PHYSICAL REVIEW D VOLUME 8, NUMBER 8 15 OCTOBER 1973

Stability of the Vacuum and Quantization of the Electron-Positron Field for Strong External Fields*

Lewis Fulcher
Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742

Abraham Klein
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania, 19104
(Received 14 May 1973)

A method is given for quantizing the electron-positron field in the external field of a superheavy nucleus. In essence we redefine the vacuum to be the state in which all negative-energy solutions of the Dirac equation, including bound states, are occupied. Among other desirable features, this provides a natural description of the instability of the state conventionally labeled as the vacuum and a continuity in the physical description of the system as a function of the nuclear charge.

In the Coulomb field of a superheavy nucleus, the most deeply bound electron orbit ($1S_{1/2}$) of the Dirac equation reaches an energy of $-mc^2$ for nuclear charge number $Z_1 \sim 169$ and with further increase of charge becomes unstable against single-positron emission. *A fortiori,* the normally considered vacuum state in the presence of such an external field is unstable against the emission of two positrons. These phenomena have been predicted in the literature from several distinct viewpoints.[1-4] Furthermore, starting from the well-known treatment of Kroll and Wichmann (KW),[5] all authors agree that the above conclusions will not be qualitatively modified by the effect of vacuum polarization. However, the formulation of a consistent quantum electrodynamics for $Z > Z_1$ has so far not been given, and as long as this has not been done there must remain some lingering doubts concerning the above conclusions. It is the purpose of the present note to suggest the essential elements of such a theory. Not only will this yield a cogent view of the aforementioned instabilities which have so far been the center of discussion, but it will also turn out that the previous arguments concerning vacuum polarization require modification and are strictly correct only when formulated in terms of the new picture.

We wish to quantize the conventional quadratic Dirac Hamiltonian ($\hbar = c = 1$)

$$\hat{H} = \int \psi^\dagger_\alpha(\vec{r})\,[\vec{\alpha}\cdot\vec{p} + \beta m + V(r)]_{\alpha\beta}\psi_\beta(\vec{r})\, d^3r$$

$$\equiv \int \psi^\dagger \mathcal{H} \psi\,, \tag{1}$$

$$\{\psi_\alpha(\vec{r}), \psi^\dagger_\beta(\vec{r}\,')\} = \delta_{\alpha\beta}\delta(\vec{r} - \vec{r}\,')\,, \tag{2}$$

where, however, $V(r)$ may be a superstrong (basically Coulomb) potential and may also, as stipulated, contain contributions from vacuum polarization. We start with the situation where $V(r)$ is the Coulomb field of the extrapolated charge distribution of the nucleus

$$V(r) = ZU(r) \tag{3}$$

and $137 < Z < Z_1$. Then the spectrum of $\mathcal{H}$ consists of the usual two continua $E > m$ and $E < -m$, an infinite number of bound states with $0 < E < m$, and a few bound states for $-m < E < 0$. Let us imagine that in the interval considered only the pair of $1S$ states belong in the latter category.[6]

It is now embarrassingly evident that as soon as the field is strong enough to yield bound eigenstates of $\mathcal{H}$ of negative energy, one gains energy by filling these states, and thus, in the example considered, the eigenstate of $\hat{H}$ in which the two $1S$ orbits are occupied has a lower energy than the conventional vacuum. Consequently, in our view, it represents the most natural choice as a reference state for excitations. To develop this standpoint, let $u^{(\nu)}_\alpha(r)$ be the one-particle eigenstates of $\mathcal{H}$ with positive energy E_ν and let $v^{(\lambda)}_\alpha(r)$ be those for negative energy $(-)E_\lambda$ including bound states, $v^{(B)}_\alpha(r)$. We shall simply extend the usual picture in which unoccupied negative-energy states represent positrons. We thus define the vacuum so that an unoccupied *bound* state of negative energy represents a bound positron of positive energy. Formally we expand

$$\psi_\alpha(\vec{r}) = \sum_\nu a^{(\nu)} u^{(\nu)}_\alpha(\vec{r}) + \sum_\lambda b^{(\lambda)\dagger} v^{(\lambda)}_\alpha(\vec{r})\,, \tag{4}$$

and the new vacuum is defined by the conditions

$$a^{(\nu)}\,|0\rangle = b^{(\lambda)}\,|0\rangle = 0\,. \tag{5}$$

Up to a discarded additive constant, $\hat{H}$ takes the form

$$\hat{H} = \sum_\nu a^{(\nu)\dagger}a^{(\nu)} E_\nu + \sum_\lambda b^{(\lambda)\dagger}b^{(\lambda)} E_\lambda\,, \tag{6}$$

