi^\mu, \quad \delta\xi^\mu = \epsilon^{\mu\nu}\xi_\nu\,. \tag{2.31}$$

A coordinate variation of the action functional will thereby contain new ingredients not considered before. Take, e.g., the action $W_I = \int_{\sigma_1}^{\sigma_2} d^4x\, \mathcal{L}_I(x)$, where $\mathcal{L}_I(x)$ is the Lagrangian (2.14). A coordinate variation of W_I will then contain at least two boundary variations, namely

$$B_\varphi W_I = \int_{\sigma_1}^{\sigma_2} \delta(d^4x)\mathcal{L}_I(x) + \int_{\sigma_1}^{\sigma_2} d^4x \int \delta(d^4\xi) f_\alpha(x,\xi)\phi^\alpha(x+\xi)\,.$$

[If $f_\alpha(x,\xi)$ contains integrals over other relative coordinates, we have boundary variations of these as well.]

We shall always require that the boundary variations of integrals over relative coordinates vanish. This requirement leads to the condition that the form functions in the Lagrangians have to vanish with a sufficient rapidity for infinite values of the relative coordinates.

Another implication of (2.31) is that [cf. (A11)]

$$\delta_\varphi \phi^\alpha(x+\xi) = -(\delta x^\mu + \delta\xi^\mu)\partial_\mu\phi^\alpha(x+\xi) + \tfrac{1}{2}\epsilon_{\mu\nu}\Sigma^{\mu\nu}_{\alpha\beta}\phi^\beta(x+\xi)\,. \tag{2.32}$$

But making use of (2.6) we get

$$\begin{aligned}\delta_\varphi\phi^\alpha(x+\xi) &= \sum_{s=0}^{\infty} \frac{\xi^{\nu_1}\cdots\xi^{\nu_s}}{s!}\delta_\varphi \partial_{\nu_1}\cdots\partial_{\nu_s}\phi^\alpha(x)\\ &= -\delta x^\mu \partial_\mu \phi^\alpha(x+\xi) + \tfrac{1}{2}\epsilon_{\mu\nu}\Sigma^{\mu\nu}_{\alpha\beta}\phi^\beta(x+\xi)\\ &\quad - \sum_{s=0}^{\infty}\frac{\xi^{\nu_1}\cdots\xi^{\nu_s}}{s!} s\epsilon_{\mu\nu_s}\partial^\mu\partial_{\nu_1}\cdots\partial_{\nu_{s-1}}\phi^\alpha(x)\\ &= -(\delta x^\mu + \delta\xi^\mu)\partial_\mu\phi^\alpha(x+\xi)\\ &\quad + \tfrac{1}{2}\epsilon_{\mu\nu}\Sigma^{\mu\nu}_{\alpha\beta}\phi^\beta(x+\xi)\,.\end{aligned} \tag{2.33}$$

Thus the technique with formal Taylor expansions takes account of (2.31) in the variation of the fields. Note that the term (2.23) comes from (2.31).

B. Application to a Particular Model

In order to get a better understanding of the derived formulas, we shall consider a particular nonlocal field-theory model. We choose the Kristensen-Møller [2,7] model, because it is one of the better-known nonlocal models considered in the literature. Kristensen and Møller proposed the following action functional for nuclear interaction:

$$W = W_0 + W_I\,, \tag{2.34}$$

where

$$W_0 = \int d^4x\, \mathcal{L}_0(x)\,,$$

$$\mathcal{L}_0(x) = \tfrac{1}{2}(\partial_\mu\phi\partial^\mu\phi - \mu^2\phi^2) + i\tfrac{1}{2}\overline{\psi}\gamma^\mu\overleftrightarrow{\partial}_\mu\psi - m\overline{\psi}\psi\,,$$

and

$$W_I = -\int\int\int d^4x_1 d^4x_2 d^4x_3 F(x_1, x_2, x_3) \times g\bar{\psi}(x_1) i\gamma_5 \phi(x_2)\psi(x_3) .$$

ψ represents the proton and ϕ the neutral pion. (For simplicity we do not consider the full isospin-invariant interaction. γ conventions are in accordance with Ref. 12.)

The equations of motion are

$$\begin{aligned} (\Box + \mu^2)\phi(x) &= -g\int\int d^4x_1 d^4x_3 F(x_1, x, x_3) \times \bar{\psi}(x_1) i\gamma_5 \psi(x_3), \\ (i\not\partial - m)\psi(x) &= g\int\int d^4x_2 d^4x_3 F(x, x_2, x_3) \times i\gamma_5 \phi(x_2)\psi(x_3), \\ -i\partial_\mu \bar{\psi}(x)\gamma^\mu - m\bar{\psi}(x) &= g\int\int d^4x_1 d^4x_2 F(x_1, x_2, x) \times \bar{\psi}(x_1) i\gamma_5 \phi(x_2) . \end{aligned} \tag{2.35}$$

The Hermiticity condition on W implies that

$$F^*(x_1, x_2, x_3) = F(x_3, x_2, x_1) . \tag{2.36}$$

By use of the method described in Sec. I, one may derive Lagrangian densities out of W whose equations of motion are (2.35). A particular (in fact the simplest) choice is [cf. (2.48)]

$$\mathcal{L}(x) = \mathcal{L}_0(x) - g\int\int d^4\xi d^4\eta F(\xi, \eta) \times \bar{\psi}(x+\eta) i\gamma_5 \phi(x)\psi(x+\xi), \tag{2.37}$$

where $\xi = x_3 - x_2$ and $\eta = x_1 - x_2$, $x = x_2$.

Lorentz invariance requires

$$F(\xi, \eta) = f(\xi^2, (\xi\eta), \eta^2) . \tag{2.38}$$

Hermiticity requires furthermore [cf. (2.36)]

$$F^*(\xi, \eta) = F(\eta, \xi) . \tag{2.39}$$

The Lagrangian (2.37) gives the equations of motion

$$\begin{aligned} (\Box + \mu^2)\phi(x) &= -g\int\int d^4\xi d^4\eta F(\xi, \eta) \times \bar{\psi}(x+\eta) i\gamma_5 \psi(x+\xi), \\ (i\not\partial - m)\psi(x) &= g\int\int d^4\xi d^4\eta F(\xi, \eta) \times i\gamma_5 \phi(x-\eta)\psi(x-\eta+\xi), \\ -i\partial_\mu \bar{\psi}(x)\gamma^\mu - m\bar{\psi}(x) &= g\int\int d^4\xi d^4\eta F(\xi, \eta) \times \bar{\psi}(x-\xi+\eta) i\gamma_5 \phi(x-\xi) . \end{aligned} \tag{2.40}$$

By a change of integration variables one may easily check that (2.40) is equivalent to (2.35).

Introduce now the action functional with boundaries, $W_{21} = \int_{\sigma_1}^{\sigma_2} d^4x\, \mathcal{L}(x)$. A functional variation δ_0 of W_{21} gives the equations of motion (2.40) and the generator [cf. (2.19)]

$$F[\sigma] = -\int_\sigma d\sigma_\mu \Big\{ \Pi^\mu_\alpha \delta_0 \phi^\alpha - g\int\int d^4\xi d^4\eta F(\xi, \eta) \times \Big[\eta^\mu \int d^4x' D(x-x', \tfrac{1}{2}\eta) \delta_0 \bar{\psi}(x'+\tfrac{1}{2}\eta) i\gamma_5 \phi(x'-\tfrac{1}{2}\eta)\psi(x'-\tfrac{1}{2}\eta+\xi) + \xi^\mu \int d^4x' D(x-x', \tfrac{1}{2}\xi) \bar{\psi}(x'-\tfrac{1}{2}\xi+\eta) i\gamma_5 \phi(x'-\tfrac{1}{2}\xi) \delta_0 \psi(x'+\tfrac{1}{2}\xi) \Big] \Big\} , \tag{2.41}$$

where

$$\Pi^\mu_\alpha \delta_0 \phi^\alpha \equiv \tfrac{1}{2}(\partial^\mu \phi \delta_0 \phi + \delta_0 \phi \partial^\mu \phi + i\bar{\psi}\gamma^\mu \delta_0 \psi - i\delta_0 \bar{\psi}\gamma^\mu \psi) .$$

(Even if in this section we only consider classical fields, we use the symmetrized expressions in the hope that they will hold in the quantized case as well. Unfortunately, this hope will not be fulfilled in Sec. IV.)

As an explicit example, one may take infinitesimal gauge transformations of the first kind:

$$\delta_0 \psi(x) = i\lambda\psi(x), \quad \delta_0 \bar{\psi}(x) = -i\lambda\bar{\psi}(x), \quad \delta_0 \phi(x) = 0$$

and $F[\sigma] = \lambda Q[\sigma]$, where Q is the total charge and λ an infinitesimal real parameter. One finds

$$Q[\sigma] = \int_\sigma d\sigma_\mu j^\mu(x), \tag{2.42}$$

where $j^\mu(x)$ is the electromagnetic current given by

$$j^\mu(x) = \bar{\psi}(x)\gamma^\mu \psi(x) + g\int\int d^4\xi d^4\eta F(\xi, \eta) \int d^4x' [\eta^\mu D(x-x', \tfrac{1}{2}\eta) \bar{\psi}(x'+\tfrac{1}{2}\eta)\gamma_5 \phi(x'-\tfrac{1}{2}\eta)\psi(x'-\tfrac{1}{2}\eta+\xi) - \xi^\mu D(x-x', \tfrac{1}{2}\eta) \bar{\psi}(x'-\tfrac{1}{2}\xi+\eta)\gamma_5 \phi(x'-\tfrac{1}{2}\xi)\psi(x'+\tfrac{1}{2}\xi)] . \tag{2.43}$$

As W is Hermitian, Q is a conserved quantity, which means that $Q[\sigma]$ in (2.42) is independent of the spacelike surface σ. This implies that $\partial_\mu j^\mu(x)=0$, which one may easily check by an explicit calculation.

A coordinate variation of W_{21} induced by the infinitesimal Poincaré transformation $x_\mu \to x_\mu+\delta x_\mu$, where $\delta x_\mu=\epsilon_\mu+\epsilon_{\mu\nu}x^\nu$, gives the energy-momentum and angular momentum tensors. Using the formulas (2.4), etc., one finds the canonical energy-momentum tensor to be

$$T_c^{\mu\nu}=-\mathcal{L}g^{\mu\nu}+\tfrac{1}{2}(\partial^\mu\phi\partial^\nu\phi+\partial^\nu\phi\partial^\mu\phi+i\bar{\psi}\gamma^\mu\overleftrightarrow{\partial}^\nu\psi)$$
$$-g\int\int d^4\xi d^4\eta\, F(\xi,\eta)\int d^4x'[\eta^\mu D(x-x',\tfrac{1}{2}\eta)\partial'^\nu\bar{\psi}(x'+\tfrac{1}{2}\eta)i\gamma_5\phi(x'-\tfrac{1}{2}\eta)\psi(x'-\tfrac{1}{2}\eta+\xi)$$
$$+\xi^\mu D(x-x',\tfrac{1}{2}\xi)\bar{\psi}(x'-\tfrac{1}{2}\xi+\eta)i\gamma_5\phi(x'-\tfrac{1}{2}\xi)\partial'^\nu\psi(x'+\tfrac{1}{2}\xi)]\ . \tag{2.44}$$

