ing, since the transformations of Ref. 2 indicate that the general bound x-space potential $a_\mu(x)$ transforms into an unbounded hyperspherical potential $A_a(\eta)$. Note that although the upper bound of Eq. (112) diverges for $l=2$, exact calculation for this case, taking gauge invariance properly into account, indicates that $\tilde{W}_4^{(0)}[A]$ is convergent.

[9]A. Salam and P. T. Matthews, Phys. Rev. 90, 690 (1953); J. Schwinger, *ibid.* 93, 615 (1954). These authors actually make fewer subtractions than we do in the argument of Eqs. (116)–(118), because they claim that $\mathrm{Tr}(KK^\dagger)$ is finite. However, although $\mathrm{Tr}(KK^\dagger)$ is formally a four-point function, it does not have the propagator and vertex factors arranged in the correct order to be the usual gauge-invariant four-point function. Since the four-point function is only conditionally and not absolutely convergent, this suggests that $\mathrm{Tr}(KK^\dagger)$ will be divergent, and that the Fredholm argument of Matthews and Salam and Schwinger will need additional subtractions to be made precise.

[10]B. Ja. Levin, *Distribution of Zeros of Entire Functions,* Translations of Mathematical Monographs Vol. 5 (American Mathematical Society, Providence, R. I., 1964), p. 17. I wish to thank A. S. Wightman for conversations which suggested the argument of property (iv).

[11]We note that our argument fails when the number of modes kept is infinite. As $N_k\to\infty$, the factor $\Gamma(n+\frac{1}{2}N_k)/\Gamma(\frac{1}{2}N_k)$ in Eq. (26) approaches unity, and so Eq. (126) becomes

$$w_\infty[0]=\sum_{n=1}^{\infty} w_\infty^{(n)}(2e^2)^n .$$

Although this equation no longer contains a combinatoric factor which grows as $n!$ for large n, we can draw no conclusion about its radius of convergence, because the estimate of Eq. (124) for an upper bound $R_{\min}{}^{-1}$ on

$$\lim_{n\to\infty}\sup|w_k^{(n)}|^{1/n}$$

diverges as $k\to\infty$. Nonanalyticity results related to ours are given by E. R. Caianiello, A. Campolattaro, and M. Marinaro [Nuovo Cimento 38, 1777 (1965)] and D. Kershaw (unpublished).

[12]The imaginary contour of course lies outside the sector of Eq. (129), and hence cannot be developed in an asymptotic expansion in e^2 by direct term-by-term integration. It may still have an asymptotic development agreeing with perturbation theory after an appropriate analytic continuation.

[13]S. Coleman and S. B. Treiman (unpublished). Their argument takes into account the presence of the asymptotically subdominant parts of w_k, which give rise to a "background" amplitude integral which is analytic at e_k.

[14]This clustering is implied by Thm. 11 on p. 21 of Levin, Ref. 10.

[15]F. J. Dyson, Phys. Rev. 85, 631 (1952).

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Approach to Scaling in Renormalized Perturbation Theory

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The consequences of the large magnitude of the bare coupling constant in Wilson's theory of critical phenomena are examined for renormalized field theory in $4-\epsilon$ dimensions. The scaling behavior of the correlation functions, the relations among critical exponents, and the existence of a scaling equation of state, regular in the temperature around T_c, are then obtained in this framework. Some corrections to the scaling laws are also discussed and are shown to be dependent on another exponent.

I. INTRODUCTION

In a previous work[1] we have studied the existence of asymptotic scaling forms for the correlation functions near the critical point within the framework of Wilson's theory[2] in $d=4-\epsilon$ dimensions. The main tool was the use of renormalized perturbation theory and of the Callan-Symanzik equations.[3] All renormalizations were performed at zero momenta. It was shown then that the renormalized coupling constant was fixed at a nontrival solution of an eigenvalue condition simply because one lets the bare coupling constant go to infinity. This corresponds to the physical situations in which the bare coupling is measured in units of the inverse lattice spacing a^{-1}, whereas the masses and relevant momenta are proportional to the inverse correlation length ξ^{-1}, and to the fact that, in the vicinity of the critical point, ξ is much greater than a.

However, the scaling behavior of the correlation functions is not sufficient to obtain all the scaling laws. In addition there are the Widom-Kadanoff[4] relations among the critical exponents; there is also an equation of state, i.e., a relation between the applied field H, the magnetization M, and the temperature T, in scaling form:

$$\frac{H}{M^\delta}=f\left[\frac{(T-T_c)}{M^{1/\beta}}\right]. \tag{1}$$

In Refs. 5 and 6 the exponents β and δ, as well as the function f, were calculated in powers of ϵ, using Wilson's original bare perturbation theory with a momentum cutoff Λ. It was then verified that β and δ were in agreement through the scaling laws, with the value of η and γ previously calculated by Wilson.[2,7] Here we try to understand in the framework of renormalized perturbation theory, where no cutoff is needed, why these relations are satisfied, and why there exists an equation of state in the scaling form (1). All the static scaling laws can, in fact, be derived in this formalism. The assumptions which enter in these derivations are justified order by order in ϵ.

Furthermore, it turns out that the same methods are useful to look for possible deviations to the scaling laws.[8] Let us mention the first correction in a/r to the spatial correlation functions, corrections to the power behavior in $T-T_c$ of the correlation length and of the susceptibility, corrections to the equation of state in stronger fields, etc. These deviations are governed by a "sub-critical" exponent[9] ω, which is calculated here up to order ϵ^3.

Section II is a summary of earlier results. In Sec. III a more general renormalization scheme is introduced, which turns out to be useful in the following. It also enables us to make the connection with the approach of Mack[10] and Schroer[11] to the same problem.[12] Section IV is devoted to the derivation of the existence of a scaling equation of state[5,6] and of its properties. In Sec. V, corrections to scaling laws are discussed. Some of the calculations are given in the appendixes. In particular, Appendix B contains a derivation of the connection between the correlation length and the temperature, and Appendix D is devoted to the logarithmic corrections in four dimensions.

II. SUMMARY OF PREVIOUS RESULTS

We shall first review the notations and results of a previous study[1] of scaling laws in $4-\epsilon$ dimensions.

The renormalized Lagrangian[13] density is

$$\mathcal{L}(x) = -\tfrac{1}{2}\left[Z_3 \sum_{i=1}^{d} (\partial_i \vec{\phi})^2 + m^2 \vec{\phi}^2 \right] - \tfrac{1}{2}(Z_3 m_0^2 - m^2)\vec{\phi}^2 - \frac{u m^\epsilon}{4!} Z_1 (\vec{\phi}^2)^2 , \tag{2}$$

where $\vec{\phi}^2 = \sum_1^N [\phi^\alpha(x)]^2$, N being the number of components of the order parameter, and u is a dimensionless coupling constant.

