[4]E. P. Wigner, Ann. Math. 40, 149 (1939).

[5]See, for instance, L. O'Raifeartaigh, Phys. Rev. Lett. 14, 575 (1965); Phys. Rev. 139, B1052 (1965).

[6]This possibility suggested itself during a conversation with H. van Dam.

[7]L. P. Staunton and H. van Dam, Nuovo Cimento Lett. (to be published).

[8]We use $g_{00}=-g_{11}=-g_{22}=-g_{33}=1$. Also,

$$[x_\mu, P_\nu] = -ig_{\mu\nu}\,,$$
$$[M_{\mu\nu}, P_\sigma] = i(g_{\mu\sigma}P_\nu - g_{\nu\sigma}P_\mu)\,,$$

and

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(g_{\mu\rho}M_{\nu\sigma} - g_{\mu\sigma}M_{\nu\rho} + g_{\nu\sigma}M_{\mu\rho} - g_{\nu\rho}M_{\mu\sigma})\,.$$

The angular momentum and boost generators are defined to be $J_i \equiv -\frac{1}{2}\epsilon_{ijk}M^{jk}$ and $K_i \equiv M_{i0}$.

[9]We define the oscillator variable operators:

$$j_1 \equiv -S_{23} = \tfrac{1}{2}(q_1q_2 + \pi_1\pi_2)\,,$$
$$j_2 \equiv -S_{31} = \tfrac{1}{4}(q_1^2 + \pi_1^2 - q_2^2 - \pi_2^2)\,,$$
$$j_3 \equiv -S_{12} = \tfrac{1}{2}(q_2\pi_1 - q_1\pi_2)\,,$$
$$k_1 \equiv S_{10} = \tfrac{1}{4}(q_1^2 - \pi_1^2 - q_2^2 + \pi_2^2)\,,$$
$$k_2 \equiv S_{20} = \tfrac{1}{2}(\pi_1\pi_2 - q_1q_2)\,,$$
$$k_3 \equiv S_{30} = \tfrac{1}{2}(q_1\pi_1 + \pi_2q_2)\,,$$
$$S_{50} = \tfrac{1}{4}(q_1^2 + q_2^2 + \pi_1^2 + \pi_2^2)\,,$$
$$S_{51} = \tfrac{1}{2}(-q_1\pi_1 + q_2\pi_2)\,,$$
$$S_{52} = \tfrac{1}{2}(q_1\pi_2 + q_2\pi_1)\,,$$
$$S_{53} = \tfrac{1}{4}(q_1^2 + q_2^2 - \pi_1^2 - \pi_2^2)\,.$$

[10]P. A. M. Dirac, J. Math. Phys. 4, 901 (1963).

[11]The oscillator part of the eigenfunctions is $\psi^{(n)}(q,p) = Q^{(n)}(q)\psi_0(q,p)$, where

$$\psi_0(q,p) = \exp\left\{-\frac{\beta}{2M}\left[(q_1^2+q_2^2) + i\frac{p_1}{m}(q_1^2-q_2^2) - 2i\frac{p_2}{m}q_1q_2\right]\right\}$$

and the functions $Q^{(n)}$ satisfy the differential equation

$$\left[\frac{M}{2\beta}(\pi_1^2+\pi_2^2) + i(q_1\pi_1+q_2\pi_2)\right]Q^{(n)} = nQ^{(n)}\,.$$

[12]This includes, of course, the operators in Eq. (15). Each set of different operators closes only on the appropriate wave function. Of course, within each Hilbert space, they may be replaced with their form within the space, i.e., by using Eq. (10).

[13]Note that we are discussing the same set of functions, the $\Psi^{(n)}$.

[14]It is amusing to note some similarities between this effect and the appearance of rigged Hilbert spaces in the usual treatment of chiral symmetries.

[15]L. Susskind, Phys. Rev. 165, 1535 (1968); 165, 1547 (1968). Comparison with this work, which involves a singular limit, is difficult.

[16]K. Bardakci and M. B. Halpern, Phys. Rev. 176, 1686 (1968). Note that the realization prescription given by these authors cannot be applied to the H_G considered in Ref. 1.