The canonical angular momentum tensor is given in turn by [see (2.9)]

$$M_c^{\mu\nu\rho}=x^\rho T_c^{\mu\nu}-x^\nu T_c^{\mu\rho}-A^{\mu\nu\rho}, \tag{2.45}$$

where

$$A^{\mu\nu\rho}=i\tfrac{1}{2}\bar{\psi}\{\gamma^\mu,\Sigma^{\nu\rho}\}\psi+g\int\int d^4\xi d^4\eta\, F(\xi,\eta)\int d^4x'[\eta^\mu D(x-x',\tfrac{1}{2}\eta)\bar{\psi}(x'+\tfrac{1}{2}\eta)i\gamma_5\phi(x'-\tfrac{1}{2}\eta)\Sigma^{\nu\rho}\psi(x'-\tfrac{1}{2}\eta+\xi)$$
$$-\xi^\mu D(x-x',\tfrac{1}{2}\xi)\bar{\psi}(x'-\tfrac{1}{2}\xi+\eta)i\gamma_5\phi(x'-\tfrac{1}{2}\xi)\Sigma^{\nu\rho}\psi(x'+\tfrac{1}{2}\xi)]$$
$$+g\tfrac{1}{2}\int\int d^4\xi d^4\eta\, F(\xi,\eta)\int d^4x'\{\xi^\mu\xi^\rho[\bar{D}(x-x',\tfrac{1}{2}\xi)\bar{\psi}(x'-\tfrac{1}{2}\xi+\eta)i\gamma_5\phi(x'-\tfrac{1}{2}\xi)\partial'^\nu\psi(x'+\tfrac{1}{2}\xi)$$
$$+D(x-x',\tfrac{1}{4}\xi)\bar{\psi}(x'-\tfrac{1}{4}\xi+\eta)i\gamma_5\phi(x'-\tfrac{1}{4}\xi)\partial'^\nu\psi(x'+\tfrac{3}{4}\xi)]-(\rho\leftrightarrow\nu)$$
$$+\eta^\mu\eta^\rho[\bar{D}(x-x',\tfrac{1}{2}\eta)\partial'^\nu\bar{\psi}(x'+\tfrac{1}{2}\eta)i\gamma_5\phi(x'-\tfrac{1}{2}\eta)\psi(x'-\tfrac{1}{2}\eta+\xi)$$
$$+D(x-x',\tfrac{1}{4}\eta)\partial'^\nu\bar{\psi}(x'+\tfrac{3}{4}\eta)i\gamma_5\,\phi(x'-\tfrac{1}{4}\eta+\xi)]-(\rho\leftrightarrow\nu)\}\ . \tag{2.46}$$

As W is Poincaré-invariant we have that $\partial_\mu T_c^{\mu\nu}=0$ and $\partial_\mu M_c^{\mu\nu\rho}=0$, which may be checked by an explicit calculation. Note that if one calculates $\partial_\mu M_c^{\mu\nu\rho}$ one has to make use of (2.38) and the assumption that $F(\xi,\eta)$ vanishes with sufficient rapidity as $\xi_\mu\to\infty$ and $\eta_\mu\to\infty$ so that $\int d^4\xi\, \partial/\partial\xi^\mu[F(\xi,\eta)\cdots]=0$, etc., in order to get that $\partial_\mu M_c^{\mu\nu\rho}=0$.

One may also calculate the corresponding Belinfante tensors, $T_B^{\mu\nu}$ and $M_B^{\mu\nu\rho}$, by use of (2.10), (2.12), and (2.13). They also fulfill $\partial_\mu T_B^{\mu\nu}=0$ and $\partial_\mu M_B^{\mu\nu\rho}=0$.

If one lets $F(\xi,\eta)\to\delta^4(\xi)\delta^4(\eta)$, one recovers the local Lagrangian

$$\mathcal{L}(x)=\mathcal{L}_0(x)-g\bar{\psi}(x)i\gamma_5\phi(x)\psi(x)\ . \tag{2.47}$$

Furthermore, the corresponding conserved quantities as, e.g., $j^\mu(x)=\bar{\psi}(x)\gamma^\mu\psi(x)$ are identical to the ones one would get if one applied the action principle directly to (2.47). Note that this property holds for general nonlocal field theories of the type considered in this paper.

Out of the action (2.34) we may choose other Lagrangian densities than (2.37). From (1.11) we have that the general Lagrangian density for the Kristensen-Møller model may be written

$$\mathcal{L}_{\alpha,\beta}(x)=\mathcal{L}_0(x)-g\int\int d^4\xi d^4\eta\, F(\xi,\eta)\bar{\psi}(x+\eta+\rho)i\gamma_5\phi(x+\rho)\psi(x+\xi+\rho), \tag{2.48}$$

where $-\rho\equiv\alpha\xi+\beta\eta$ and α,β are arbitrary real constants. (Hermiticity requires $\alpha=\beta$.) Instead of the generator (2.41), we get

$$F_{\alpha,\beta}[\sigma]=-\int_\sigma d\sigma_\mu\Big\{\Pi_\alpha^\mu\delta_0\phi^\alpha-g\int\int d^4\xi d^4\eta$$
$$\times F(\xi,\eta)\Big[(\eta^\mu+\rho^\mu)\int d^4x' D(x-x',\tfrac{1}{2}(\eta+\rho))$$
$$\times\delta_0\bar{\psi}(x'+\tfrac{1}{2}\eta+\tfrac{1}{2}\rho)i\gamma_5\phi(x'-\tfrac{1}{2}\eta+\tfrac{1}{2}\rho)\psi(x'-\tfrac{1}{2}\eta+\tfrac{1}{2}\rho+\xi)$$
$$+(\xi^\mu+\rho^\mu)\int d^4x' D(x-x',\tfrac{1}{2}(\xi+\rho))\bar{\psi}(x'-\tfrac{1}{2}\xi+\tfrac{1}{2}\rho+\eta)$$
$$\times i\gamma_5\phi(x'-\tfrac{1}{2}\xi+\tfrac{1}{2}\rho)\delta_0\psi(x'+\tfrac{1}{2}\xi+\tfrac{1}{2}\rho)$$
$$+\rho^\mu\int d^4x' D(x-x',\tfrac{1}{2}\rho)\bar{\psi}(x'+\eta+\tfrac{1}{2}\rho)i\gamma_5\delta_0\phi(x'+\tfrac{1}{2}\rho)\psi(x'+\xi+\tfrac{1}{2}\rho)\Big]\Big\}. \tag{2.49}$$

$F_{\alpha,\beta}[\sigma]$ is all we need in order to write down the conserved quantities considered in this paper. The electromagnetic current density is, e.g., given by [cf. (2.43)]

$$j^{\mu}_{\alpha,\beta}(x)=\bar{\psi}(x)\gamma^{\mu}\psi(x)+g\int\int d^4\xi\, d^4\eta\, F(\xi,\eta)\int d^4x'[(\eta^{\mu}+\rho^{\mu})D(x-x',\tfrac{1}{2}(\eta+\rho))\bar{\psi}(x'+\tfrac{1}{2}\eta+\tfrac{1}{2}\rho)\gamma_5$$
$$\times\phi(x'-\tfrac{1}{2}\eta+\tfrac{1}{2}\rho)\bar{\psi}(x'+\xi-\tfrac{1}{2}\eta+\tfrac{1}{2}\rho)$$
$$-(\xi^{\mu}+\rho^{\mu})D(x-x',\tfrac{1}{2}(\xi+\rho))\bar{\psi}(x'+\eta-\tfrac{1}{2}\xi+\tfrac{1}{2}\rho)\gamma_5$$
$$\times\phi(x'-\tfrac{1}{2}\xi+\tfrac{1}{2}\rho)\psi(x'+\tfrac{1}{2}\xi+\tfrac{1}{2}\rho)]\ . \tag{2.50}$$

The canonical energy-momentum tensor is, in turn, given by

$$T^{\mu\nu}_{\alpha,\beta}(x)=-\mathcal{L}_{\alpha,\beta}(x)g^{\mu\nu}+\tfrac{1}{2}(\partial^{\mu}\phi\,\partial^{\nu}\phi+\partial^{\nu}\phi\,\partial^{\mu}\phi+i\bar{\psi}\gamma^{\mu}\overleftrightarrow{\partial}^{\nu}\psi)$$
$$-g\int\int d^4\xi\, d^4\eta\, F(\xi,\eta)\int d^4x'[(\eta^{\mu}+\rho^{\mu})D(x-x',\tfrac{1}{2}(\eta+\rho))\partial'^{\nu}\bar{\psi}(x'+\tfrac{1}{2}\eta+\tfrac{1}{2}\rho)i\gamma_5\phi(x'-\tfrac{1}{2}\eta+\tfrac{1}{2}\rho)\psi(x'-\tfrac{1}{2}\eta+\xi+\tfrac{1}{2}\rho)$$
$$+(\xi^{\mu}+\rho^{\mu})D(x-x',\tfrac{1}{2}(\xi+\rho))\bar{\psi}(x'+\eta-\tfrac{1}{2}\xi+\tfrac{1}{2}\rho)i\gamma_5\phi(x'-\tfrac{1}{2}\xi+\tfrac{1}{2}\rho)\,\partial'^{\nu}\psi(x'+\tfrac{1}{2}\xi+\tfrac{1}{2}\rho)$$
$$+\rho^{\mu}D(x-x',\tfrac{1}{2}\rho)\bar{\psi}(x'+\eta+\tfrac{1}{2}\rho)\,i\gamma_5\partial'^{\nu}\phi(x'+\tfrac{1}{2}\rho)\psi(x'+\xi+\tfrac{1}{2}\rho)]\ . \tag{2.51}$$

From these expressions one sees how the general nonuniqueness of conserved densities looks explicitly. We shall now investigate the corresponding properties of the integral conserved quantities.

The total charge is given by

$$Q(t)=\int d^4x\,\delta(t-x_0)\,j^0_{\alpha,\beta}(x)$$
$$=\int d^3x\,\bar{\psi}\gamma^0\psi+g\int\int d^4\xi\, d^4\eta\, F(\xi,\eta)\int d^4x'\int d^4x\,\delta(t-x_0)[(\eta^0+\rho^0)D(x-x',\tfrac{1}{2}(\eta+\rho))(\cdots)+\cdots]$$
$$=\int d^3x\,\bar{\psi}\gamma^0\psi-g\int\int d^4\xi\, d^4\eta\, F(\xi,\eta)\int d^4x\,\theta(t-x_0)[\bar{\psi}(x)\gamma_5\phi(x-\eta)\psi(x+\xi-\eta)-\bar{\psi}(x+\eta-\xi)\gamma_5\phi(x-\xi)\psi(x)]\,, \tag{2.52}$$

where use has been made of

$$g\int\int d^4\xi\, d^4\eta\, F(\xi,\eta)\int d^4x'\int d^4x\,\delta(t-x_0)(\eta^0+\rho^0)D(x-x',\tfrac{1}{2}(\eta+\rho))\cdots$$
$$=g\int\int d^4\xi\, d^4\eta\, F(\xi,\eta)\int d^4x'[\theta(t-x_0)(\eta^{\mu}+\rho^{\mu})\partial_{\mu}D(x-x',\tfrac{1}{2}(\eta+\rho))\cdots$$
$$-\partial_{\mu}\{\theta(t-x_0)(\eta^{\mu}+\rho^{\mu})D(x-x',\tfrac{1}{2}(\eta+\rho))\cdots\}]\,, \tag{2.53}$$

where the last term vanishes as $F(\xi,\eta)$ vanishes when $\xi^{\mu}\to\infty$ or $\eta^{\mu}\to\infty$. The energy-momentum vector is given by

$$P^{\nu}(t)=\int d^4x\,\delta(t-x^0)T^{0\nu}_{\alpha,\beta}(x)$$
$$=\int d^3x[-\mathcal{L}_0(x)g^{0\nu}+\tfrac{1}{2}(\partial^0\phi\,\partial^{\nu}\phi+\partial^{\nu}\phi\,\partial^0\phi+i\bar{\psi}\gamma^0\overleftrightarrow{\partial}^{\nu}\psi)]$$
$$+g\int\int d^4\xi\, d^4\eta\, F(\xi,\eta)\int d^4x\,\theta(t-x_0)[\partial^{\nu}\bar{\psi}(x)i\gamma_5\phi(x-\eta)\psi(x-\eta+\xi)+\bar{\psi}(x+\eta)i\gamma_5\partial^{\nu}\phi(x)\psi(x+\xi)$$
$$+\bar{\psi}(x+\eta-\xi)i\gamma_5\phi(x-\xi)\partial^{\nu}\psi(x)]\,, \tag{2.54}$$

where we also have made use of (2.53).