When d is smaller than four, this theory is super-renormalizable and Z_1 and Z_3 are finite quantities. Nevertheless, in order to obtain finite limits when $\epsilon = 4-d$ goes to zero, we perform all the renormalizations needed in 4 dimensions. If we call $\Gamma^{(n)}(p_1, \ldots, p_n; m, u)$ the one-particle irreducible vertex functions[14] ($\Gamma^{(2)}$ is the inverse propagator, $\Gamma^{(4)}$ the amputated 4-point function, etc.), we choose the renormalization conditions to be

$$\Gamma^{(2)}(p, -p; m, u)|_{p^2=0} = m^2 , \tag{3a}$$

$$\frac{\partial}{\partial p^2} \Gamma^{(2)}(p, -p; m, u)|_{p^2=0} = 1 , \tag{3b}$$

$$\Gamma^{(4)}(0,0,0,0; m, u) = u m^\epsilon . \tag{3c}$$

The vertex functions $\Gamma^{(n)}$ satisfy the Callan-Symanzik (CS) equations[3]:

$$\left[m \frac{\partial}{\partial m} + \beta(u) \frac{\partial}{\partial u} - \frac{n}{2} \gamma_3(u) \right] \Gamma^{(n)}(p_i; m, u) = [2 - \gamma_3(u)] m^2 \Delta\Gamma^{(n)}(p_i; m, u) , \tag{4a}$$

where the $\Delta\Gamma^{(n)}$ are n-point vertex functions with one mass insertion normalized by

$$\Delta\Gamma^{(2)}(p=0; m, u) = 1 . \tag{4b}$$

The $\Delta\Gamma^{(n)}$ are negligible, order by order in perturbation theory, with respect to $\Gamma^{(n)}$ when all the momenta are much larger than m. The functions $\beta(u)$ and $\gamma(u)$ are finite even when ϵ vanishes, and they are defined by

$$\beta(u) = -\epsilon \left\{ \frac{d}{du} \ln\left[\frac{u Z_1(u)}{Z_3^2(u)} \right] \right\}^{-1} \tag{5}$$

$$\gamma_3(u) = \beta(u) \frac{d \ln Z_3}{du} . \tag{6}$$

It is important to keep in mind the statistical-mechanical origin of the problem. In particular, the bare coupling constant

$$g_0 = m^\epsilon u Z_1(u)/Z_3^2(u) \tag{7}$$

is given as $g_0 = u_0 \Lambda^\epsilon$, where u_0 is dimensionless and Λ is a mass. This mass is of order a^{-1}, where a is the lattice spacing, and it is then very large compared to the mass m and to the physical momenta, which are all of order ξ^{-1}. In Ref. 1 it was shown that the fact that Λ is much greater than m fixes u to a value u_∞ which is the solution of $\beta(u)=0$ and $\beta'(u)>0$.

Then the vertex functions can be obtained as power series in u_∞, which is of order ϵ. In addition, their asymptotic behavior for $p \gg m$, i.e., distances much smaller than the correlation length ξ, is governed by the CS equations (4a) with a right-hand side which may be neglected in this limit. We thus obtain

$$\left(m \frac{\partial}{\partial m} - \frac{n}{2} \eta \right) \Gamma^{(n)}_{\text{as}}(p_i; m, u_\infty) = 0 , \tag{8}$$

with

$$\eta \equiv \gamma_3(u_\infty) . \tag{9}$$

Integrating Eq. (8) we find

$$\Gamma^{(n)}_{\text{as}}(\lambda p_i; m, u_\infty) \underset{\lambda\to\infty}{\sim} \lambda^{d-n(d-2+\eta)/2} . \tag{10}$$

Similarly it will be necessary to consider the vertex functions, with one ϕ^2 insertion, of the type $\langle \phi^2(y)\phi(x_1)\cdots\phi(x_n)\rangle$. The corresponding one-particle irreducible vertex functions $\Gamma^{(1,n)}(q; p_i, \ldots, p_n; m, u)$, which appeared already for $q=0$ in the right-hand side of (4a), are renormalized with the condition

$$\Gamma^{(1,2)}(0;0,0;m,u) = \Delta\Gamma^{(2)}(0,0;m,u) = 1 . \tag{11}$$

This means that an additional wave-function renormalization $Z_4(u)$ (finite in $d<4$ dimensions) has been introduced. To be precise this means, in terms of the bare vertex functions, that $\Gamma^{(1,n)} = Z_3^{n/2}(Z_4/Z_3)\Gamma^{(1,n)}_{\text{bare}}$. These functions satisfy the CS equations:

$$\left[m\frac{\partial}{\partial m} + \beta(u)\frac{\partial}{\partial u} - \left(\frac{n}{2}-1\right)\gamma_3(u) - \gamma_4(u)\right]\Gamma^{(1,n)} = \Delta\Gamma^{(1,n)} , \tag{12}$$

where the right-hand side contains an extra mass insertion and is then also negligible, when the momenta are large, with respect to the left-hand side. The function $\gamma_4(u)$ is computed from

$$\gamma_4(u) = \beta(u)\frac{d\ln Z_4(u)}{du} . \tag{13}$$

For $u=u_\infty$ and when the momenta are large, the integration of (12) gives

$$\Gamma^{(1,n)}(\lambda q; \lambda p_i; m, u_\infty) \underset{\lambda\to\infty}{\sim} \lambda^{(d-2+\eta)(1-n/2)-\gamma_4(u_\infty)} . \tag{14}$$

It will be shown in Appendix B that $\gamma_4(u_\infty)$ is related to physical properties of the system through

$$\frac{2-\eta}{\gamma} = \nu^{-1} = 2-\eta+\gamma_4(u_\infty) , \tag{15}$$

where ν is the critical exponent which governs the divergence of the correlation length when T goes to T_c. The results of the ϵ expansion of these formulas are[15]:

$$\gamma_4(u_\infty) = -\frac{\epsilon(N+2)}{N+8}\left[1+\frac{6(N+3)}{(N+8)^2}\epsilon + \frac{36(3N+14)(N+3)}{(N+8)^4}\epsilon^2 - \frac{12(5N+22)}{(N+8)^3}\zeta(3)\epsilon^2 - \frac{(N^2+222N+560)}{4(N+8)^3}\epsilon^2\right] + O(\epsilon^4) , \tag{16}$$

$$\eta = \gamma_3(u_\infty) = \frac{\epsilon^2(N+2)}{2(N+8)^2}\left\{1+\left[\frac{6(3N+14)}{(N+8)^2}-\frac{1}{4}\right]\epsilon + \left[\frac{1}{(N+8)^4}\left(-\tfrac{5}{16}N^4 - \tfrac{115}{8}N^3 + \tfrac{281}{4}N^2 + 1120N + 2884\right) - \frac{24(5N+22)}{(N+8)^3}\zeta(3)\right]\epsilon^2\right\} + O(\epsilon^5) .$$