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Symmetry Breaking in a Field Theory of Currents. II

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We apply the formalism of our previous paper to the symmetry breaking of a field theory of currents with an underlying SU(2) × SU(2) structure. We find that the form of the symmetry breaking is in one-to-one correspondence with the nonlinear realization of the pseudoscalar fields; in particular, the requirement of vanishing of exotic commutators uniquely leads to the nonlinear realization of the σ model. The existence of canonical momentum for the pseudoscalar fields is established, and several examples are given; one of them is the momentum of the σ model. The problem of extending this model to SU(3) × SU(3) is also discussed.

INTRODUCTION

The idea that weak- and electromagnetic-interaction currents govern strong-interaction dynamics has received considerable impetus from the success of the SU(3)-symmetry scheme[1] and current algebras.[2] Carried to the extreme, this idea would imply that *all* strong-interaction effects can be understood in terms of the properties of those currents. There is, as yet, no compelling evidence for or against this possibility; it is, however, interesting to investigate whether it is theoretically possible to implement it, in particular, whether one can construct a field theory, in which the basic fields are currents. This problem becomes more attractive because those fields (or at least some of them) have directly measurable matrix elements, and therefore are closer to observation than conventional fields. One hopes, for instance, that some of the difficulties of conventional field theory might be absent in a field theory of currents.

A very elegant theory of this type has been presented by Sugawara.[3] In his model, one postulates a set of commutation relations among the currents, which are a generalization of the current algebras proposed by Gell-Mann[1] to a unitary group SU(n) or SU(n)⊗SU(n). A notable feature of this algebra is the absence of q-number Schwinger terms and the commutativity of the space components of the currents with each other. Then it turns out that, unless delicate cancellations occur, the requirement of relativistic invariance puts severe restrictions on the possible form of the energy-momentum tensor $\Theta^s_{\mu\nu}$ in terms of the current variables: it must be either bilinear or an infinite polynomial in the currents, and *it must be symmetric with respect to the underlying SU(n) or SU(n)⊗SU(n) group.*[4] The resulting form of $\Theta^s_{\mu\nu}$, in the case of an underlying SU(n)×SU(n) algebra generated by vector currents $V^i_\mu(x)$ and axial-vector currents $A^i_\mu(x)$ is[3]

$$\Theta^s_{\mu\nu} = -\frac{1}{2c}\left(\{V^i_\mu(x), V^i_\nu(x)\} + \{A^i_\mu(x), A^i_\nu(x)\} - \delta_{\mu\nu}[V^i_\rho(x)V^i_\rho(x) + A^i_\rho(x)A^i_\rho(x)]\right), \quad (1)$$

where curly brackets denote anticommutators. This form is seen to be consistent with relativistic invariance and leads to conserved vector and axial-vector currents.

Since many of the currents involved in (1) are not conserved in nature, it is desirable to modify the model in order to allow for this nonconservation. This modification is not unique; in the present paper we will follow the prescription developed by us in the general case.[4,5] In our model, the set of dynamical variables is enlarged to include the divergences of the nonconserved vector and axial-vector currents, denoted by $S^i(x)$ and $P^i(x)$, respectively. Furthermore, we assume that the symmetry-breaking term in $\Theta_{\mu\nu}$ is a world scalar and a function of $S^i(x)$ and $P^i(x)$ only. Thus our energy-momentum tensor has the form

$$\Theta_{\mu\nu} = \Theta^s_{\mu\nu} + \delta_{\mu\nu}\phi(S, P), \quad (2)$$

where $\Theta^s_{\mu\nu}$ is given by (1). We further assume that the S's and P's commute among themselves and with the space components of currents, while their commutators with the time components of currents are functions of S's and P's alone. This latter assumption, along with the PCAC-PCVC (partial conservation of axial-vector and vector currents) condition that identifies the S's and P's as Heisenberg (interpolating) fields for scalar and pseudoscalar particles, leads directly to the notion of nonlinear realizations[6] for such fields. By extensive use of Jacobi identities, we establish a set of functional differential equations relating the form of the symmetry breaking to the nonlinear realization for S's and P's. In the case of SU(2)(with conserved I_3) and SU(3) [with conserved SU(2)], which we examined in detail, we found that the equations determine the breaking term uniquely, once the form of the nonlinear realization is given, and vice versa. A detailed account of the model can be found in I.