and the vacuum expectation value of the charge-density operator can be written

$$\rho_{\rm vac}(\vec{r}) = \langle 0|\hat{\rho}(\vec{r})|0\rangle = \langle 0|\tfrac{1}{2}|e|[\psi_\alpha(\vec{r}), \psi_\alpha^\dagger(\vec{r})]|0\rangle = \tfrac{1}{2}|e|\left\{\sum_{\alpha,\nu}|u_\alpha^{(\nu)}(\vec{r})|^2 - \sum_{\alpha,\lambda}|v_\alpha^{(\lambda)}(\vec{r})|^2\right\}. \quad (7)$$

It is important to notice that in (7) all positive-energy solutions contribute with one sign and all negative-energy solutions with the other. This is a sufficient condition for the formal applicability of the calculational procedure of KW. Applied to a "reasonable" nuclear charge distribution of even such large Z as presently considered, one obtains vacuum polarization corrections which are relatively minor emendations[7] to the given charge density. This is self-consistent in the sense that we may thus view $ZU(r)$ in the conventional sense as the potential measured in a *gedanken* electron-scattering experiment. From the point of view of the old picture, however, we are dealing with a nucleus of charge $Z+2$ to which two $1S$ electrons are strongly bound.

What other advantages accrue to the new choice of vacuum? First, it gives continuity of description as Z increases through the value Z_1 beyond which the old choice and language are inapplicable. Moreover, as additional negative-energy bound states of $\mathcal{H}$ occur, we may again modify the definition of the vacuum.[6] Second, the old vacuum state for $137 < Z < Z_1$ appears as an excited state of the system, i.e., two positrons are bound to the potential $ZU(r)$ with positive total energy. As Popov has shown,[3] the negative-energy bound electron moves in an effective potential which is nonmonotonic. It is easy to see that the bound positron feels this same potential, whose attractive part becomes shallower as Z increases, thus decreasing the binding energy of the positron. Thus in the present picture, the phenomena of instability of the vacuum and one-electron states for $Z = Z_1$ are reinterpreted by the statements that the positron bound-state wave function becomes unbound for this value of the charge and there can be either two or one such bound positrons. Third, the entire machinery of the Feynman-Dyson-Schwinger version of field theory can be applied for $Z > Z_1$ with the new definition of the vacuum.

Even though for $Z > Z_1$ the new choice of vacuum becomes mandatory, for $Z < Z_1$ one can if one wishes maintain the old description. In terms of the latter, however, previous discussions of the vacuum polarization based on the KW result are incomplete. Let us consider the old vacuum state. We try to describe it by using a charge distribution extrapolated from observable values of Z. We would then naturally say that we are talking about a nucleus of charge $Z+2$. With this description, in which the "Fermi level" is put at energy $-m$, the old and new expressions for the vacuum polarization can be related by a simple rearrangement of terms. Thus, when the definition (7) is applied to the old vacuum, the result can be rewritten in the form

$$\rho_{\rm vac}^{(0)}(Z+2, \vec{r}) = \rho_{\rm vac}(Z+2, \vec{r}) + 2|e|\sum_\alpha |v_\alpha^{(B)}(\vec{r})|^2, \quad (8)$$

where the first term on the right-hand side has the same formal structure as Eq. (7) except that Z has been replaced by $(Z+2)$. This means that the finite part of the first term alone will yield the KW result.[8] There is, however, an additional aspect to the argument. Since the total vacuum charge must vanish, the first term on the right-hand side of (8), besides effecting this minor structural redistribution of the given nuclear charge, must also renormalize that charge from $Z+2$ back to Z. Thus, roughly speaking, the old vacuum now appears as a nucleus of charge Z to which two positrons are bound [the second term of (8)]. But this is precisely our new view of the old vacuum. It is perhaps fair to say that although the effect of vacuum polarization in the domain $137 < Z < Z_1$ is not decisive quantitatively, it is essential qualitatively from the point of view of the old definition of the vacuum. If we could invent a system with otherwise similar properties in which the lowest bound level had a degeneracy comparable with the inducing nuclear charge number, the quantitative effects of vacuum polarization would also be decisive.

We have implied above that quantization with the new vacuum is *de rigeur* for $Z > Z_1$. This is not strictly correct since we can in principle use any complete set of states. Thus we can expand any eigenstate for $Z > Z_1$ in terms of those for $Z < Z_1$, and the latter can be constructed using the old vacuum. Such a description underlies the treatment of the instability of the one-electron state in the neighborhood of $Z = Z_1$ given recently by Muller *et al.*[4] and can also be used to extend their results. This method turns out to be mathematically convenient for a restricted class of problems as long as $Z - Z_1$ is not too great.