III. RENORMALIZED PERTURBATION THEORY WITH A LARGE BARE COUPLING CONSTANT

Previously[1] we have discussed what happens in the massive theory with subtractions made at zero momentum. This was sufficient to describe the scaling laws in the critical region above T_c. In some cases, and in particular for the equation of state in the critical region, it is necessary to go through T_c. This corresponds to a correlation length which becomes infinite, and the mass m vanishes (Appendix B). Therefore it is necessary to make the subtractions at an external mass μ in order to avoid infrared divergences. This will also allow us to make the connection with a different approach followed in Refs. 10 and 11. Specifically, the renormalization conditions will be

$$\bar\Gamma^{(2)}(p,-p;\bar m,\bar u,\mu)|_{p^2=0} = \bar m^2 , \tag{17a}$$

$$\bar\Gamma^{(2)}(p,-p;\bar m,\bar u,\mu)|_{p^2=\mu^2} = \bar m^2+\mu^2 , \tag{17b}$$

$$\bar\Gamma^{(4)}(p_1,\ldots,p_4;\bar m,\bar u,\mu)|_S = \bar u(\bar m^2+\mu^2)^{\epsilon/2} , \tag{17c}$$

where S is the symmetry point $p_i\cdot p_j = \frac{1}{3}\mu^2(4\delta_{ij}-1)$.

The new vertex functions $\bar\Gamma^{(n)}$ are related to the previous ones (which correspond to the choice $\mu=0$) by

$$\bar\Gamma^{(n)}(p_i;\bar m,\bar u,\mu) = [Z(\mu/m,u)]^{n/2}\,\Gamma^{(n)}(p_i;m,u) . \tag{18}$$

Once μ is chosen, the conditions (17) and the relation (18) determine $\bar m, \bar u$, and the function Z in terms of m, u, and μ. The results are:

$$\bar m^2 = Z(\mu/m,u)m^2 , \tag{19}$$

$$Z(\mu/m, u) = \mu^2[\Gamma^{(2)}(p, -p; m, u)|_{p^2=\mu^2} - m^2]^{-1}, \tag{20}$$

$$(\bar{m}^2 + \mu^2)^{\epsilon/2}\,\bar{u} = \Gamma^{(4)}(p_i; m, u)|_S \times [\Gamma^{(2)}(p; m, u)|_{p^2=\mu^2} - m^2]^{-2}\mu^4. \tag{21}$$

The bare coupling constant $g_0 = u_0\Lambda^\epsilon$ may be expressed in two different ways, in terms of the renormalization constants of the theory

$$u_0\Lambda^\epsilon = m^\epsilon u Z_1(u) Z_3^{-2}(u) = (\bar{m}^2 + \mu^2)^{\epsilon/2}\bar{u}\,\bar{Z}_1(\bar{Z}_2)^{-2}. \tag{22}$$

And now we remember that Λ is much larger than m, (or equivalently $a \ll \xi$). This then fixes u to some value u_∞, up to a term which vanishes with m/Λ. It was shown in Ref. 1 that $u - u_\infty$ is proportional to $(m/\Lambda)^\omega$, where

$$\omega = \beta'(u_\infty).$$

But, from Eq. (22), it is clear that concerning the theory renormalized with the conditions (17), two possibilities are still opened:

(i) μ is taken much smaller than Λ; then the situation is similar to the previous one, i.e., $\bar{u}$ is fixed at some value $\bar{u}_\infty$ (which depends on the ratio m/μ).

(ii) μ is taken of order Λ; then $\bar{u}$ remains a function of u_0 and is not fixed to any special value. When $\bar{m} = 0$, this is how one recovers the case studied in Refs. 10 and 11.

Let us examine separately what happens to the vertex-functions in these various situations.

(a) If $\mu = 0$ (or $\mu \ll \Lambda$), for large momenta $p_i \gg m$ the right-hand side of Eq. (4a) is negligible and for $u = u_\infty$ the asymptotic behavior is given by Eq. (10). In this case, it is instructive, if Λ is not strictly infinite, to compute the first correction in m/Λ (Sec. V), which gives for $\Gamma^{(2)}$

$$\Gamma^{(2)}_{\text{as}}(p; m, u) \simeq \Gamma^{(2)}_{\text{as}}(p; m, u_\infty) \times \left[1 + A(u_0)\left(\frac{m}{\Lambda}\right)^\omega + B(u_0)\left(\frac{p}{\Lambda}\right)^\omega\right]. \tag{23}$$

The value $u = u_\infty$ of the coupling constant is unstable for the short-distance limit since $\omega = \beta'(u_\infty)$ is positive.[1] This fact is reflected here by the appearance of a correction term which, for $p \gg \Lambda$, is larger than the first one. Nevertheless it may indeed be neglected in the range $m \ll p \ll \Lambda$.

(b) The other possibility is to take μ of order Λ. For $\bar{m} = 0$ it has been shown in Refs. 10, 11 that scaling laws hold in the long-distance limit $p \ll \mu$ for an arbitrary value of the coupling constant $\bar{u}$ (in some range around $\bar{u}_\infty$). Let us note that the condition $p \ll \mu$ is indeed satisfied by the physical momenta since μ is of order Λ.

If $\bar{m}$ does not vanish, the easiest way of obtaining the vertex functions $\bar{\Gamma}^{(n)}$ is to relate them to the $\Gamma^{(n)}$ through Eq. (18). In particular, this equation enables us to determine the behavior of $\bar{\Gamma}^{(n)}$ for the range of momenta m, $\bar{m} \ll p \ll \mu$, for which the $\Gamma^{(n)}$ may be replaced by their asymptotic form at short distance. Furthermore Eq. (20) shows that the large-(μ/m) limit of $Z(\mu/m, u)$ is also related to the short-distance behavior of $\Gamma^{(2)}$.

In addition, from Eq. (21), we shall derive in Sec. V that, when μ/m is large and $\bar{u}$ fixed, u approaches u_∞ according to

$$u - u_\infty \underset{\mu/m \gg 1}{\simeq} A(\bar{u})\left(\frac{m}{\mu}\right)^\omega. \tag{24}$$

In the same limit the corresponding behavior of $Z(\mu/m, u)$ is established in Appendix A, where it is shown that it behaves as $B(\bar{u})(\mu/m)^\eta$.