In the present paper, we apply our model to the physically more interesting case of SU(2)⊗SU(2), with the vector SU(2) subgroup (isospin) conserved. We find again that the constraint equations uniquely determine the breaking once the nonlinear realization is given and vice versa. Our main result is that the vanishing of exotic commutators (i.e., commutators carrying isospin larger than unity) fixes the form of the breaking term and *leads uniquely to the nonlinear realization of SU(2) ⊗SU(2) characteristic of the σ model.*[7] The plan of the paper is as follows: In Sec. I, we apply the model to the case of SU(2)⊗SU(2), establish the differential equations involved, and solve these differential equations for the case of vanishing exotic commutators. In Sec. II, the problem of the existence of a canonical momentum to the P's, consistent with the equations of motion, is examined. In Sec. III, the formalism is applied to the case of SU(2)⊗SU(2), with only the third component of isospin conserved, and of SU(3)⊗SU(3) with isospin conserved. The difficulties in solving the problem for those cases are pointed out.

I. THE BREAKING OF CHIRAL SU(2)⊗SU(2)

In this section, we apply the formalism of Ref. 4 to the case of chiral SU(2)⊗SU(2) with conserved isospin. We have three vector currents $V^i_\mu(x)$ and three axial-vector currents $A^i_\mu(x)$ ($i = 1, 2, 3$) satisfying the Sugawara algebra.[3,8]

$$\begin{aligned}[V^i_\mu(x), V^j_\nu(y)] &= [A^i_\mu(x), A^j_\nu(y)] \\ &= \epsilon_{ijk}\{\delta_{\mu4}V^k_\nu(x) + \tfrac{1}{2}\delta_{\nu4}[V^k_\mu(x) + g_{\mu\sigma}V^k_\sigma(x)]\} \\ &\quad \times \delta(\vec{x} - \vec{y}) \\ &\quad + c\delta^{ij}(\delta_{\mu4}\partial^x_\nu + \delta_{\nu4}\partial^x_\mu)\delta(\vec{x} - \vec{y}).\end{aligned} \quad (3)$$

We assume that the vector SU(2) subgroup is conserved, i.e.,

$$\begin{aligned}\partial_\mu V^i_\mu(x) &= 0, \\ \partial_\mu A^i_\mu(x) &\equiv P^i.\end{aligned} \quad (4)$$

Furthermore,

$$[P^i(x), P^j(y)] = [P^i(x), V_a^j(y)] = [P^i(x), A_a^j(y)] = 0 \quad (a=1,2,3), \tag{5}$$

$$[V_4^i(x), P^j(y)] = i\epsilon^{ijk}P^k(y)\delta(\vec{x}-\vec{y}), \tag{6}$$

$$[P^i(x), A_4^j(y)] = iR^{ij}(x)\delta(\vec{x}-\vec{y}). \tag{7}$$

Equations (5) constitute one of the assumptions of our model.[4] Equation (6) follows from (4), and we further assume that $R^{ij}(x)$ in (7) is a function of the P's only.[4] (R^{ij} is symmetric and Hermitian.) *Knowledge of $R^{ij}(x)$ will fix the nonlinear transformation law of the P's.*

Our energy-momentum tensor is given by (2):

$$\Theta_{\mu\nu} = -\frac{1}{2c}\left(\{V_\mu^i(x), V_\nu^i(x)\} + \{A_\mu^i(x), A_\nu^i(x)\} - \delta_{\mu\nu}[V_\rho^i(x)V_\rho^i(x) + A_\rho^i(x)A_\rho^i(x)]\right) + \delta_{\mu\nu}\phi(P). \tag{8}$$

As pointed out in I, the equations of motion of the currents remain the same as in the symmetric theory (see Eq. (14) of I). The equation of motion of the new variables $P^i(x)$ is:

$$\partial_\mu P^i(x) = \frac{1}{2c}\left(\{R^{ij}(x), A_\mu^j(x)\} + \epsilon_{ijk}\{P^k(x), V_\mu^j(x)\}\right). \tag{9}$$