Note added in proof. In addition to the references cited in the text J. Rafelski has called to our notice more recent references. Thus the results of Ref. 4 have been elaborated by B. Müller, J. Rafelski, and W. Greiner [Z. Phys. 257, 183 (1972)]. These authors have, moreover, obtained independently results in essential agreement with those given here: cf. J. Rafelski, B. Müller, and W. Greiner [Nucl. Phys. (to be published)].

The authors are deeply indebted to Professor W. Greiner for his hospitality at the Institute for Theoretical Physics of the University of Frankfurt, where their interest in this problem was aroused. One of the authors (L.F.) acknowledges the Alexander Von Humboldt Stiftung for the support which made his stay in Frankfurt possible. We are grateful to W. Greiner, J. Rafelski, G. E. Brown, and L. Wilets for stimulating and helpful remarks.

---

*Work supported in part by the U. S. Atomic Energy Commission.

[1]W. Pieper and W. Greiner, Z. Phys. 218, 327 (1969).

[2]V. S. Popov, Yad. Fiz. 12, 429 (1970) [Sov. J. Nucl. Phys. 12, 235 (1971)].

[3]V. S. Popov, Zh. Eksp. Teor. Fiz. 59, 965 (1970) [Sov. Phys.—JETP 32, 526 (1972)].

[4]B. Müller, H. Peitz, J. Rafelski, and W. Greiner, Phys. Rev. Lett. 28, 1235 (1972).

[5]E. H. Wichmann and N. M. Kroll, Phys. Rev. 101, 843 (1956).

[6]From the calculations of Ref. 1, it appears that also the $2P_{1/2}$ states have negative energies for $Z = Z_1$. This requires no modification in principle of the arguments which follow, but only one of detail.

[7]This is substantiated by the numerical calculations of Ref. 1 and the observations in Ref. 3. We do not wish, however, to gainsay the essential quantitative role of vacuum polarization for the spectra of μ-mesic atoms. See in this regard M. S. Dixit, H. L. Anderson, C. K. Hargrove, R. J. McKee, D. Kessler, H. Mes, and A. C. Thompson, Phys. Rev. Lett. 27, 878 (1971).

[8]This statement is accurate to the order Z^{-1} only. We would need a fully self-consistent treatment of vacuum polarization to make up this small discrepancy.

PHYSICAL REVIEW D VOLUME 8, NUMBER 8 15 OCTOBER 1973

SU(N)-Singlet Gluon Model in Less-Than-Four Dimensions*

Sun-Sheng Shei and Tung-Mow Yan

Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14850

(Received 5 June 1973)

The interaction $g_0 \bar{\psi}\gamma_\mu \psi\phi^\mu$ for an SU(N)-singlet gluon and an N-component fermion is studied in a space-time of dimension $d = 4 - \epsilon$, slightly less than four. Wilson's prescription for coupling-constant renormalization is applied so that the theory is scale-invariant at short distances with anomalous dimensions. General properties of the model at high energies are discussed by using Callan-Symanzik equations. Anomalous dimensions of various operators including the symmetric and traceless tensors are calculated to order ϵ and are generally large except for the energy-momentum tensor. A large correction to order ϵ is also found for the parton-model prediction of the "e^+-e^- annihilation" total cross section. These results are qualitatively different from those in $\lambda\phi^4$ scalar field theory in which these corrections are found to be very small. We also discuss briefly how a normal scaling Ward identity may become anomalous in the limit $\epsilon \to 0$.

I. INTRODUCTION

Wilson[1] has emphasized the relevance of the Gell-Mann–Low renormalization-group method[2] for strong interactions. Its importance lies in the fact that short-distance behavior of operator products is controlled by the properties of the underlying field theory near the fixed point which is a zero of the Gell-Mann–Low function even when the physical coupling constant is distinct from the eigenvalue.[1] Based on the general properties of the renormalization-group equations Wilson[3] has proposed that strong interactions are scale-invariant at short distances with anomalous dimensions. These anomalous dimensions will, for example, manifest themselves in deep-inelastic electron and neutrino scattering. However, the SLAC-MIT experiment on deep-inelastic electron scattering[4,5] suggests that if anomalous dimensions exist at all they are small.

Unfortunately none of the field theories in four dimensions is known to possess a nontrivial eigen-