For u close to u_∞ we may expand $\bar{\Gamma}^{(n)}$ as:

$$\bar{\Gamma}^{(n)}(p_i; \bar{m}, \bar{u}, \mu) \underset{\bar{m} \ll p \ll \mu}{=} \bar{\Gamma}^{(n)}_{\text{as}}\left[1 + (u - u_\infty)\frac{\partial}{\partial u}\ln\bar{\Gamma}^{(n)}|_{u_\infty}\right], \tag{25}$$

where

$$\bar{\Gamma}^{(n)}_{\text{as}} = Z_{\text{as}}^{\,n/2}(\mu/m, u)\Gamma^{(n)}_{\text{as}}(p_i; m, u_\infty). \tag{26}$$

The asymptotic behavior of $\Gamma^{(n)}$ is given by Eq. (10). For $(\partial/\partial u)\ln\Gamma^{(n)}|_{u_\infty}$, we obtain it from the CS equation (4), after differentation with respect to u. The combined results yield

$$\bar{\Gamma}^{(n)}(p_i; \bar{m}, \bar{u}, \mu) \underset{\bar{m} \ll p \ll \mu}{=} B^{n/2}(\bar{u})\mu^{n\eta/2}a^{(n)}(p_i) \times \left[1 + \frac{n}{2}\frac{\gamma'(u_\infty)}{\omega}A(\bar{u})\left(\frac{m}{\mu}\right)^\omega + \frac{A(\bar{u})}{\mu^\omega}b^{(n)}(p_i)\right]. \tag{27}$$

The functions $b^{(n)}$ have the dimension of a momentum raised to the power ω, and thus represent a small correction only if $p \ll \mu$ [which is the equivalent of the condition $p \ll \Lambda$ of case (a)]. The mass m is related to $\bar{m}$ and μ by (Appendix A)

$$\frac{m^2}{\bar{m}^2} \underset{\mu/m \gg 1}{=} B(\bar{u})(\mu/m)^\eta. \tag{28}$$

In the leading term of Eq. (27), $\bar{m}$ has disappeared and the result coincides with the result of the massless theory. This was expected since the $\bar{\Gamma}^{(n)}$ functions are infrared-convergent when $\bar{m}$ goes to zero.

This shows that, from both points of view

(i) $\mu = 0$, $m/\Lambda \ll 1$

(ii) $\mu/\Lambda \sim 1$, $\bar{m}/\mu \ll 1$

we can derive the same scaling behavior of the correlation functions, and obtain the first corrections. In the next sections, the second possibility will be exploited, because we need the equivalent of a cutoff to fix the scale and the renormalization-group equations are much simpler with that scheme than with a momentum cutoff.

IV. EQUATION OF STATE

The equation of state relates the magnetization M to the applied field H and the temperature T. In the vicinity of the critical point, scaling laws postulate that the system is characterized by a single thermodynamic parameter, for instance $(T-T_c)M^{1/\beta}$, and the equation of state takes the form

$$H/M^{\delta}=f\big((T-T_c)/M^{1/\beta}\big).$$

In order to study this relation, a source term $\vec{H}\vec{\phi}$ is added to the Lagrangian (2). We are thus in a situation identical to that of the linear σ model,[16] with the symmetry broken linearly. The renormalization of this theory has been discussed in great detail,[17] and some of the results obtained there will be used in the following. The expectation value

$$\vec{M}=\langle\vec{\phi}\rangle \tag{29}$$

is then computed in terms of $H, \overline{m}, \overline{u}$, and μ. If we then eliminate $\overline{m}$ in terms of the temperature (Appendix B), we shall obtain the equation of state.

In a first step, we want to show that the scaling relation

$$H/M^{\delta}=\mu^{(d+2)/2-\delta(d-2)/2}\times F(M\overline{m}^{\,2/(1-\delta)}\mu^{2/(\delta-1)-(d-2)/2}) \tag{30}$$

is satisfied.

The renormalization-group equations (18)–(21), together with the CS equations, will enable us to derive the relation (30).

We consider the Legendre transform of the connected vacuum-to-vacuum amplitude (or free energy) in presence of a uniform source whose expansion is

$$\overline{\Gamma}(M;\overline{m},\overline{u},\mu)=\sum_2^{\infty}\frac{M^n}{n!}\,\overline{\Gamma}^{(n)}(p_i=0;\overline{m},\overline{u},\mu), \tag{31}$$

in terms of which the field is obtained as

$$H_{\alpha}=\frac{\partial}{\partial M_{\alpha}}\,\overline{\Gamma}(M;\overline{m},\overline{u},\mu). \tag{32}$$

In these formulas, the $\overline{\Gamma}^{(n)}$ still refer to the symmetric theory without source. Now, using Eq. (18), we can replace (31) by

$$\overline{\Gamma}(M;\overline{m},\overline{u},\mu)=\sum_2^{\infty}\frac{M^n}{n!}Z^{n/2}(u;\mu/m)\times\Gamma^{(n)}(p_i=0,m,u). \tag{33}$$

By dimensional arguments we know that

$$\Gamma^{(n)}(p_i=0,m,u)=m^{d-(n/2)(d-2)}C^{(n)}(u). \tag{34}$$

This, together with Eq. (33), implies that

$$\overline{\Gamma}(M;\overline{m},\overline{u},\mu)=m^2M^2Z(u;\mu/m)\Phi\left(\frac{MZ^{1/2}}{m^{1-\epsilon/2}},u\right), \tag{35}$$

where m and u are related to $\overline{m},\overline{u}$, and μ by Eqs. (19)–(21).

We are now again interested in the limit $m/\mu \ll 1$, with fixed $\overline{u}$. The behavior of Z is known (Appendix A) and in the function Φ, provided one considers magnetizations in the range $M/\mu^{1-\epsilon/2}\ll 1$, the coupling constant u may be replaced by u_∞ as discussed in Sec. V. Thus in this limit, Eq. (35) becomes

$$\overline{\Gamma}(M;\overline{m},\overline{u},\mu)\sim B(\overline{u})m^2M^2(\mu/m)^{\eta}\times\Phi\left(\frac{B^{1/2}(\overline{u})M}{m^{1-\epsilon/2}}(\mu/m)^{\eta},u_\infty\right). \tag{36}$$

We can now eliminate m in terms of $\overline{m}$ by Eq. (28), and Eq. (36) takes the form of Eq. (30) provided we identify δ with

$$\delta=1+\frac{2(2-\eta)}{d-2+\eta}, \tag{37}$$

which is one of the Widom-Kadanoff scaling laws.

In order to obtain the equation of state, it remains now to eliminate the mass $\overline{m}$ in terms of the temperature $t=T-T_c$. In Appendix B, it is shown that $\overline{m}^2$ is proportional to the inverse of the magnetic susceptibility and therefore $\overline{m}^2$ is proportional to t^{γ}. Similarly, it is shown that m, which is proportional to the inverse of the correlation length ξ, behaves like t^{ν}, where $\nu=\gamma/(2-\eta)$.