Following I, we choose a spherical basis in isospin space:

$$V_\mu^\pm \equiv V_\mu^1 \pm iV_\mu^2, \quad V_\mu^0 \equiv V_\mu^3$$

with similar definitions for $A_\mu^\pm$, $P^\pm$, A_μ^0, and P^0, and define (we drop the δ functions from the right-hand side for simplicity):

$$[P^+, A^+] \equiv (P^+)^2 F, \tag{10a}$$

$$[P^+, A^-] \equiv H, \tag{10b}$$

$$[P^+, A^0] \equiv P^+ K, \tag{10c}$$

$$[P^0, A^0] \equiv L. \tag{10d}$$

F, H, K, and L are functions of the P's only, and they are Hermitian.[4] Our main task is to connect those functions, which govern the nonlinear transformation of the P's, to the symmetry-breaking term ϕ. The equations relating them can be obtained from Appendix B of I. It turns out that they take a particularly simple form when expressed in terms of the variables:

$$\omega \equiv P^i P^i, \quad t \equiv P^0. \tag{11}$$

Then ϕ and F are functions of ω alone (i.e., isoscalars), and t dependence of the other functions becomes trivial:

$$H(\omega, t) = -t^2 F(\omega) + \tilde{H}(\omega), \tag{12a}$$

$$L(\omega, t) = t^2 F(\omega) + \tilde{L}(\omega), \tag{12b}$$

$$K(\omega, t) = tF(\omega). \tag{12c}$$

$\tilde{H}(\omega)$ and $\tilde{L}(\omega)$ are defined through Eqs. (12a) and (12b). They are related to $F(\omega)$ and $\phi(\omega)$ through the set of differential equations:

$$\tilde{H}(\omega) = 2\tilde{L}(\omega) + \omega F(\omega), \tag{13a}$$

$$2\tilde{H}(\omega)F(\omega) + [\tilde{H}(\omega) + \omega F(\omega)][\omega F'(\omega) - \tilde{H}'(\omega)] = 2, \tag{13b}$$

$$\phi'(\omega)[\tilde{H}(\omega) + \omega F(\omega)] = 1, \tag{13c}$$

where a prime denotes differentiation with respect to ω. These are the basic equations of our model. *Knowledge of any one of the functions H, L, F, or ϕ completely determines all the others through Eqs. (13)*; to see this explicitly, define new functions:

$$2G \equiv \tilde{H} + \omega F,$$

$$2D \equiv \tilde{H} - \omega F.$$

Then Eqs. (13) have the form

$$D = \tilde{L},$$

$$G = (2\phi')^{-1},$$

$$D(G-D) - 2\omega G D' = \omega.$$

for which the above statement becomes obvious.

As a physically interesting example, let us solve equations (13) for the case of *vanishing exotic commutators*. From equation (10a), this requirement implies:

$$F(\omega) = 0. \tag{14a}$$

Then Eqs. (13), along with Eqs. (12), yield the unique solution:

$$K(\omega) = 0, \tag{14b}$$

$$H(\omega) = 2L(\omega) = 2(\alpha-\omega)^{1/2}, \tag{14c}$$

$$\phi(\omega) = -(\alpha-\omega)^{1/2} + \sqrt{\alpha}, \tag{14d}$$

where α is an integration constant, and in (14d) we used the requirement the $\phi(0) = 0$ (symmetry limit). Notice that $\phi(\omega)$ is positive definite as given by (14d).

Going back to (7), we find that

$$R^{ij}(x) = \delta^{ij}(\alpha-\omega)^{1/2} \tag{15}$$

so that the nonlinear transformation of the P's becomes

$$[P^i(x), A_4^j(y)] = i\delta^{ij}(\alpha-\omega)^{1/2}. \tag{16}$$

This is precisely the transformation obtained in the σ model.[7] *Our model, along with the assumption of vanishing exotic commutators, leads uniquely to the σ-model realization for the pseudoscalar field.*