Note that in (2.53) one may replace the step function on the left-hand side by $\theta(t-x_0)+C$, where C is an arbitrary constant. This implies that one may get additional terms in the expressions for the integral conserved quantities. Such a term may look like [in (2.52)]

$$Cg\int\int d^4\xi d^4\eta F(\xi,\eta)\int d^4x[\bar\psi(x)\gamma_5\phi(x-\eta)\psi(x+\xi-\eta) -\bar\psi(x+\eta-\xi)\gamma_5\phi(x-\xi)\psi(x)]. \tag{2.55}$$

However, a change of integration variables shows that (2.55) vanishes. One realizes easily that this is the case in general, as all additional terms coming from the nonuniqueness in (2.53) will only contain differences between terms, where the integration variables are shifted relative to each other due to the property (2.22).

As the expressions (2.52) and (2.54) for the charge and energy-momentum vector do not depend on the parameters α and β, we get the result that they are independent of which Lagrangian density we have chosen to start with. Taking account of the arguments above, we finally reach the conclusion that (2.52) and (2.54) are uniquely determined by the action functional (2.34) or, which is equivalent, they are uniquely determined by the equations of motion. (2.52) and (2.54) should therefore be equal to the expressions earlier derived by Pauli.[7] This is also the case.

This result should hold for any integral conserved quantity in the considered model as well as in other similar models.

III. FURTHER NONUNIQUENESS AND CONSEQUENCES FOR LOCAL FIELD THEORIES

For the Kristensen-Møller model we have that the interaction part of the action integrand looks like (1.2) with $N=p=3$. In this section we shall look into theories with $N>p$. This case will be shown to have interesting consequences both for nonlocal and local theories.

By simply having the expression (1.1) for the action functional in memory, one may without proof state the following theorem: For any action integrand of the type (1.2) where $N>p$ and the form functions are integrable, one can define a new action integrand with $N=p$ whose action gives the same equations of motion as the original one. The new form function is thereby given by the following expression of the old form function:

$$E'_{\alpha_1\cdots\alpha_M}(x_1,\ldots,x_p)=\int\cdots\int d^4x_{p+1}\cdots d^4x_N \times E_{\alpha_1\cdots\alpha_M}(x_1,\ldots,x_N)$$

Note, however, that from the point of view of conserved quantities there is no equivalence between the new and old action integrands. The old one gives rise to a larger class of possible Lagrangian densities than the new one, as (1.11) then contains $N-p$ more parameters.

One now realizes with a bit of uneasiness that to a given action integrand there corresponds many more Lagrangian densities than we thought of in Sec. I. Thus, the nonuniqueness of the Lagrangian has become worse and is now represented by an infinite number of parameters in (1.11).

One notes also that this holds for local field theories as well. Local theories are here represented by the case $p=1$ in (1.2). Consider, e.g., a local Lagrangian density $\mathcal{L}(x)$ like (2.47). Furthermore let $f(\xi)$ be an arbitrary (Lorentz scalar), integrable function of the four-vector ξ^μ, normalized so that $\int d^4\xi f(\xi)=1$. Then one may define a new nonlocal Lagrangian out of $\mathcal{L}(x)$ by

$$\mathcal{L}'(x)=\int d^4\xi f(\xi)\mathcal{L}(x+\xi). \tag{3.1}$$

$$W\equiv\int d^4x\,\mathcal{L}'(x)=\int d^4x\,\mathcal{L}(x)$$

by a change of variables. This means that $\mathcal{L}'(x)$ gives the same equations of motion as $\mathcal{L}(x)$. $\mathcal{L}(x)$ represents the case $N=p=1$ in (1.2) while $\mathcal{L}'(x)$ represents $N=2$, $p=1$. One may, of course, write down other Lagrangians representing $N=3$, $p=1$ like

$$\mathcal{L}''(x)=\int\int d^4\xi d^4\eta f(\xi,\eta)\mathcal{L}(x+\xi+\eta),\quad \text{etc.}$$

In general we have [$\mathcal{L}(x)$ may be local or not]

$$\mathcal{L}'_{a_1,\ldots,a_n}(x)=\int\cdots\int d^4\xi_1\cdots d^4\xi_n f(\xi_1,\ldots,\xi_n) \times\mathcal{L}(x+\rho), \tag{3.2}$$

where $-\rho\equiv\sum_{i=0}^n a_i\xi_i$ and a_i are arbitrary parameters and n is an arbitrary number. However, one may not introduce different form functions in the different parts of $\mathcal{L}(x)$.

Let us consider a particular model in order to get an idea of what may happen. Take, e.g., the Lagrangian

$$\mathcal{L}(x)=\int d^4\xi f(\xi)[\partial_\mu\phi^*(x+\xi)\partial^\mu\phi(x+\xi) -m^2\phi^*(x+\xi)\phi(x+\xi)], \tag{3.3}$$

where $f(\xi)$ fulfills $\int d^4\xi f(\xi)=1$. In Appendix B we have extended the application of the action principle to nonlocal Lagrangians containing derivatives of the field operators. By use of (B2) and (2.15) we find that (3.3) gives the equations of motion

$$(\Box+m^2)\phi(x)=0,\qquad(\Box+m^2)\phi^*(x)=0. \tag{3.4}$$

Thus, $\phi(x)$ represents a charged free boson.

By putting $\delta_0\phi(x)=i\lambda\phi(x)$ and $\delta_0\phi(x)=-i\lambda\phi(x)$ (λ is a real, infinitesimal constant) into the formulas (2.19) and (B5) one finds the conserved electromagnetic current density to be

$$j^\mu(x)=i\int d^4\xi f(\xi)\int d^4x' D(x-x',\tfrac{1}{2}\xi)\times\phi^*(x'+\tfrac{1}{2}\xi)\overleftrightarrow{\partial}'^\mu\phi(x'+\tfrac{1}{2}\xi)\,. \tag{3.5}$$

All the other terms in (B5) vanish after use of (3.4).

The canonical energy-momentum tensor is in turn found by putting $\delta_\rho\phi^*(x)=-\epsilon^\mu\partial_\mu\phi^*(x)$ into (2.10) and (B5), and by making use of (2.8):

$$\begin{aligned}T_c^{\mu\nu}(x)=-\mathcal{L}(x)g^{\mu\nu}+\int d^4\xi f(\xi)\int d^4x'\{&D(x-x',\tfrac{1}{2}\xi)[\partial'^\mu\phi^*(x'+\tfrac{1}{2}\xi)\partial'^\nu\phi(x'+\tfrac{1}{2}\xi)+(\mu\leftrightarrow\nu)]\\&+\tfrac{1}{2}\xi^\mu D(x-x',\tfrac{1}{4}\xi)\partial'^\nu[\partial'^\rho\phi^*(x'+\tfrac{3}{4}\xi)\partial'_\rho\phi(x'+\tfrac{3}{4}\xi)+m^2\phi^*(x'+\tfrac{1}{4}\xi)\phi(x'+\tfrac{1}{4}\xi)]\\&-\xi^\mu D(x-x',\tfrac{1}{2}\xi)m^2\partial'^\nu[\phi^*(x'+\tfrac{1}{2}\xi)\phi(x'+\tfrac{1}{2}\xi)]\\&+\tfrac{1}{2}\xi^\mu\overline{D}(x-x',\tfrac{1}{2}\xi)\partial'_\rho[\partial'^\rho\phi^*(x'+\tfrac{1}{2}\xi)\partial'^\nu\phi(x'+\tfrac{1}{2}\xi)+(\rho\leftrightarrow\nu)]\}\,.\end{aligned} \tag{3.6}$$

The charge may formally be expressed in the following form:

$$Q(t)=\int d^3x\, j_I^0(x)+\sum_{n=1}^{\infty}a_{\nu_1\cdots\nu_n}\int d^3x\,\partial^{\nu_1}\cdots\partial^{\nu_n}j_I^0(x), \tag{3.7}$$

where $j_I^0(x)=i\phi^*(x)\overleftrightarrow{\partial}^0\phi(x)$. Since we formally have that

$$\int d^3x\partial^{\nu_1}\cdots\partial^{\nu_n}j_I^0(x)=\begin{cases}\int d^3x\,\partial^{\nu_1}\cdots\partial^{\nu_n-1}\partial^0 j_I^0(x)=\int d^3x\,\partial^{\nu_1}\cdots\partial^{\nu_n-1}\partial_\mu j_I^\mu(x)=0 & \text{if } \nu_n=0\,,\\ \int d^3x\,\partial^i\partial^{\nu_1}\cdots\partial^{\nu_n-1}j_I^0(x)=0 & \text{if } \nu_n=i=1,2,3\,,\end{cases} \tag{3.8}$$

we get

$$Q(t)=\int d^3x\, j_I^0(x). \tag{3.9}$$

Thus, the current density (3.5) gives the usual charge operator.

By use of (2.53) one may show that the energy-momentum vector $P^\nu=\int d^3x\,T_c^{0\nu}$ is also given by an expression of the form (3.7), which means that the energy-momentum tensor (3.6) gives the usual energy-momentum vector.

It seems, thus, as if the extra nonuniqueness introduced in this section as well as the non-uniqueness in Secs. I and II does not affect the integral conserved quantities.

IV. QUANTIZATION

Quantization of nonlocal theories is of course a problem in itself. In spite of the fact that we have generalized with advantage the variation method for higher-order Lagrangians to infinite order, one cannot generalize the quantization method of higher-order Lagrangians (see Appendix A) to infinite order, because in that limit this quantization method implies that all quantities commute.

There are several ways to approach this problem of quantization, but we shall not discuss them here. Instead we shall make use of the only quantization method which has been considered so far, namely the Yang-Feldman method.[9] All field theories with nonlocal interaction like the Kristensen-Møller model may be quantized by this method.

We shall here quantize just the Kristensen-Møller model previously considered in Sec. II B. This quantization will be shown to have an unexpected consequence; the energy-momentum tensor (2.44) which was conserved for classical fields will no longer be conserved. Yet it will be shown that integral conserved quantities are still derivable from the action principle.

Our starting equations are

$$\phi(x)=\phi_{\text{in}}(x)-g\int d^4y\,\Delta_R(x-y)\rho(y), \tag{4.1}$$

$$\psi(x)=\psi_{\text{in}}(x)+g\int d^4y\,S_R(x-y)f(y), \tag{4.2}$$

$$\overline{\psi}(x)=\overline{\psi}_{\text{in}}(x)+g\int d^4y\,\overline{f}(y)S_A(y-x), \tag{4.3}$$

$$\rho(y)=\int\int d^4\xi d^4\eta\,F(\xi,\eta)\overline{\psi}(y+\eta)i\gamma_5\psi(y+\xi), \tag{4.4}$$

$$f(y)=\int\int d^4\xi d^4\eta\,F(\xi,\eta)i\gamma_5\phi(y-\eta)\psi(y-\eta+\xi), \tag{4.5}$$

$$\overline{f}(y)=\int\int d^4\xi d^4\eta\,F(\xi,\eta)\overline{\psi}(y-\xi+\eta)i\gamma_5\phi(y-\xi), \tag{4.5'}$$

and the convention for the invariant functions is in accordance with Ref. 12. (Note that in an actual application we have to deal with the normal ordered expressions. However, here we avoid the properly symmetrized Lagrangian [see (4.49)] for simplicity.)