We shall now derive some properties of the function F, which entered in Eq. (30).

(i) Let us first notice that the equation of state (30) obviously satisfies Griffiths's property[18] which requires that the expansion

$$H/M^{\delta}=\sum_1^{\infty}a_n(t/M^{1/\beta})^{\gamma-2(n-1)\beta} \tag{38}$$

exists for $t/M^{1/\beta}$ large, with

$$\beta=\gamma/(\delta-1). \tag{39}$$

This means that there exists an expansion of the form

$$H/\mu^{(d+2)/2}=(\bar{m}/\mu)^{2\delta/(\delta-1)}\times\sum_1^\infty a_n\left[\left|\frac{M}{\bar{m}^{2/(\delta-1)}}\right|\mu^{2/(\delta-1)-(d-2)/2}\right]^{2n-1} \tag{40}$$

for $M\bar{m}^{2/(\delta-1)}$ small. But this is an immediate consequence of Eqs. (31)–(32) for M small.

(ii) At $T=T_c$, where $\bar{m}$ vanishes, it is known that H is proportional to M^δ where, according to the scaling laws, δ is given by Eq. (37). Technically it requires to show that the function $F(y)$ of Eq. (30) has a finite limit where y goes to infinity. A more explicit derivation will be given in the next paragraph but it is clear that this limit exists since it corresponds to the existence of the zero-mass theory renormalized at an external mass μ.

(iii) An important physical property of the equation of state is to be regular around $x=0$, where $x=t/M^{1/\beta}$, in such a way that a single function describes the two regions $T>T_c$ and $T<T_c$. This is not obviously true since the limit x goes to zero now corresponds to a large M limit, and the rest of this section is devoted to the discussion of this property.

We first start with the CS equation (4) at $u=u_\infty$:

$$\left(m\frac{\partial}{\partial m}-\frac{n}{2}\eta\right)\Gamma^{(n)}(p_i;m,u_\infty)=(2-\eta)m^2\Delta\Gamma^{(n)}(p_i;m,u_\infty)\,,$$

and introduce the generating functionals:

$$\Gamma(M;m,u)=\sum_2^\infty\frac{M^n}{n!}\,\Gamma^{(n)}(p_i=0;m,u) \tag{41a}$$

$$\Delta\Gamma(M;m,u)=\sum_2^\infty\frac{M^n}{n!}\,\Delta\Gamma^{(n)}(p_i=0;m,u)\,. \tag{41b}$$

These functionals are related by a CS equation which, for $u=u_\infty$, reads

$$\left(m\frac{\partial}{\partial m}-\frac{\eta}{2}M\frac{\partial}{\partial M}\right)\Gamma(M;m)=(2-\eta)m^2\Delta\Gamma(M;m)\,. \tag{42}$$

When $M/m^{1-\epsilon/2}$ is large, the right-hand side of (42) is negligible. This property is not obvious since each term of the expansions (41) diverges when m vanishes. In order to verify it, one has to perform a loop-wise summation, i.e., an expansion in powers of u with fixed uM^2, which, for instance, is generated by the steepest-descent method applied on the Feynman path integral (see Appendix C). The effect of this summation is to add to the mass squared in the propagators a term proportional to M^2 in such a way that the limit where $M/m^{1-\epsilon/2}$ goes to infinity can be taken in $\Delta\Gamma$ by letting m go to zero. A simple dimensional argument gives in addition

$$\Gamma(M,m)=m^2M^2h(y)\,, \tag{43}$$

with

$$y=M\,m^{\epsilon/2-1}\,,$$

and for y large Eq. (42) gives

$$(2-\eta)h_{\rm as}(y)-(1-\tfrac{1}{2}\epsilon+\tfrac{1}{2}\eta)h'_{\rm as}(y)=0\,. \tag{44}$$

The solution of Eq. (44) is

$$h_{\rm as}(y)=\text{const}\times y^{\delta-1}\,, \tag{45}$$

with δ related to ϵ and η as indicated in Eq. (37). This means that

$$\Gamma_{\rm as}(M;m)\underset{Mm^{\epsilon/2-1}\gg1}{\sim}m^2M^2(Mm^{\epsilon/2-1})^{\delta-1}\,, \tag{46}$$

and therefore for the function $\bar{\Gamma}$ from Eq. (33)

$$\bar{\Gamma}_{\rm as}(M;\bar{m},\bar{u},\mu)\underset{\mu^{1-\epsilon/2}\gg M\gg\bar{m}^{1-\epsilon/2}}{\sim}(M\mu^{\eta/2})^{\delta+1}\,. \tag{47}$$

Taking the derivative of Eq. (47) with respect to M, we obtain for the magnetic field, under the same conditions,

$$H\mu^{\epsilon/2-3}\sim(M\mu^{\epsilon/2-1})^\delta\,, \tag{48}$$

which establishes the result stated in the previous paragraph.

We now have to go one step further and compute the corrections to (48) when the ratio $\bar{m}^{1-\epsilon/2}/M$ is no longer neglected. We start with Eq. (13) for $u=u_\infty$, and the corresponding generating functionals satisfy

$$\left\{m\frac{\partial}{\partial m}-\frac{\eta}{2}M\frac{\partial}{\partial M}-[\gamma_4(u_\infty)-\eta]\right\}\Delta\Gamma(M;m)=(2-\eta)m^2\Delta^2\Gamma(M;m)\,. \tag{49}$$

Again the same argument indicates that the right-hand side may be neglected when $Mm^{\epsilon/2-1}$ is large. The dimensions are such that

$$\Delta\Gamma(M;m)=M^2\Delta h(y),\qquad y=Mm^{\epsilon/2-1} \tag{50}$$

and from (49), we obtain when y is large

$$(1-\tfrac{1}{2}\epsilon+\tfrac{1}{2}\eta)y\frac{d}{dy}\Delta h_{\rm as}(y)+\gamma_4(u_\infty)\Delta h_{\rm as}(y)=0\,. \tag{51}$$

Integrating Eq. (51), one finds

$$\Delta h_{\rm as}(y)=\text{const}\times y^{-\gamma_4(u_\infty)/(1-\epsilon/2+\eta/2)}=\text{const}\times y^{\delta-1-1/\beta}\,, \tag{52}$$

where β is defined by

$$\frac{1}{\beta}=\frac{2-\eta+\gamma_4(u_\infty)}{1-\frac{1}{2}\epsilon+\frac{1}{2}\eta}\,. \tag{53a}$$