II. EXISTENCE OF CANONICAL MOMENTUM FOR P'S

As is well known, the symmetric Sugawara theory [Eq. (1)] is not canonical, i.e., there exists no canonical momentum for the currents $V^i_\mu(x)$ and $A^i_\mu(x)$ consistent with the equations of motion.[3] Since, in our model, the equations of motion of the currents remain the same, the above result still holds[9]; in this section, we want to investigate the possibility that the P's are canonical, i.e., that there exist operators $\pi^i(x)$ such that[8]

$$[\pi^i(x), P^j(y)] = -\delta^{ij}\delta(\vec{x}-\vec{y}). \tag{17}$$

We will restrict ourselves to the solution found for the case of vanishing exotic commutators, i.e., the transformation properties of the P's are given by Eqs. (6) and (16). The main task is to find a $\pi^i(x)$ such that (17) is compatible with the equation of motion (9); in fact, taking space derivatives of (17) on both sides and substituting (9) on the left-hand side, we get the constraint equation:

$$\begin{aligned}-c\delta^{ij}\,\partial^y_a\delta(\vec{x}-\vec{y}) = {} & [\pi^i(x), A^j_a(y)](\alpha-\omega)^{1/2} \\ & + \frac{P^i(y)A^j_a(x)}{(\alpha-\omega)^{1/2}}\,\delta(\vec{x}-\vec{y}) \\ & - \epsilon_{ijm}V^m_a(y)\delta(\vec{x}-\vec{y}) \\ & + \epsilon_{jmk}P^k(y)[\pi^i(x), V^m_a(y)],\end{aligned} \tag{18}$$

where a is a Lorentz index running from 1 to 3. We first look for a canonical momentum of the general form:

$$\pi^i(x) = F^{ij}(\omega)V^j_4(x) + G^{ij}(\omega)A^j_4(x), \tag{19}$$

where F^{ij} and G^{ij} are functions of ω only. Equation (17) then gives the restriction:

$$G^{ij} = \frac{1}{(\alpha-\omega)^{1/2}}[\delta^{ij} + \epsilon_{kjl}F^{ik}(x)P^l(x)]. \tag{20}$$

When we substitute (19) into (18), and use the relations

$$[\partial^{(x)}_a P^i(x), V^j_4(y)] = \epsilon_{ijk}P^k(y)\partial^{(x)}_a\delta(\vec{x}-\vec{y}), \tag{21a}$$

$$[\partial^{(x)}_a P^i(x), A^j_4(y)] = \delta^{ij}[\alpha-\omega(y)]^{1/2}\partial^{(x)}_a\delta(\vec{x}-\vec{y}), \tag{21b}$$

where $\partial^{(x)}_a \equiv \partial/\partial x_a$, $a = 1, 2, 3$,[10] we find that (18) is satisfied if (20) holds. Therefore *(19) and (20) define an acceptable canonical momentum.* A particularly simple special case of (19) corresponds to $F^{ij}=0$:

$$\pi^i(x) = \frac{1}{(\alpha-\omega)^{1/2}}A^i_4(x). \tag{22}$$

Another possible form of the canonical momentum is:

$$\pi^i(x) = \frac{c}{\alpha}\left[\delta^{ij} + \frac{1}{\alpha-\omega}P^i(x)P^j(x)\right]\partial_4 P^j(x). \tag{23}$$

One easily verifies, using (21a)–(21b), that the form (23) satisfies both requirements (17) and (18). This is precisely the form of the canonical momentum in the nonlinear σ model. Thus the analogy between the σ model and our formalism is quite far-reaching.

In conclusion, we found that although the Sugawara theory is still noncanonical (since the V's and A's have no canonical conjugate) it is possible to find canonical conjugates to the pseudoscalar fields, of which (19), (22), and (23) are special examples.

III. EXTENSIONS OF THE MODEL

In view of the success of the model thus far in obtaining unique nonlinear realizations, it is tempting to try to apply it to more complicated cases. A physically very interesting case is that of SU(3)⊗SU(3), with only the vector isospin subgroup conserved. Such extensions have been attempted without much success. To understand the main difficulty involved, let us consider the simpler case of SU(2)⊗SU(2) with only the third component of isospin conserved.