We require now the following free-field commutation relations:

$$[\phi_{\rm in}(x'),\phi_{\rm in}(x)]=i\Delta(x'-x)\,, \tag{4.6}$$

$$\{\psi_\alpha^{\rm in}(x'),\bar\psi_\beta^{\rm in}(x)\}=-iS_{\alpha\beta}(x'-x)\,, \tag{4.7}$$

$$\{\psi_\alpha^{\rm in}(x'),\psi_\beta^{\rm in}(x)\}=\{\bar\psi_\alpha^{\rm in}(x'),\bar\psi_\beta^{\rm in}(x)\}=0\,. \tag{4.8}$$

The solutions of the interpolating fields (3.1), (3.2), and (3.3) may be written in terms of power expansions in the coupling constant g:

$$\phi(x)=\phi_{\rm in}(x)+\sum_{n=1}^{\infty}g^n\phi_n(x)\,, \tag{4.9}$$

$$\psi(x)=\psi_{\rm in}(x)+\sum_{n=1}^{\infty}g^n\psi_n(x)\,. \tag{4.10}$$

The commutation relations of the fields may now be calculated. Up to first order in g, we get

$$[\phi(x'),\phi(x)]=i\Delta(x'-x)+O(g^2)\,, \tag{4.11}$$

$$[\phi(x'),\psi(x)]=ig\int d^4y\int\int d^4\xi\, d^4\eta\, F(\xi,\eta)[\theta(x_0-y_0)\Delta(y-x'-\eta)S(x-y)i\gamma_5\psi(y-\eta+\xi)$$
$$-\theta(x_0'-y_0)\Delta(y-x')S(x-y-\eta)i\gamma_5\psi(y+\xi)]+O(g^2)\,, \tag{4.12}$$

$$\{\psi(x'),\bar\psi(x)\}=-iS(x'-x)+ig\int d^4y\int\int d^4\xi\, d^4\eta\, F(\xi,\eta)\{\theta(x_0'-y_0)\phi(y-\eta)S(x'-y)i\gamma_5S(y-x+\xi-\eta)$$
$$-\theta(x_0-y_0)\phi(y-\xi)S(x'-y-\eta+\xi)i\gamma_5S(y-x)\}+O(g^2)\,, \tag{4.13}$$

$$[\bar\psi(x'),\phi(x)]=ig\int d^4y\int\int d^4\xi\, d^4\eta\, F(\xi,\eta)\{\theta(x_0'-y_0)\Delta(y-x-\xi)\bar\psi(y-\xi+\eta)i\gamma_5S(y-x')$$
$$-\theta(x_0-y_0)\Delta(y-x)\bar\psi(y+\eta)i\gamma_5S(y-x'+\xi)\}+O(g^2)\,. \tag{4.14}$$

From (4.12) and (4.14) it follows that ϕ and ψ in general do not commute, which means that they are no longer canonical variables. The question of whether it is possible to define new fields which can be canonically quantized was dwelled upon by Pauli.[7] He suggested that one should replace $\psi(x)$ by the field

$$\mathfrak{u}(x)\equiv\psi(x)+g\int\int d^4\xi\, d^4\eta\, F(\xi,\eta)\int d^4x'\theta(x_0-x_0')[S(x-x')i\gamma_5\phi(x'-\eta)\psi(x'-\eta+\xi)-S(x-x'-\eta)i\gamma_5\phi(x')\psi(x'+\xi)]\,, \tag{4.15}$$

which together with $\phi(x)$ then should form the canonical variables. However, this is true only in the first-order approximation in the coupling constant g. Equation (4.11), e.g., gives a second-order contribution, which one may easily check. $\phi(x)$ and $\psi(x)$ seem, though, to become canonical variables if one chooses a form function of the following noncovariant form:

$$F(\xi,\eta)=\tilde F(\xi,\eta)\delta(\xi_0)\delta(\eta_0)\,, \tag{4.16}$$

which means that we make all expressions local in time. Such a manifestly noncovariant form function does perhaps not necessarily lead to a S matrix which violates Lorentz invariance. However, we shall not consider this possibility any further here.

We have called $F[\sigma]$ in the action principle (1.13) the infinitesimal generator of the variation in question. But in what sense $F[\sigma]$ actually has this property is not clear beforehand, especially as we no longer have any canonical theory. However, it seems natural to impose the condition that any type of quantization must be such that $F[\sigma]$ gets the right generator property. Thus, if we quantize the fields by use of the Yang-Feldman method, which is a noncanonical quantization method, we have to check if this method is consistent with the expected generator property of $F[\sigma]$. The charge operator, e.g., has to fulfill

$$\begin{aligned}&[Q(t),\bar\psi(\vec{x},t)]=\bar\psi(\vec{x},t)\,,\\ &[Q(t),\psi(\vec{x},t)]=-\psi(\vec{x},t)\,,\\ &[Q(t),\phi(\vec{x},t)]=0\,,\end{aligned} \tag{4.17}$$

where $Q(t)$ is the total charge given by (2.52). But in order to show what the equal-time commutators between the current density and the fields look like, we represent the charge operator by

$$Q(t)=\int d^3x'\,j^0(\vec{x}',t)\,, \tag{4.18}$$

where we in turn let $j^\mu(x')$ be given by (2.43).

Now we get in first order in g

$$[Q(t),\bar\psi(\vec{x},t)]=\int d^3x'[j^0(\vec{x}',t),\bar\psi(\vec{x},t)]\,, \tag{4.19}$$

where the equal-time commutator is

$$[j^0(x'), \bar\psi(x)]_{\text{ET}} = \bar\psi_{\text{in}}(\vec{x}\,', t)\delta^3(\vec{x}\,'-\vec{x})$$
$$- ig\int d^4y \int\int d^4\xi\, d^4\eta\, F(\xi,\eta)[\bar\psi(x')\gamma^0 S_R(x'-y) i\gamma_5\phi(y-\eta)S(y-x-\eta+\xi)$$
$$+\bar\psi(y-\xi+\eta)i\gamma_5\phi(y-\xi)S_A(y-x')\gamma^0 S(x'-x)$$
$$+\bar\psi(x')\gamma^0 S(x'-y+\xi-\eta)i\gamma_5\phi(y-\xi)S_A(y-x)$$
$$+\eta^0 D(x'-y,\tfrac12\eta)\bar\psi(y+\tfrac12\eta)\gamma_5\phi(y-\tfrac12\eta)S(y-x-\tfrac12\eta+\xi)$$
$$-\xi^0 D(x'-y,\tfrac12\xi)\bar\psi(y-\tfrac12\xi+\eta)\gamma_5\phi(y-\tfrac12\xi)S(y-x+\tfrac12\xi)]|_{x_0'=x_0}$$
$$+O(g^2)\,.$$

From (4.3) and (4.5) we get the requirement

$$[Q(t), \bar\psi(\vec{x},t)] = \bar\psi_{\text{in}}(\vec{x},t) + g\int d^4y\int\int d^4\xi\, d^4\eta\, F(\xi,\eta)\bar\psi(y-\xi+\eta)i\gamma_5\phi(y-\xi)S_A(y-x)|_{x_0=t} + O(g^2)\,. \tag{4.20}$$

Identification yields ($x_0'=x_0=t$)

$$\int d^3x'\int d^4y\int\int d^4\xi\, d^4\eta\, F(\xi,\eta)[\bar\psi_{\text{in}}(x')\gamma^0\theta(t-y_0)S(x'-y)i\gamma_5 S(y-x-\eta+\xi)\phi_{\text{in}}(y-\eta)$$
$$-\bar\psi_{\text{in}}(x')\gamma^0\theta(t-y_0)S(x'-y+\xi-\eta)i\gamma_5 S(y-x)\phi_{\text{in}}(y-\xi)$$
$$-\eta^0 D(x'-y,\tfrac12\eta)\bar\psi_{\text{in}}(y+\tfrac12\eta)\gamma_5 S(y-x-\tfrac12\eta+\xi)\phi_{\text{in}}(y-\tfrac12\eta)$$
$$+\xi^0 D(x'-y,\tfrac12\xi)\bar\psi_{\text{in}}(y-\tfrac12\xi+\eta)\gamma_5 S(y-x+\tfrac12\xi)\phi_{\text{in}}(y-\tfrac12\xi)] = 0\,. \tag{4.21}$$

Notice that this condition is satisfied in the local limit $[F(\xi,\eta)=\delta^4(\xi)\delta^4(\eta)]$, as it should be, but it is also satisfied as it stands. This may be seen in the following way. Integrate the first two terms in (4.21) over x'. This yields

$$\int d^4y\int\int d^4\xi\, d^4\eta\, F(\xi,\eta)\theta(t-y_0)$$
$$\times[\bar\psi_{\text{in}}(y-\xi+\eta)\gamma_5 S(y-x)\phi_{\text{in}}(y-\xi)$$
$$-\bar\psi_{\text{in}}(y)\gamma_5 S(y-x-\eta+\xi)\phi_{\text{in}}(y-\eta)]\,, \tag{4.22}$$

where use has been made of

$$i\int d^3x'\bar\psi_{\text{in}}(x')\gamma^0 S(x'-z) = -\bar\psi_{\text{in}}(z)\,. \tag{4.23}$$

If one then makes use of (2.53) and the property (2.22), and changes variables ($y \leftrightarrow x'$), one finds that the last two terms in (4.21) are equal to (4.22) with opposite sign.

In fact, Pauli[7] has already checked the consistency of (4.17) by use of the "canonical" field operator (4.15) in the first-order approximation. But as $\mathfrak{u}(x)$ in (4.15) is canonical only up to the first order, the consistency of (4.17) in higher orders of the perturbation expansion was left uncertain. We shall, therefore, check (4.17) beyond this first-order approximation.

The unique charge operator (2.52) may be written in terms of the in-fields by use of the perturbation expansions,

$$Q = Q_0(\text{in}) + \sum_{n=1}^{\infty} g^n Q_n(\text{in})\,. \tag{4.24}$$

Under the assumption of no bound states, we should have that

$$Q = Q_0(\text{in}),\ \text{or equivalently}\ Q_n(\text{in}) = 0\ \text{for}\ n \geq 1\,. \tag{4.25}$$

Thus (4.17) should hold even if we replace Q by $Q_0(\text{in})$.

We have checked that $Q_n(\text{in})=0$ for $n=1, 2, 3$ and that $Q_0(\text{in})$ fulfills (4.17) up to the same order of approximation. Thus (4.17) holds at least up to the third order in the coupling constant g.

We should, furthermore, have that

$$Q = Q_0(\text{in}) = Q_0(\text{out})\,. \tag{4.26}$$

This property can be checked by expanding the free out-fields in terms of the in-fields according to the following equations:

$$\phi_{\text{out}}(x) = \phi_{\text{in}}(x) + g\int d^4y\,\Delta(x-y)\rho(y)\,,$$
$$\psi_{\text{out}}(x) = \psi_{\text{in}}(x) - g\int d^4y\, S(x-y)f(y)\,, \tag{4.27}$$
$$\bar\psi_{\text{out}}(x) = \bar\psi_{\text{in}}(x) + g\int d^4y\,\bar f(y)S(y-x)\,.$$

This has been done up to the same order as (4.25). Hence, (4.26) is true at least up to the third order in g.