From the relations (15) and (37), one sees that (53a) establishes the scaling law

$$\delta = \gamma/\beta + 1 . \tag{53b}$$

The behavior of $\Delta h(y)$ given by Eq. (52) is compatible with the fact that the right-hand side in Eq. (42) had been neglected. If we now keep in Eq. (42) the right-hand side in its asymptotic form, we shall obtain the first correction to $\Gamma(M, m)$:

$$(2-\eta)h(y) - (1 - \tfrac{1}{2}\epsilon + \tfrac{1}{2}\eta) y h'(y) \underset{y \text{ large}}{\sim} y^{\delta - 1 - 1/\beta} . \tag{54}$$

Integration of Eq. (54) gives

$$h(y) \underset{y \text{ large}}{\sim} y^{\delta-1}(1 + \text{const} \times y^{-1/\beta}) . \tag{55}$$

More generally, we can write a similar equation for $\Delta^p \Gamma$, the functional associated to the vertex functions with p mass-insertions:

$$\left\{ m \frac{\partial}{\partial m} - \frac{\eta}{2} M \frac{\partial}{\partial M} - p[\gamma_4(u_\infty) - \eta] \right\} \Delta^p \Gamma(M; m) = (2-\eta) m^2 \Delta^{p+1} \Gamma . \tag{56}$$

It is important to note that for $p > 1$, a mass insertion does not create any new divergence and therefore no new renormalization constant is needed for $p > 1$.

If the right-hand side of (56) is neglected, in the limit where $Mm^{\epsilon/2-1}$ is large, we can build successive corrections to $\Gamma(M; m)$ in powers of $y^{-1/\beta}$, of the form

$$\Gamma(M, m) = m^2 M^2 y^{\delta - 1} \sum_{p=0}^{\infty} a_p y^{-p/\beta} . \tag{57}$$

From this equation, we obtain an expansion for $\overline{\Gamma}(M; \overline{m}, \mu)$ and differentiation with respect to M gives the magnetic field. It is straightforward to verify that Eq. (57) means that H/M^δ is a regular function of $[(\mu/m)^{\eta/2} M m^{\epsilon/2-1}]^{-1/\beta}$, which is proportional to $[Mt^{-\gamma/(\delta-1)}]^{-1/\beta} = t/M^{1/\beta}$. This is just what we wanted to prove.

In Appendix C, the equation of state is calculated up to order ϵ with these methods. The result agrees with Refs. 5 and 6.

V. CORRECTIONS TO SCALING LAWS

We have already mentioned that the scaling limit was obtained by neglecting terms which were indeed small in the vicinity of the critical point, but that we want now to discuss by themselves. We shall only study the corrections which appear in the framework of a ϕ^4 theory with cutoff, which is equivalent here to a theory with μ/m, large, but finite. Other corrections might be considered as well, for instance those coming from interaction terms of higher order in the field. These effects have to be studied outside the framework of renormalizable field theories. The related problem of corrections to the Gell-Mann–Low limit in four dimensions has been considered by Mack and Symanzik[19] and analogous methods will be used here too.

Let us first justify that u approaches u_∞ when m/μ goes to zero, with fixed $\overline{u}$, as indicated in Eq. (24). First we consider Eq. (21) in its asymptotic form

$$\overline{u} \mu^{\epsilon - 4} = \frac{\Gamma^{(4)}_{\text{as}}(p_i; m, u)|_S}{[\Gamma^{(2)}_{\text{as}}(p, m, u)|_{p^2 = \mu^2}]^2} . \tag{58}$$

Then we replace in (58) $\Gamma^{(4)}_{\text{as}}$ and $\Gamma^{(2)}_{\text{as}}$ by the solutions of the CS equations for an arbitrary coupling constant u:

$$\frac{\Gamma^{(4)}_{\text{as}}}{(\Gamma^{(2)}_{\text{as}})^2} = \mu^{\epsilon - 4} \Phi \left(\int^u \frac{du'}{\beta(u')} - \ln(m/\mu) \right), \tag{59}$$

where Φ is an arbitrary function. From Eqs. (58)–(59) we see that one can compute $\int^u du'/\beta(u') - \ln(m/\mu)$ in terms of $\overline{u}$. If, for fixed $\overline{u}$, one lets m/μ go to zero, the only way to keep fixed the difference $\int^u du'/\beta(u') - \ln(m/\mu)$ is to let u go to a zero u_∞ of $\beta(u)$, where $\beta(u_\infty) = 0$ and $\omega = \beta'(u_\infty) > 0$. It is then simple to obtain the m/μ dependence of u in the form

$$u - u_\infty = A(\overline{u})(m/\mu)^\omega, \quad m/\mu \ll 1 . \tag{24'}$$

This exponent ω will be present in all our corrections to scaling laws. One can remark that the derivative of $\beta(u)$, at a point where β vanishes, is independent of the renormalization scheme. In particular let us show that

$$\frac{\partial}{\partial \overline{u}} \overline{\beta}(\overline{u}, \overline{m}/\mu) \Big|_{\overline{u} = \overline{u}_\infty(\overline{m}/\mu)} = \omega . \tag{60}$$

From Eq. (21), and from the definition of $\overline{\beta}$, it is easy to show that

$$\overline{\beta}(\overline{u}, \overline{m}/\mu) = \frac{\beta(u)}{\partial u/\partial \overline{u} - \beta(u)(\partial/\partial \overline{u}) \ln(m/\mu)} , \tag{61}$$

where m/μ is related to $\overline{u}$ and $\overline{m}/\mu$ by Eqs. (19)–(20). From Eq. (31) we first verify that $\overline{\beta}$ does vanish for some $\overline{u} = \overline{u}_\infty$ corresponding to $u = u_\infty$. In addition, in the neighborhood of u_∞ Eq. (61) gives

$$\overline{\beta}(\overline{u}, \overline{m}/\mu) \simeq \frac{\beta(u)}{\partial u/\partial \overline{u}} ,$$

and differentiation with respect to $\overline{u}$ leads to Eq. (60). In the massless theory, the function β has been calculated[20] up to order u^4 and from this we obtain:

$$\omega = \epsilon - \frac{3(3N+14)}{(N+8)^2} \epsilon^2 + [\tfrac{33}{4} N^2 + \tfrac{461}{2} N + 740 + 24(5N+22)\zeta(3)] \frac{\epsilon^3}{(N+8)^3} - 18 \frac{(3N+14)^2}{(N+8)^4} \epsilon^3 + O(\epsilon^4) , \tag{62a}$$

$$\omega\nu = \tfrac{1}{2}\epsilon - \frac{(-N^2+8N+68)}{4(N+8)^2}\epsilon^2 + \frac{(N+2)(N+3)(N+20)}{8(N+8)^3}\epsilon^3 - \frac{9(3N+14)^2}{(N+8)^4}\epsilon^3$$
$$+\frac{1}{2(N+8)^3}\left[\tfrac{15}{4}N^2+\tfrac{401}{2}N+698+24(5N+22)\zeta(3)\right]\epsilon^3+O(\epsilon^4). \quad (62b)$$

The convergence of this expansion for $\epsilon = 1$ is very poor. For $N=1, 2, 3$, Padé approximants give the estimates $\omega \simeq 0.8$ and $\omega\nu \sim 0.45$. We shall now discuss a few examples where corrections may be computed.