In this case, the current algebra is still given by (3), but, besides the three pseudoscalar fields $P^i(x)$ defined by (4), there will also be two scalar fields $S^+(x)$ and $S^-(x)$ corresponding to the nonconserved vector currents:

$$S^\pm(x) = \partial_\mu V^\pm_\mu(x). \tag{24}$$

The commutation relation (6) will not hold anymore; in fact the S's and P's will transform nonlinearly under both the vector and axial-vector SU(2) subgroups [except for the third component of isospin, $V^3_4(x)$]. By following the general techniques described in I, we can show that ϕ can be written in two different ways; either as

$$\phi = F[X, \bar{X}, Y, P^3] + F^*[\bar{X}, X, \bar{Y}, -P^3] \tag{25a}$$

or

$$\phi = G[X, Y, \bar{Y}, P^3] + G^*[\bar{X}, \bar{Y}, Y, -P^3], \tag{25b}$$

where

$$\begin{aligned} X &\equiv (S^+ + P^+)(S^- + P^-), \\ \overline{X} &\equiv (S^+ - P^+)(S^- - P^-), \\ Y &\equiv (S^+ + P^+)(S^- - P^-), \\ \overline{Y} &\equiv (S^+ - P^+)(S^- + P^-), \\ X\overline{X} &= Y\overline{Y}, \end{aligned} \tag{26}$$

(notice that all P's commute with all S's as well as among themselves). All these variables have zero third component of isospin, as they should. Because of this ambiguity and the fact that the variables are related by the nonlinear relation (26), the form of ϕ is not well defined. In particular, the relation between ϕ and the nonlinear transformation properties of the scalar and pseudoscalar fields may not be one-to-one. The same kind of ambiguity appears in the case of SU(3) $\otimes$ SU(3) when only the vector SU(2) is conserved, and have been noticed by the authors of Ref. 11.

In conclusion, we found that the formalism can be applied to the case of chiral SU(2)$\otimes$ SU(2) with conserved isospin; in this case a one-to-one connection between the form of the symmetry-breaking term and the nonlinear transformation property of the pseudoscalar fields is established. In particular, imposing the vanishing of exotic commutators leads unambiguously to the commutation relations of the nonlinear σ model. The existence of a canonical conjugate to the pseudoscalar fields was established in Sec. II, and several examples were exhibited. One of them was identical to the canonical momentum in the σ model.

Extension of our model to more complex cases, like that of SU(3)$\otimes$ SU(3) with only the vector SU(2) conserved, leads to ambiguities and it is not clear whether it provides a one-to-one connection between the form of the symmetry breaking and the nonlinear realization of the scalar and pseudoscalar fields. It may be that the ambiguity is connected to the fact that, e.g., the vanishing of exotic commutators does not fix uniquely the representation of the symmetry-breaking term in this case. [In SU(3)$\times$SU(3) vanishing exotic commutators allow a symmetry breaking which can be either $(3, 3^*) + (3^*, 3)$ or $(8, 1) + (1, 8)$ or a mixture of both.] These questions are being further investigated.

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[1]M. Gell-Mann, Phys. Rev. 125, 1067 (1962); Y. Ne'eman Nucl. Phys. 26, 222 (1961).

[2]See the book by S. Adler and R. Dashen, *Current Algebras* (Benjamin, New York, 1968).

[3]H. Sugawara, Phys. Rev. 170, 1659 (1968).

[4]Ch. Le Monnier de Gouville, N. Papastamatiou, and H. Umezawa, Phys. Rev. D 4, 2966 (1971), in particular Appendix A. This paper will be referred to as I in the following.

[5]References to other models for breaking the symmetry can be found in Ref. 4.

[6]S. Weinberg, Phys. Rev. 166, 1568 (1968).

[7]M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960).

[8]All commutators in this paper are at equal times, unless explicitly stated otherwise.

[9]The lack of canonical momentum should be considered as an advantage of the model, rather than as a liability, since it makes the theory different from a conventional vector field theory. See Ref. 3.

[10]These relations follow directly from (6) and (16). They can also be explicitly verified by using the equation of motion (9) for the P's.

[11]M. Ahmed and M. Taha, Phys. Rev. 188, 2517 (1969).