Turning to the energy-momentun vector P^ν, we encounter a problem briefly mentioned in the beginning of this section. Notice that the general formulas in Sec. II A were derived under the assumption of c-number variations. These formulas are thus true at least for a classical theory. In conventional local quantum field theories, such formulas are still used but with a bit of care, as one often has to put in the variations in their proper positions. Here, however, we get the remarkable result that in spite of the fact that in Sec. II B we have been very careful about where we put in the variations of the fields, the energy-momentum tensor is no longer conserved in the quantized case. One finds

$$\partial_\mu T_c^{\mu\nu}(x) = -\tfrac{1}{2}g\int\int d^4\xi\, d^4\eta\, F(\xi,\eta) \times\{\bar\psi(x+\eta)i\gamma_5[\psi(x+\xi),\partial^\nu\phi(x)] +[\partial^\nu\phi(x),\bar\psi(x+\eta)]i\gamma_5\psi(x+\xi)\}, \quad (4.28)$$

from which it is clear that it is the fact that ϕ and ψ no longer commute which causes the nonconservation of $T_c^{\mu\nu}$.

Furthermore, because of (4.28) the energy-momentum vector P^ν is neither conserved nor has the right generator property. In fact, an expansion corresponding to (4.24),

$$P^\nu = P_0^\nu(\text{in}) + \sum_{n=1}^{\infty} g^n P_n^\nu(\text{in}), \quad (4.29)$$

yields $P_1^\nu(\text{in})=0$ but $P_2^\nu(\text{in})\neq 0$.

Thus, there must be something wrong with the use of c-number variations in the quantized version of nonlocal field theories. Consequently, we have to consider the action principle with q-number variations in order to derive the conserved quantities.

Consider, therefore, the following general functional, q-number variation of the Lagrangian density (2.37):

$$\delta_0\mathcal{L}(x) = \tfrac{1}{2}(\partial_\mu\delta_0\phi\,\partial^\mu\phi + \partial_\mu\phi\,\partial^\mu\delta_0\phi - \mu^2\delta_0\phi\,\phi - \mu^2\phi\,\delta_0\phi + i\delta_0\bar\psi\gamma^\mu\overleftrightarrow{\partial}_\mu\psi + i\bar\psi\gamma^\mu\overleftrightarrow{\partial}_\mu\delta_0\psi)$$
$$- m\delta_0\bar\psi\psi - m\bar\psi\delta_0\psi - g\int\int d^4\xi\, d^4\eta\, F(\xi,\eta)[\delta_0\bar\psi(x+\eta)i\gamma_5\phi(x)\psi(x+\xi) + \bar\psi(x+\eta)i\gamma_5\delta_0\phi(x)\psi(x+\xi)$$
$$+\bar\psi(x+\eta)i\gamma_5\phi(x)\delta_0\psi(x+\xi)], \quad (4.30)$$

which may be written

$$\delta_0\mathcal{L}(x) = \delta_0\bar\psi\left[i\partial\!\!\!/\psi - m\psi - g\int\int d^4\xi\, d^4\eta\, F(\xi,\eta)i\gamma_5\phi(x-\eta)\psi(x-\eta+\xi)\right]$$
$$+\left[-i\partial_\mu\bar\psi\gamma^\mu - m\bar\psi - g\int\int d^4\xi\, d^4\eta\, F(\xi,\eta)\bar\psi(x-\xi+\eta)i\gamma_5\phi(x-\xi)\right]\delta_0\psi$$
$$+\tfrac{1}{2}\partial_\mu(\partial^\mu\phi\,\delta_0\phi + \delta_0\phi\,\partial^\mu\phi - i\delta_0\bar\psi\gamma^\mu\psi + i\bar\psi\gamma^\mu\delta_0\psi)$$
$$+g\int\int d^4\xi\, d^4\eta\, F(\xi,\eta)[\delta_0\bar\psi(x)i\gamma_5\phi(x-\eta)\psi(x-\eta+\xi) - \delta_0\bar\psi(x+\eta)i\gamma_5\phi(x)\psi(x+\xi)$$
$$+\bar\psi(x-\xi+\eta)i\gamma_5\phi(x-\xi)\delta_0\psi(x) - \bar\psi(x+\eta)i\gamma_5\phi(x)\delta_0\psi(x+\xi)]$$
$$-\tfrac{1}{2}(\Box+\mu^2)\phi\,\delta_0\phi - \tfrac{1}{2}\delta_0\phi(\Box+\mu^2)\phi - g\int\int d^4\xi\, d^4\eta\, F(\xi,\eta)\bar\psi(x+\eta)i\gamma_5\delta_0\phi(x)\psi(x+\xi). \quad (4.31)$$

The last three terms may in turn be rewritten as

$$-\tfrac{1}{2}\left[(\Box+\mu^2)\phi(x) + g\int\int d^4\xi\, d^4\eta\, F(\xi,\eta)\bar\psi(x+\eta)i\gamma_5\psi(x+\xi)\right]\delta_0\phi(x)$$
$$-\tfrac{1}{2}\delta_0\phi(x)\left[(\Box+\mu^2)\phi(x) + g\int\int d^4\xi\, d^4\eta\, F(\xi,\eta)\bar\psi(x+\eta)i\gamma_5\psi(x+\xi)\right]$$
$$+g\int\int d^4\xi\, d^4\eta\, F(\xi,\eta)[\tfrac{1}{2}\delta_0\phi(x)\bar\psi(x+\eta)i\gamma_5\psi(x+\xi) + \tfrac{1}{2}\bar\psi(x+\eta)i\gamma_5\psi(x+\xi)\delta_0\phi(x) - \bar\psi(x+\eta)i\gamma_5\delta_0\phi(x)\psi(x+\xi)]. \quad (4.32)$$

The fourth term in (4.31) can, furthermore, be transformed into a total divergence

$$-g\int\int d^4\xi\, d^4\eta\, F(\xi,\eta)\int d^4x'\,\partial_\mu[\eta^\mu D(x-x',\tfrac{1}{2}\eta)\delta_0\bar\psi(x'+\tfrac{1}{2}\eta)i\gamma_5\phi(x'-\tfrac{1}{2}\eta)\psi(x'-\tfrac{1}{2}\eta+\xi)$$
$$+\xi^\mu D(x-x',\tfrac{1}{2}\xi)\bar\psi(x'-\tfrac{1}{2}\xi+\eta)i\gamma_5\phi(x'-\tfrac{1}{2}\xi)\delta_0\psi(x'+\tfrac{1}{2}\xi)]. \quad (4.33)$$

Application of the action principle (1.13) gives the following:

(i) The two first terms in (4.31) yield two of the equations of motion (2.40).

(ii) The third and fourth terms in (4.31) yield the generator (2.41).

(iii) The last three terms in (4.31) yield, according to (4.32), the first equation of motion in (2.40) *and* an additional term to the generator (2.41). We get, e.g., the following new generator ($F_{\text{new}}[\sigma]=F_{\text{old}}[\sigma]$ at $\sigma=-\infty$):

$$F_{\text{new}}[\sigma]=F_{\text{old}}[\sigma]-g\int_{-\infty}^{\sigma}d^4x\int\int d^4\xi\, d^4\eta\, F(\xi,\eta)[\tfrac{1}{2}\delta_0\phi(x)\bar{\psi}(x+\eta)i\gamma_5\psi(x+\xi)+\tfrac{1}{2}\bar{\psi}(x+\eta)i\gamma_5\psi(x+\xi)\delta_0\phi(x)$$
$$-\bar{\psi}(x+\eta)i\gamma_5\delta_0\phi(x)\psi(x+\xi)], \tag{4.34}$$

from which one sees that if $\delta_0\phi(x)$ commutes with ψ and $\bar{\psi}$, then $F_{\text{new}}[\sigma]=F_{\text{old}}[\sigma]$. Furthermore, we get an explanation of why the charge (2.52) is conserved even in the quantized case: because $\delta_0\phi=0$ there.

The new conserved energy-momentum vector is

$$P^\nu(t)=\int d^3x[-\mathcal{L}_0(x)g^{0\nu}+\tfrac{1}{2}(\partial^0\phi\partial^\nu\phi+\partial^\nu\phi\partial^0\phi+i\bar{\psi}\gamma^0\overleftrightarrow{\partial}^\nu\psi)]$$
$$+g\int\int d^4\xi\, d^4\eta\, F(\xi,\eta)\int d^4x\,\theta(t-x_0)[\partial^\nu\bar{\psi}(x)i\gamma_5\phi(x-\eta)\psi(x-\eta+\xi)+\tfrac{1}{2}\partial^\nu\phi(x)\bar{\psi}(x+\eta)i\gamma_5\psi(x+\xi)$$
$$+\tfrac{1}{2}\bar{\psi}(x+\eta)i\gamma_5\psi(x+\xi)\partial^\nu\phi(x)+\bar{\psi}(x+\eta-\xi)i\gamma_5\phi(x-\xi)\partial^\nu\psi(x)], \tag{4.35}$$

which does not differ much from the old one (2.54).

Checking the generator properties

$$[P^\nu,\phi(x)]=-i\partial^\nu\phi(x),$$
$$[P^\nu,\psi(x)]=-i\partial^\nu\psi(x), \tag{4.36}$$

one finds now by making use of the expansion (4.29) that $P_n^\nu(\text{in})=0$ for $n=1, 2, 3$. Thus we examine (4.36) up to the same order by replacing P^ν by $P_0^\nu(\text{in})$, and we get that (4.36) is true at least up to the third order in the coupling constant g.

$$P_0^\nu(\text{in})=P_0^\nu(\text{out}) \tag{4.37}$$

has also been checked up to the third order in g by use of (4.27).

We now make the hypothesis that

$$P^\nu=P_0^\nu(\text{in}) \tag{4.38}$$

and that P^ν satisfies (4.36) when one expands in in-fields. This hypothesis seems to the author very plausible. [In $P_n^\nu(\text{in})$, $n=1, 2, 3$ the cancellations between the different terms were straightforward, in contradistinction to, e.g., the case (4.37).] The hypothesis is, furthermore, in agreement with Pauli's[7] belief according to the quotation in the Introduction.

So far so good. However, the new generator (4.34) is not uniquely determined from the action principle. One also gets the following alternative one ($F'_{\text{new}}[\sigma]=F_{\text{old}}[\sigma]$ at $\sigma=+\infty$):

$$F'_{\text{new}}[\sigma]=F_{\text{old}}[\sigma]+g\int_{\sigma}^{+\infty}d^4x\int\int d^4\xi\, d^4\eta\, F(\xi,\eta)[\tfrac{1}{2}\delta_0\phi(x)\bar{\psi}(x+\eta)i\gamma_5\psi(x+\xi)+\tfrac{1}{2}\bar{\psi}(x+\eta)i\gamma_5\psi(x+\xi)\delta_0\phi(x)$$
$$-\bar{\psi}(x+\eta)i\gamma_5\delta_0\phi(x)\psi(x+\xi)]\,. \tag{4.39}$$

(Other alternatives are not equal to the free generator either in the limit $\sigma\to-\infty$ or when $\sigma\to+\infty$ and are therefore excluded.) (4.39) yields the following conserved energy-momentum vector as an alternative to (4.35):

$$P'^\nu(t)=\int d^3x[-\mathcal{L}_0(x)g^{0\nu}+\tfrac{1}{2}(\partial^0\phi\partial^\nu\phi+\partial^\nu\phi\partial^0\phi+i\bar{\psi}\gamma^0\overleftrightarrow{\partial}^\nu\psi)]$$
$$-g\int d^4x\,\theta(x_0-t)\int\int d^4\xi\, d^4\eta\, F(\xi,\eta)[\partial^\nu\bar{\psi}(x)i\gamma_5\phi(x-\eta)\psi(x-\eta+\xi)+\tfrac{1}{2}\partial^\nu\phi(x)\bar{\psi}(x+\eta)i\gamma_5\psi(x+\xi)$$
$$+\tfrac{1}{2}\bar{\psi}(x+\eta)i\gamma_5\psi(x+\xi)\partial^\nu\phi(x)+\bar{\psi}(x+\eta-\xi)i\gamma_5\phi(x-\xi)\partial^\nu\psi(x)]\,. \tag{4.40}$$

In order to have a consistent quantized nonlocal field theory we must have that $F'_{\text{new}}[\sigma]=F_{\text{new}}[\sigma]$. This requirement gives for the Kristensen-Møller model the consistency condition

$$\tfrac{1}{2}g\int_{-\infty}^{\infty} d^4x \int\int d^4\xi\, d^4\eta\, F(\xi,\eta) \times\{[\delta_0\phi(x), \bar\psi(x+\eta)]i\gamma_5\psi(x+\xi) - \bar\psi(x+\eta)i\gamma_5[\delta_0\phi(x), \psi(x+\xi)]\} = 0 \quad (4.41)$$

for *any* functional variation δ_0.