(i) Correlation function at $T=T_c$. The results have already been mentioned in Eqs. (23) and (27), and we shall now derive them.

From (24) we know that u is close to u_∞ and therefore

$$\Gamma^{(n)}(p_i;m,u) \simeq \Gamma^{(n)}(p_i;m,u_\infty) + (u-u_\infty)\frac{\partial\Gamma^{(n)}}{\partial u}(p_i;m,u)\bigg|_{u_\infty}. \quad (63)$$

For $p_i \gg m$, the asymptotic form of $\Gamma^{(n)}(u_\infty)$ is given by Eq. (10); it remains to find the behavior of $(\partial\Gamma^{(n)}/\partial u)|_{u_\infty}$. We take the derivative of Eq. (4a), set $u=u_\infty$, and neglect the right-hand side. This gives

$$\left[m\frac{\partial}{\partial m} - \frac{n}{2}\eta + \omega\right]\frac{\partial\Gamma^{(n)}_{\text{as}}}{\partial u}\bigg|_{u_\infty} = \tfrac{1}{2}n\gamma_3'(u_\infty)\Gamma^{(n)}_{\text{as}}(u_\infty), \quad (64)$$

and the integration of (64) leads to

$$\frac{\partial\Gamma^{(n)}_{\text{as}}}{\partial u}\bigg|_{u_\infty} = \Gamma^{(n)}_{\text{as}}(p_i;m,u_\infty)\left[\frac{n}{2}\frac{\gamma_3'(u_\infty)}{\omega} + b^{(n)}(p_i)m^{-\omega}\right]. \quad (65)$$

Finally, this gives

$$\Gamma^{(n)}(p_i,m,u) \underset{\mu\gg p\gg m}{\simeq} \Gamma^{(n)}_{\text{as}}(p_i,m,u_\infty)\left[1+\frac{n}{2}\frac{\gamma_3'(u_\infty)}{\omega}A(\bar{u})\left(\frac{m}{\mu}\right)^\omega + \frac{A(\bar{u})}{\mu^\omega}b^{(n)}(p_i)\right], \quad (66)$$

the corrections to the $\bar{\Gamma}$ functions are of the same form from Eq. (18). In particular, for the two-point function, Eq. (66) reads

$$\Gamma^{(2)}(p,m,u) \underset{\mu\gg p\gg m}{\simeq} p^{2-\eta}\left[1+\frac{\gamma_3'(u_\infty)}{\omega}A(\bar{u})\left(\frac{m}{\mu}\right)^\omega + A(\bar{u})b_2\left(\frac{p}{\mu}\right)^\omega\right]. \quad (67)$$

In position space, Eq. (67) implies for the spin-spin correlation function that

$$\langle\phi(X)\phi(0)\rangle \underset{a\ll|X|\ll\xi}{\sim} \frac{1}{X^{d-2+\eta}}\left[1+\text{const}\left(\frac{a}{X}\right)^\omega\right]. \quad (68)$$

In (68) the power ω of a/X is universal, but not the constant which is in front.

(ii) Corrections to power behavior in $t=T-T_c$. The corrections in powers of $u-u_\infty$ are related to powers of $(m/\mu)^\omega$, and since the correlation length is both proportional to $1/m$ and to $t^{-\nu}$, finally this yields corrections in powers of $t^{\nu\omega}$. For instance, the magnetic susceptibility behaves as

$$\chi \propto t^{-\gamma}(1+\text{const}\times t^{\omega\nu}), \quad (69)$$

and the correlation length as

$$\xi \propto t^{-\nu}(1+\text{const}\times t^{\omega\nu}). \quad (70)$$

(iii) Corrections to the equation of state. We want to examine what happens to the equation of state when the field is such that $M/\mu^{1-\epsilon/2}$ is no longer neglected. If $M/t^{1/\beta}$ remains small, we see through the mass corrections (69) and vertex corrections (27) that

$$H/M^\delta = f\left(\frac{t}{M^{1/\beta}}\right)(1+\text{const}\times t^{\omega\nu}). \quad (71)$$

But if $M/t^{1/\beta}$ is large, another correction will appear.

This requires us to write the CS equation (42), for arbitrary u, as

$$\left[m\frac{\partial}{\partial m}+\beta(u)\frac{\partial}{\partial u}-\frac{\gamma_3(u)}{2}M\frac{\partial}{\partial M}\right]\Gamma(M;m,u) = [2-\gamma_3(u)]m^2\Delta\Gamma(M;m,u). \quad (72)$$

For $M/t^{1/\beta} \gg 1$ one has also $M/m^{1-\epsilon/2} \gg 1$ and the right-hand side is negligible. For u close to u_∞,

$$\Gamma(M;m,u) \simeq \Gamma(M;m,u_\infty)+(u-u_\infty)\frac{\partial\Gamma}{\partial u}\bigg|_{u_\infty} = M^2m^2[h(y)+(u-u_\infty)k(y)], \quad (73)$$

where $h(y)$ was introduced in Eq. (43), $y \equiv Mm^{\epsilon/2-1}$.

As for the corrections to the correlation functions, we get an equation for $k(y)$ by taking the derivative of Eq. (72) with respect to u. For y large, this gives

$$\left[(1-\tfrac{1}{2}\epsilon+\tfrac{1}{2}\eta)y\frac{d}{dy}+\eta-2-\omega\right]k(y)$$
$$=\tfrac{1}{2}[\gamma_3'(u_\infty)][(1-\tfrac{1}{2}\epsilon)yh'(y)-2h(y)]\,. \quad (74)$$

Equation (45) gives $h(y)$ for large y, and from (73)

$$\Gamma(M;m,u)\underset{y\text{ large}}{\sim} M^2m^2y^{\delta-1}[1+\text{const}(u-u_\infty)\times y^{\omega/(1-\epsilon/2+\eta/2)}]\,. \quad (75)$$

This gives also $\overline{\Gamma}$, and then the field H, and finally

$$\frac{H}{M^\delta}\sim 1+\text{const}\left(\frac{M}{\mu^{1-\epsilon/2}}\right)^{\omega\nu/\beta}, \quad (76)$$

where ν and β are defined in (16) and (39).