This consistency condition may be derived frcm another point of view according to the following considerations: In local quantum field theory one requires that the action shall be stationary, i.e., $\delta_0 W_{21}=0$, when the variations δ_0 of the field operators vanish on the surfaces σ_1 and σ_2. The corresponding condition for nonlocal quantum field theories is somewhat weaker, namely that only when $\sigma_1=-\infty$ and $\sigma_2=-\infty$ does one require $\delta_0 W_{21}=0$ when the variations of the field operators vanish on these surfaces. This is due to the fact that here the Lagrangian densities are expressed in field operators depending not just on the space-time point on which the Lagrangian explicitly depends, but on a space-time domain around this point. This in turn implies that integral conserved quantities $F[\sigma]$ are expressed by field operators depending not just on the spacelike surface σ, but on a space-time domain around σ.

This stationary condition is equivalent to the uniqueness requirement $F'_{\text{new}}[\sigma]=F_{\text{new}}[\sigma]$, i.e., (4.41) for the Kristensen-Møller model.

Notice that every field theory model with nonlocal interaction of the nonlinear type (like the Kristensen-Møller model) yields a consistency condition similar to (4.41), and derivable by use of a q-number variation of the Lagrangian in question. This is easily realized, as one then will always have a variation of a field operator sandwiched between other field operators. Furthermore, if the resulting consistency condition is not fulfilled by the imposed quantization, some generators are not uniquely determined, and especially the energy-momentum vector is not unique. The last point is exactly what happens in the present case.

Consider the consistency condition (4.41) for the energy-momentum vector, i.e.,

$$P^\nu - P'^\nu = \tfrac{1}{2}g\int d^4x \int\int d^4\xi\, d^4\eta\, F(\xi,\eta) \times\{[\partial^\nu\phi(x), \bar\psi(x+\eta)]i\gamma_5\psi(x+\xi) - \bar\psi(x+\eta)i\gamma_5[\partial^\nu\phi(x), \psi(x+\xi)]\} = 0\,. \quad (4.42)$$

Then by making use of the perturbation expansions in in-fields, one finds the following violation of (4.42):

$$\begin{aligned} P^\nu - P'^\nu = g^4 i \int\int\int\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \int\int d^4\xi_1 d^4\eta_1 F(\xi_1,\eta_1)\int\int d^4\xi_2 d^4\eta_2 F(\xi_2,\eta_2)\int\int d^4\xi_3 d^4\eta_3 F(\xi_3,\eta_3) \\ \times\int\int d^4\xi_4 d^4\eta_4 F(\xi_4,\eta_4)\theta(x_1+\xi_1-x_2-\eta_2)\theta(x_2-x_4)\theta(x_4+\eta_4-x_3-\xi_3)\theta(x_3-x_1)\partial_4^\nu\Delta(x_4-x_2) \\ \times\Delta(x_1-x_3)\bar\psi_{\text{in}}(x_1+\eta_1)\bar\psi_{\text{in}}(x_3+\eta_3)i\gamma_5 S(x_3+\xi_3-x_4-\eta_4)i\gamma_5\psi_{\text{in}}(x_4+\xi_4)i\gamma_5 \\ \times S(x_1+\xi_1-x_2-\eta_2)i\gamma_5\psi_{\text{in}}(x_2+\xi_2) + \text{H.c.} + O(g^5)\,. \end{aligned} \quad (4.43)$$

In order to find out the meaning of the alternative energy-momentum vector (4.40), we shall expand the field operators in terms of out-fields according to

$$\begin{aligned} \phi(x) &= \phi_{\text{out}}(x) - g\int d^4y\, \Delta_A(x-y)\rho(y)\,, \\ \psi(x) &= \psi_{\text{out}}(x) + g\int d^4y\, S_A(x-y)f(y)\,, \\ \bar\psi(x) &= \bar\psi_{\text{out}}(x) + g\int d^4y\, \bar f(y) S_R(y-x)\,. \end{aligned} \quad (4.44)$$

One now finds by requiring that the free out-fields fulfill the free field commutation relations (4.6)–(4.8) that

$$P'^\nu = P_0^\nu(\text{out}) \quad (4.45)$$

if one accepts the hypothesis (4.38) [in any case, (4.45) is true at least up to the third order in the coupling constant g]. This one realizes since the only difference between the expansions in out-fields and those in in-fields is that every step function $\theta(x_j-x_k)$ is replaced by $-\theta(x_k-x_j)$. The hypothesis (4.38) implies, therefore, that

$$P_0^\nu(\text{out}) - P_0^\nu(\text{in}) = P'^\nu - P^\nu, \quad (4.46)$$

which according to (4.43) is different from zero. (4.37) seems, therefore, to be violated in the fourth order.

(4.45) was arrived at by the requirement that the out-fields fulfill the commutation relations (4.6)–(4.8). However, the quantization of the out-fields may be calculated by use of (4.27). This has also been done up to the fourth order in the coupling constant by several persons (see, e.g., the discussions in Ref. 3). They found that the out-fields did not satisfy the same quantization as the in-fields. The violation is of the fourth order in g. Hayashi[13,14] has published such calculations for a similar model (scalar fields).

There is a proof by Bloch[6] that the out-fields fulfill (4.6)–(4.8) under the assumption that $P_0^\nu(\text{out}) = P_0^\nu(\text{in})$. According to the above calculations his result must be wrong, and (4.43) and (4.46) tell us why; the assumption is wrong (as expected[14]). The violation of the assumption is, furthermore, of the same order as the violation of the property Bloch was to prove.

Now the conditions (4.37) and

$$[\phi_{\text{out}}(x), \phi_{\text{out}}(y)] = [\phi_{\text{in}}(x), \phi_{\text{in}}(y)], \quad \text{etc.} \tag{4.47}$$

are necessary conditions for the existence of a unitary S operator. The violations of (4.37) and (4.47), therefore, imply that a unitary S operator does not exist, i.e., an operator S fulfilling

$$\phi_{\text{out}} = S^\dagger \phi_{\text{in}} S, \quad \text{etc.} \tag{4.48}$$

does not exist. In consistency with the above reasoning, Imamura *et al.*[15] have derived formal relations involving the S operator which was shown to be violated by an explicit calculation. The violation was in the same order as the preceding violations.

Hayashi[14] showed now that the relations (4.47) could be fulfilled up to the fourth order by imposing the following conditions (assuming that his results are applicable to the Kristensen-Møller model):

(i) The properly symmetrized Lagrangian density should be used instead of (2.37), i.e.,

$$\mathcal{L}(x) = \tfrac{1}{2}(\partial_\mu\phi\partial^\mu\phi - \mu^2\phi^2) + \tfrac{1}{4}[\bar\psi, (i\gamma^\mu\partial_\mu - m)\psi] - \tfrac{1}{4}[i\partial_\mu\bar\psi\gamma^\mu + m\bar\psi, \psi] - \tfrac{1}{4}g\iint d^4\xi\, d^4\eta\, F(\xi,\eta)\{\phi(x), [\bar\psi(x+\eta), i\gamma_5\psi(x+\xi)]\}. \tag{4.49}$$

Notice that the corresponding action integral is CPT-invariant.

(ii) The form function $F(\xi, \eta)$ must be real.

The Lagrangian (4.49) yields the following consistency condition [cf. (4.41)]:

$$\int_{-\infty}^{\infty} d^4x \iint d^4\xi\, d^4\eta\, F(\xi,\eta)\{-[\phi(x-\xi), \delta_0\psi^T(x)]i\gamma_5\bar\psi^T(x-\xi+\eta) + \bar\psi(x-\xi+\eta)i\gamma_5[\delta_0\psi(x), \phi(x-\xi)]$$
$$-\psi^T(x-\eta+\xi)i\gamma_5[\delta_0\bar\psi^T(x), \phi(x-\eta)] + [\phi(x-\eta), \delta_0\bar\psi(x)]i\gamma_5\psi(x-\eta+\xi)\} = 0 \tag{4.50}$$

for *any* functional variation δ_0. If this condition in its particular form $P^\nu - P'^\nu = \cdots = 0$ [cf. (4.42)] is violated, then the violation must be a c number if the condition (ii) above is to be useful.

V. SUMMARY AND FURTHER COMMENTS

The action principle has been applied to classical and quantum nonlocal field theories. This has been made possible by the introduction of Lagrangian densities for such theories. In contradistinction to local field theories, the classical results are in general not valid for the corresponding quantum case. We, therefore, distinguish between the two cases below and start with the classical one.

The motivation for the introduction of Lagrangian densities was in fact twofold; besides making the action principle applicable, we should be able to derive conserved densities. Our results are as follows: It has been shown that there exist an infinity of different Lagrangian densities to a given set of equations of motion derivable from an action functional. The action principle has thereafter been applied to such Lagrangians by use of a generalized variation method, which yielded general formulas for conserved densities. These formulas were derived under the assumption of c-number variations and hold, therefore, at least for a classical nonlocal field theory. In an actual application one has to choose a particular Lagrangian density, e.g., by simplicity. It is thereby noticed that if the conserved densities have a physical interpretation which is different for different choices of the Lagrangian densities, then one may say that such a choice is a dynamical choice even if all the possible choices yield the same equations of motion.

In distinction from the conserved densities, the integral conserved quantities are uniquely determined from the given set of equations of motion by the action principle. For the Kristensen-Møller model, the unique charge and energy-momentum vector (and angular momentum tensor) have been derived explicitly, and shown to be equal to the expressions given by Pauli.[7] However, Pauli used another method of derivation, which may be described as follows:

(i) One derives the integral conserved quantity $F_0(t)$ for the free fields as usual.

(ii) The time derivative $dF_0(t)/dt$ is then calculated by use of the equations of motion. The resulting expression is thereafter integrated and called $-F_I(t)$. It now follows that $F(t)=F_0(t)+F_I(t)$ is conserved.

(iii) The integration constant is then determined by the condition $F_I(t)\to 0$, when $t\to\pm\infty$.

As every density in general depends on all space-time, or effectively on a space-time domain (determined by the form function) around the point on which it is given, every infinitesimal generator $F[\sigma]$ is not just depending on the spacelike surface σ, but effectively on a space-time domain around σ. The consequence is that the interpretation of the action principle is here somewhat changed in general, since one cannot derive the equations of motion by putting the variations of the fields equal to zero just on the spacelike surfaces σ_1 and σ_2. One exception is when $\sigma_1=-\infty$ and $\sigma_2=+\infty$, because in this limit the space-time domains around σ_1 and σ_2 are in effect separated from the finite space-time. There exist, however, sometimes variations (not induced by space-time transformations) of fields, which yield generators $F[\sigma]$ only depending on σ [put, e.g., $F(\xi,\eta)=f(\xi)\delta^4(\xi-\eta)$ into the current density (2.43), or directly into the charge (2.52)].

In this connection one may ask in what sense $F[\sigma]$ are generators in the classical nonlocal case. According to Pauli[7] there should exist canonical variables, which means that $F[\sigma]$ could be generators in the usual Poisson bracket sense, but the situation seems not completely clear.

Finally, we have seen in Sec. III that as soon as we permit integrals in the Lagrangian densities the action principle yields an infinity of different conserved densities all giving one and the same integral conserved quantity even for local field theories. Note that this result is not connected to Kibble's[16] result that any free local field theory permits an infinity of different local currents for every invariance, though his result is also related to nonlocality[17] and may be investigated by use of the general formulas in Sec. II A. That it is so may be seen as follows: Take, e.g., the Lagrangian density

$$\mathcal{L}(x)=\int d^4\xi\, f(\xi)[\partial_\mu\phi^*(x+\tfrac{1}{2}\xi)\partial^\mu\phi(x-\tfrac{1}{2}\xi)-\mu^2\phi^*(x+\tfrac{1}{2}\xi)\phi(x-\tfrac{1}{2}\xi)], \quad (5.1)$$

which gives the equations of motion [use (2.15) and (B2)]

$$\int d^4\xi\, f(\xi)(\Box+\mu^2)\phi(x-\xi)=0, \quad \text{etc.} \quad (5.2)$$

Making then the following choice of $f(\xi)$:

$$f(\xi)=\delta^4(\xi-\lambda), \quad (5.3)$$

where λ^μ is an arbitrary constant four-vector. Equation (5.2) becomes now

$$(\Box+\mu^2)\phi(x-\lambda)=0, \quad \text{etc.}, \quad (5.4)$$

which by translation invariance is equal to

$$(\Box+\mu^2)\phi(x)=0, \quad \text{etc.} \quad (5.5)$$

Thus, in spite of the fact that we have split the points in the Lagrangian (5.1), the equations of motion are unchanged. This is, however, only possible for bilinear Lagrangians, i.e., essentially for free field theories.

Now derive the conserved densities by use of the formulas in Sec. II A and Appendix B. Each such density then contains infinitely many locally conserved densities which are simply derived by a Taylor expansion with respect to λ^μ. This is exactly Kibble's result (cf. Fairlie[17]).

When we quantized a field theory with nonlocal interaction by use of the Yang-Feldman method, we came across some new unexpected features, which were shown to be due to this very quantization.

The Yang-Feldman method of quantization was shown to have the following properties in the nonlocal case: It is defined within Heisenberg's picture and is applicable to all field theories with nonlocal interaction. It gives a highly noncanonical quantization in that all fields are noncommuting and all their commutators are q numbers. The quantization is perturbatively defined, and one has, therefore, to work order-by-order in the perturbation expansions and can hardly say anything about the exact commutators (up to now). In particular the noncanonical property was shown to have the following consequences:

(i) Integral conserved quantities and conserved densities, derived under the assumption of c-number variations, are in general no longer conserved in the quantized case.

(ii) The application of the action principle requires, therefore, q-number variations in this case.

(iii) By use of q-number variations one can then derive only integral conserved quantities and in general not conserved densities from a given Lagrangian (in the sense that these should be associated with this Lagrangian density and have no explicit σ dependence).

(iv) Integral conserved quantities can, furthermore, be derived essentially in two different ways. It seems very plausible that both these quantities have the right generator properties, but in different senses; one with respect to canonically quantized in-fields, the other with respect to canonically quantized out-fields. This has anyway been checked up to the third order in the perturbation expansions for the Kristensen-Møller model.

(v) We have shown how to derive a general consistency condition for every nonlocal field theory model. This is required by each of the following conditions separately: (1) uniqueness of the generators, (2) stationarity of the total action when the variations of the fields are zero at infinite past and future, (3) the existence of a unitary S operator, when the in- and out-fields fulfill the canonical free field quantization. Note that every nonlinear field theory with nonlocal interaction has such a consistency condition, which never can be said to be satisfied offhand. E.g., for the Kristensen-Møller model, this consistency condition is given by (4.41), which we showed to be violated in the fourth order in g by an explicit calculation.

It is interesting to note that Pauli[7] did not realize that his energy-momentum vector was no longer conserved in the quantized case. If he had done so, then he would still have been able to derive the conserved energy-momentum vectors by use of his rules (i)–(iii) (given earlier in this section), and would have thereby been able to discover the nonuniqueness. However, he would not have been able to derive such general consistency conditions as (4.41).

The fact that the noncanonical quantization made q-number variations necessary implies the following observation: The assumption that c-number variations are applicable even in the quantized case[4] restricts the quantization essentially to a canonical one.[18]

In connection with point (v) above, a fundamental problem has arisen: Does there exist any consistent quantum nonlocal, nonlinear field theory of the kind considered in this paper? This may be reduced to the question: What does the fulfillment of consistency conditions like (4.41) really require? The answer might be somewhere between not much more than CPT-invariance (weak locality) to completely unreasonable requirements. In order to investigate this, one has to know much more about the complete commutators between the field operators. Therefore, one must in turn develop a convenient calculational tool for this purpose. Maybe generalized retarded products[6] will do here?

To the author, the above problem is the only really serious one and therefore the most important problem in quantum nonlocal field theory. Another problem is convergence. Here it is the Lorentz invariance of the form function that causes the difficulties; the form function must be different from zero on an infinite domain of space-time. However, form functions fulfilling the macrocausality conditions[6,19] yield Lagrangian densities, etc., which are smeared effectively only over a finite domain of space-time. Now, Bloch[6,3] has given a sufficient condition for convergence which is reconcilable with these macrocausality conditions. Furthermore, there seem to exist form functions not fulfilling this condition which give convergence and macrocausality. Thus, in contradistinction to local field theory, most of the expressions presented in this paper can be given a mathematically well-defined meaning, even if the convergence problem needs further investigation.

A third problem is the initial-value problem; does it give the same answer as for free fields? Pauli[7] believed so if the form function belongs to his normal class. (See also Ref. 20.) However, Taniuti[21] requires stronger conditions here.

Further properties of quantum nonlocal field theories are displayed in Ref. 22.

Note added: The relation (4.38) can be proved up to all orders as follows: Express the in-fields in terms of the interaction fields by use of (4.1)–(4.3), then put these expressions into $P_0^\nu(\text{in})$. After some simple manipulations one arrives at P^ν. Thereby it is proved that (4.43) yields $P_0^\nu(\text{in}) \neq P_0^\nu(\text{out})$ and that (4.42) is a necessary condition for the existence of a unitary S operator. [Also the full hypothesis at (4.38) as well as the relations (4.17) can in fact be proved.]

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APPENDIX A: THE ACTION PRINCIPLE FOR HIGHER-ORDER LAGRANGIANS [23]

Let $\mathfrak{L}^{(N)}(x)$ be a local Lagrangian containing field operators and derivatives of field operators up to at most Nth order. If $N \geq 2$, one calls such Lagrangians higher-order Lagrangians.

The application of Schwinger's action principle[4] to Lagrangians of the above type offers no new problems. The operator principle of stationary action has the usual form. Introduce the action integral

$$W_{21}^{(N)} = \int_{\sigma_1}^{\sigma_2} d^4x\, \mathfrak{L}^{(N)}(x), \tag{A1}$$

where σ_2 and σ_1 are two spacelike surfaces, serving as covariant generalizations of time, such that σ_1 is "earlier" than σ_2. The action principle is then

$$\delta W_{21}^{(N)} = F[\sigma_1] - F[\sigma_2], \tag{A2}$$

where δ is a small variation generated by the infinitesimal Hermitian operator $F[\sigma]$, whose associated unitary operator is $U = e^{iF}$.

Consider first a functional variation δ_0 of the action functional $W_{21}^{(N)}$, i.e., $\delta_0 W_{21}^{(N)} = \int_{\sigma_1}^{\sigma_2} d^4x\, \delta_0 \mathfrak{L}^{(N)}(x)$. The action principle (A2) gives then

(1) the equations of motion

$$\frac{\partial \mathcal{L}^{(N)}}{\partial \phi^\alpha} - \partial_\nu \pi^\nu_\alpha = 0, \tag{A3}$$

and

(2) the infinitesimal generator

$$F[\sigma] = -\int_\sigma d\sigma_\mu \sum_{i=0}^{N-1} \pi^{\mu\nu_1\cdots\nu_i}_\alpha \delta_0 \partial_{\nu_1}\cdots\partial_{\nu_i}\phi^\alpha, \tag{A4}$$

where we have introduced the following useful quantities:

$$\pi^{\nu_1\cdots\nu_s}_\alpha \equiv \sum_{i=0}^{N-s} (-1)^i \partial_{\mu_1}\cdots\partial_{\mu_i} \frac{\partial \mathcal{L}^{(N)}}{\partial\partial_{\nu_1}\cdots\partial_{\nu_s}\partial_{\mu_1}\cdots\partial_{\mu_i}\phi^\alpha}, \qquad s = 0, 1, \ldots, N \tag{A5}$$

which are totally symmetric in the indices ν_k as $\partial\mathcal{L}^{(N)}/\partial\partial_{\nu_1}\cdots\partial_{\nu_s}\phi^\alpha$, $s \le N$, has to be symmetric in ν_k. Furthermore, they satisfy the following N identities:

$$\pi^{\nu_1\cdots\nu_s}_\alpha = \frac{\partial \mathcal{L}^{(N)}}{\partial\partial_{\nu_1}\cdots\partial_{\nu_s}\phi^\alpha} - \partial_\nu \pi^{\nu\nu_1\cdots\nu_s}_\alpha, \qquad s = 1, \ldots, N-1$$

$$\pi^{\nu_1\cdots\nu_N}_\alpha = \frac{\partial \mathcal{L}^{(N)}}{\partial\partial_{\nu_1}\cdots\partial_{\nu_N}\phi^\alpha}. \tag{A6}$$

Note that the equation of motion (A3) may be written

$$\pi_\alpha = 0 \text{ or } \sum_{i=0}^{N} (-\partial_\mu)^i \frac{\partial \mathcal{L}^{(N)}}{\partial(\partial_\mu)^i\phi^\alpha}, \tag{A7}$$

where $(\partial_\mu)^i \equiv \partial_{\mu_1}, \ldots, \partial_{\mu_i}$.

If the functional variation δ_0 is a symmetry of the action, $(\delta_0 W^{(N)}_{21} = 0)$, then the generator $F[\sigma]$ is σ-independent (independent of the spacelike surface σ). σ independence $[\delta F\sigma/\delta\sigma(x) = 0]$ in turn implies that the infinitesimal current

$$j^\mu(x) \equiv -\sum_{i=0}^{N-1} \pi^{\mu\nu_1\cdots\nu_i}_\alpha \delta_0\partial_{\nu_1}\cdots\partial_{\nu_i}\phi^\alpha$$

is locally conserved, i.e., $\partial_\mu j^\mu(x) = 0$.

If two Lagrangians $\mathcal{L}$ and $\mathcal{L}'$ differ by a four-divergence $(\mathcal{L}' - \mathcal{L} = \partial_\mu\Lambda^\mu)$, then the action principle implies that they give rise to identical equations of motion. However, they do not give the same locally conserved quantities.

Consider now coordinate variations D_T (variations induced by infinitesimal coordinate transformations T). For convenience we consider only the coordinate variation $D_\mathcal{P}$ induced by an infinitesimal Poincaré transformation $\mathcal{P}$, $\delta x_\nu = \epsilon_\nu + \epsilon_{\nu\rho}x^\rho$. $D_\mathcal{P}$ consists of two parts, a surface variation $B_\mathcal{P}$ and a functional variation $\delta_\mathcal{P}$. The surface variation $B_\mathcal{P}$ of the action is given by

$$B_\mathcal{P} W^{(N)}_{21} = \left(\int_{\sigma_2} - \int_{\sigma_1}\right) d\sigma_\mu \delta x^\mu \mathcal{L}^{(N)}(x). \tag{A8}$$

The induced functional variation $\delta_\mathcal{P}$ of the field operators is defined by either

$$\delta_\mathcal{P}\phi^\alpha(x) = {}'\phi^\alpha(x) - \phi^\alpha(x), \tag{A9}$$

where ${}'\phi^\alpha(x)$ is the field dragged along by $\mathcal{P}$, or

$$\delta_\mathcal{P}\phi^\alpha(x) = \phi^\alpha(x) - {}''\phi^\alpha(x), \tag{A10}$$

where ${}''\phi^\alpha(x)$ is the field dragged along by $\mathcal{P}^{-1}$ (infinitesimal differences). Both (A9) and (A10) give

$$\delta_\mathcal{P}\phi^\alpha(x) = -\delta x_\mu \partial^\mu\phi^\alpha(x) + \tfrac{1}{2}\epsilon_{\mu\nu}\Sigma^{\mu\nu}_{\alpha\beta}\phi^\beta(x), \tag{A11}$$

where $\Sigma^{\mu\nu}_{\alpha\beta}$ comes from the infinitesimal Lorentz transformation matrix, which is given by

$$S_{\alpha\beta}(\Lambda) = \delta_{\alpha\beta} + \tfrac{1}{2}\epsilon_{\mu\nu}\Sigma^{\mu\nu}_{\alpha\beta} \quad (\Lambda_{\mu\nu} = g_{\mu\nu} + \epsilon_{\mu\nu}).$$

The generator of the infinitesimal Poincaré transformation is known to be (by choosing a certain sign convention)

$$F[\sigma] = \epsilon_\mu P^\mu[\sigma] + \tfrac{1}{2}\epsilon_{\mu\nu}J^{\mu\nu}[\sigma], \tag{A12}$$

where P^μ is the energy-momentum operator (generator of translations) and $J^{\mu\nu}$ is the angular momentum operator (generator of Lorentz transformations). Thus $\delta_\mathcal{P}\phi^\alpha$ is also given by $\delta_\mathcal{P}\phi^\alpha = i[\phi^\alpha, F[\sigma]]$.

The action principle, $D_\mathcal{P} W^{(N)}_{21} \equiv (B_\mathcal{P} + \delta_\mathcal{P})W^{(N)}_{21} = F[\sigma_1] - F[\sigma_2]$, gives now

$$F[\sigma] = -\int_\sigma d\sigma_\mu \left(\mathcal{L}^{(N)}\delta x^\mu + \sum_{i=0}^{N-1} \pi^{\mu\nu_1\cdots\nu_i}_\alpha \delta_\mathcal{P}\partial_{\nu_1}\cdots\partial_{\nu_i}\phi^\alpha\right). \tag{A13}$$

Identification of (A12) with (A13) using (A11), generalized in an obvious manner, yields

$$P^\nu[\sigma] = \int_\sigma d\sigma_\mu T^{\mu\nu}_c, \qquad J^{\nu\rho}[\sigma] = \int_\sigma d\sigma_\mu M^{\mu\nu\rho}_c, \tag{A14}$$

where $T^{\mu\nu}_c$ is the canonical energy-momentum tensor given by

$$T^{\mu\nu}_c = -\mathcal{L}^{(N)}g^{\mu\nu} + \sum_{i=0}^{N-1} \pi^{\mu\nu_1\cdots\nu_i}_\alpha \partial^\nu\partial_{\nu_1}\cdots\partial_{\nu_i}\phi^\alpha \tag{A15}$$

and $M^{\mu\nu\rho}_c$ is the corresponding angular momentum tensor, given by

$$M^{\mu\nu\rho}_c = x^\rho T^{\mu\nu}_c - x^\nu T^{\mu\rho}_c - \sum_{i=0}^{N-1} \pi^{\mu\nu_1\cdots\nu_i}_\alpha \Sigma^{\nu\rho\mu_1\cdots\mu_i}_{\alpha\beta\nu_1\cdots\nu_i}\partial_{\mu_1}\cdots\partial_{\mu_i}\phi^\beta. \tag{A16}$$

If the action functional is Poincaré-invariant, i.e., $D_\mathcal{P} W_{21} = 0$, then the generators of Poincaré transformations, P^μ and $J^{\mu\nu}$, are σ-independent or constants of the motion. One then says that $\mathcal{P}$ is a symmetry transformation.

Introducing the quantity

$$f^{\lambda\mu\nu} \equiv \frac{1}{2}\sum_{i=0}^{N-1} \left(\pi_\alpha^{\lambda\nu_1\cdots\nu_i} \Sigma_{\alpha\beta\nu_1\cdots\nu_i}^{\mu\nu\mu_1\cdots\mu_i} + \pi_\alpha^{\nu\nu_1\cdots\nu_i} \Sigma_{\alpha\beta\nu_1\cdots\nu_i}^{\mu\lambda\mu_1\cdots\mu_i} + \pi_\alpha^{\mu\nu_1\cdots\nu_i} \Sigma_{\alpha\beta\nu_1\cdots\nu_i}^{\nu\lambda\mu_1\cdots\mu_i}\right) \partial_{\mu_1}\cdots\partial_{\mu_i}\phi^\beta \tag{A17}$$

and assuming that $\int_\sigma d\sigma_\mu \partial_\nu (\delta x_\rho f^{\mu\nu\rho}) = 0$ (which holds if $f^{\mu\nu\rho}\delta x_\rho$ approaches zero with sufficient rapidity at infinitely remote points on σ), one may define a symmetric energy-momentum tensor by

$$T_B^{\mu\nu} = T_c^{\mu\nu} + \partial_\lambda f^{\lambda\mu\nu}. \tag{A18}$$

The index B denotes the fact that (A18) is sometimes called the Belinfante energy-momentum tensor.

The corresponding angular momentum tensor is given by

$$M_B^{\mu\nu\rho} = x^\rho T_B^{\mu\nu} - x^\nu T_B^{\mu\rho}. \tag{A19}$$

Note that $\partial_\mu T_B^{\mu\nu} = 0$ implies $\partial_\mu M_B^{\mu\nu\rho} = T_B^{\rho\nu} - T_B^{\nu\rho}$. Thus if $\partial_\mu M_B^{\mu\nu\rho} = 0$, then $T_B^{\mu\nu}$ is symmetric. Note that the ordinary local case $N = 1$ is included in all the derived formulas.

Finally one should perhaps mention some of the important properties of theories constructed out of higher-order Lagrangians. Quantization offers no problems.[23] One treats $\phi(x), \ldots, (\hat\partial)^N\phi(x)$ ($\hat\partial \equiv n^\mu\partial_\mu$ is the normal derivative, the covariant generalization of the time derivative) as independent variables, i.e., they commute. The corresponding canonical momenta are given by

$$\hat\pi^{(l+1)}(x) = \sum_{i=l}^{N-1} (-1)^{i+l} \binom{i}{l} (n_\nu)^l (\vec\partial_\nu)^{i-l} n_\mu \pi^{\mu\nu_1\cdots\nu_i}(x),$$

$l = 0, \ldots, N-1$ where $\vec\partial_\nu \equiv \partial_\nu - n_\nu\hat\partial$ (the tangential derivative) and $\pi^{\mu\nu_1\cdots\nu_i}(x)$ are given by (A5). There is in fact only one main difficulty and that is that the total energy is indefinite. The usual remedy for this is to introduce indefinite-metric states. States with negative norm are then considered unphysical and projected out from the final result. The motivation for such theories is the fact that they may provide convergence (cf. Pauli-Villars regularization method). Note though that the theory is local up to the final step, that of projecting out the unphysical states, but after the final step the theory is equivalent to a nonlocal theory.[24]

One can also see the equivalence between higher-order Lagrangian theories and nonlocal theories more directly by just studying the equations of motion in p space.[6] The resulting equations will then be of the type (1.5), which here, however, are not derivable from an action functional. The resulting form function does, furthermore, not belong to Pauli's[7] normal class, and is therefore not satisfactory. See also Ref. 20.

APPENDIX B:

In this appendix we shall treat the case when the nonlocal Lagrangian density contains derivatives of the field operators. This will essentially complete the derivations in Sec. IIA.

Instead of a Lagrangian of the type (2.14), we shall consider the following one:

$$\begin{aligned}\mathcal{L}(x) &= \int d^4\xi\, f_{\alpha_k}^\mu(x,\xi)\partial_\mu\phi^{\alpha_k}(x+\xi)\\ &= \int d^4\xi\, f_{\alpha_k}^\mu(x,\xi)\sum_{l=0}^\infty \partial_\mu \frac{(\xi^\nu\partial_\nu)^l}{l!}\phi^{\alpha_k}(x),\end{aligned} \tag{B1}$$

where as before $f_\alpha^\mu(x,\xi)$ contains the form function and the other fields involved. (2.2) gives now

$$\sum_{i=0}^\infty (-\partial_\mu)(-\partial_\nu)^i \frac{\partial\mathcal{L}}{\partial\partial_\mu(\partial_\nu)^i\phi^\alpha} = -\int d^4\xi\, \partial_\mu f_\alpha^\mu(x-\xi,\xi). \tag{B2}$$

The infinitesimal current density is

$$\begin{aligned}j^\mu(x) &= -\sum_{s=0}^\infty\sum_{i=0}^\infty (-\partial_\lambda)^i \frac{\partial\mathcal{L}}{\partial\partial_\mu(\partial_\nu)^s(\partial_\lambda)^i\phi^\alpha(x)}\delta_0(\partial_\nu)^s\phi^\alpha(x)\\ &= -\sum_{s=0}^\infty\sum_{i=0}^\infty \int d^4\xi \frac{1}{(s+i+1)!}\big[(-\xi\partial)^i f_\alpha^\mu(x,\xi)\delta_0(\xi\partial)^s\phi^\alpha(x) + s(-\xi\partial)^i f_\alpha^\nu(x,\xi)\xi^\mu\delta_0(\xi\partial)^{s-1}\partial_\nu\phi^\alpha(x)\\ &\qquad - i(-\xi\partial)^{i-1}\partial_\nu f_\alpha^\nu(x,\xi)\xi^\mu\delta_0(\xi\partial)^s\phi^\alpha(x)\big].\end{aligned} \tag{B3}$$

The divergence of $j^\mu(x)$ is given by

$$\partial_\mu j^\mu(x) = -\int d^4\xi f^\mu_\alpha(x,\xi)\partial_\mu \delta_0\phi^\alpha(x+\xi) - \int d^4\xi\, \partial_\mu f^\mu_\alpha(x-\xi,\xi)\delta_0\phi^\alpha(x), \tag{B4}$$

which one may easily show by use of (B3). $j^\mu(x)$ can also be written in terms of the functions (2.20) and (2.28):

$$j^\mu(x) = -\int\int d^4\xi\, d^4x' \{ D(x-x',\tfrac{1}{2}\xi) f^\mu_\alpha(x'-\tfrac{1}{2}\xi,\xi)\delta_0\phi^\alpha(x'+\tfrac{1}{2}\xi) + \tfrac{1}{2}\xi^\mu [D(x-x',\tfrac{1}{4}\xi) f^\nu_\alpha(x'-\tfrac{1}{4}\xi,\xi)\partial'_\nu \delta_0\phi^\alpha(x'+\tfrac{3}{4}\xi) - D(x-x',\tfrac{1}{4}\xi)\partial'_\nu f^\nu_\alpha(x'-\tfrac{3}{4}\xi,\xi)\delta_0\phi^\alpha(x'+\tfrac{1}{4}\xi) + \overline{D}(x-x',\tfrac{1}{2}\xi)\partial'_\nu (f^\nu_\alpha(x'-\tfrac{1}{2}\xi,\xi)\delta_0\phi^\alpha(x'+\tfrac{1}{2}\xi))]\} . \tag{B5}$$

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