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APPENDIX A: ASYMPTOTIC BEHAVIOR OF $Z(\mu/m,u)$

When μ varies and $\bar{u}$ is fixed, both arguments of the function $Z(\mu/m,u)$ vary. If μ goes to infinity, it was shown in Sec. V that the behavior of u is

$$u-u_\infty\simeq A(\bar{u})(m/\mu)^\omega\,. \quad (A1)$$

For μ large, Eq. (20) relates Z to the asymptotic behavior of $\Gamma^{(2)}$ which is given by the homogeneous CS equation. This means that for large x, the function $Z(x,u)$ is the solution of

$$\left[-x\frac{\partial}{\partial x}+\beta(u)\frac{\partial}{\partial u}+\gamma(u)\right]Z_{\text{as}}(x,u)=0\,. \quad (A2)$$

By integration of this equation we find that

$$Z_{\text{as}}(x,u)=\Phi(x/f(u))\exp\left[-\int^u\frac{\gamma(u')du'}{\beta(u')}\right], \quad (A3)$$

where Φ is an arbitrary function, and $f(u)$ is given by

$$\ln f(u)=-\int^u\frac{du'}{\beta(u')}\,. \quad (A4)$$

When u approaches u_∞, from (A4) we obtain $\ln f(u)\sim-(1/\omega)\ln(u-u_\infty)$ and then from (A1),

$$\ln f(u)\underset{\mu/m\gg1}{\sim}-\ln(m/\mu)+\ln C(\bar{u}).$$

Therefore, for $x=\mu/m$ large,

$$x/f(u)\xrightarrow[\mu/m\gg1]{} C(\bar{u})\,.$$

Furthermore, in the same limit

$$\int^u\frac{\gamma(u')du'}{\beta(u')}\underset{u\to u_\infty}{\sim}\eta\int^u\frac{du'}{\beta(u')}=-\eta\ln f(u)\,,$$

then

$$\int^u\frac{\gamma(u')du'}{\beta(u')}\underset{\mu\to\infty}{\sim}-\eta\ln(m/\mu)+\eta\ln C(\bar{u})\,.$$

This means finally that for Z we obtain

$$Z(\mu/m,u)\underset{\mu\to\infty}{\sim}B(\bar{u})(\mu/m)^\eta\,, \quad (A5)$$

where B is some function of $\bar{u}$.

APPENDIX B: RELATION BETWEEN MASS AND TEMPERATURE

If one comes back to Wilson's original arguments to show that the Lagrangian $\mathcal{L}(X)$ given in (2) describes the critical behavior of a magnetic system, one sees that the scale of temperature is given by the bare mass m_0^2 which varies linearly with T.

Unfortunately m_0^2 diverges with the cutoff, but one can consider instead its derivative $\partial m_0^2/\partial\bar{m}^2|_{g_0}$ which is finite when the cutoff becomes infinite. In physical terms $\bar{m}^{-2}$, which is the value of the renormalized propagator at zero-momentum, coincides, for large μ, with the magnetic susceptibility χ.

We start with the relations

$$\left.\frac{\partial m_0^2}{\partial m^2}\right|_{g_0}=\left.\frac{\partial m_0^2}{\partial\bar{m}^2}\right|_{g_0}\left.\frac{\partial\bar{m}^2}{\partial m^2}\right|_{g_0} \quad (B1)$$

and

$$\bar{m}^2=Z(\mu/m,u)m^2 \underset{\mu\gg m}{=}\frac{m^2\mu^2}{\Gamma^{(2)}(\mu^2;m,u)}\,. \quad (B2)$$

From Eq. (B2) we obtain

$$\left.\frac{\partial\bar{m}^2}{\partial m^2}\right|_{g_0}=Z\left(1+\frac{m}{2}\frac{\partial}{\partial m}\ln Z\bigg|_{g_0}\right),$$

and from the CS equation for Z [Eq. (A2) of Appendix A] we find

$$\left.\frac{\partial\bar{m}^2}{\partial m^2}\right|_{g_0}=Z(1-\tfrac{1}{2}\gamma_3(u))\,. \quad (B3)$$

The right-hand side of the CS equation (4a) satisfied by $\Gamma^{(2)}$ is given at zero-momentum by the renormalization conditions. If we come back to the derivation of Eqs. (4), through a differentiation of $\Gamma^{(2)}$ with respect to m with fixed g_0, we obtain

$$[2-\gamma_3(u)]m^2=2Z_3Z_4{}^{-1}m^2\frac{\partial m_0{}^2}{\partial m^2}\bigg|_{g_0}. \quad \text{(B4)}$$

Therefore from Eqs. (B1), (B3), and (B4) one has

$$\frac{\partial m_0{}^2}{\partial \overline{m}^2}\bigg|_{g_0\ \mu/m\gg 1}=\frac{Z_4(u)}{Z_3(u)}\,Z^{-1}(\mu/m,u). \quad \text{(B5)}$$

We now have to take into account that the behavior of Z, Z_3, Z_4 is known when u approaches u_∞ as in Eq. (24):

$$\begin{aligned} Z^{-1}(\mu/m,u)&\sim(m/\mu)^\eta,\\ Z_3(u)&\sim(u-u_\infty)^{\eta/\omega}\\ &\sim(m/\mu)^\eta,\\ Z_4(u)&\sim(u-u_\infty)^{\gamma_4(u_\infty)/\omega}\\ &\sim(m/\mu)^{\gamma_4(u_\infty)}. \end{aligned} \quad \text{(B6)}$$

This with Eq. (B5) means that

$$\frac{\partial m_0{}^2}{\partial \overline{m}^2}\bigg|_{g_0}\sim\left(\frac{m}{\mu}\right)^{\gamma_4(u_\infty)},$$

or relating m to $\overline{m}$,

$$\frac{\partial m_0{}^2}{\partial \overline{m}^2}\bigg|_{g_0}\underset{\substack{\mu\to\infty\\ \overline{u}\ \text{fixed}}}{\sim}\left(\frac{\overline{m}^2}{\mu^2}\right)^{\gamma_4(u_\infty)/(2-\eta)}. \quad \text{(B7)}$$

Since, in the vicinity of T_c, the magnetic susceptibility $\overline{m}^{-2}$ is assumed to diverge as

$$\overline{m}^{-2}\underset{t\to+0}{\sim}t^{-\gamma}, \quad \text{(B8)}$$

one should have for $dt/d\overline{m}^2$, or rather for $\partial m_0{}^2/\partial\overline{m}^2|_{g_0}$, which is proportional to it,

$$\frac{\partial m_0{}^2}{\partial \overline{m}^2}\bigg|_{g_0}\sim(\overline{m}^2)^{1/\gamma-1}. \quad \text{(B9)}$$

Comparison between (B8) and (B9) yields the relation

$$\frac{1}{\gamma}=1+\frac{\gamma_4(u_\infty)}{2-\eta}. \quad \text{(B10)}$$

In addition, let us study the divergence of the correlation length in the vicinity of T_c. Since $\Gamma^{(2)}(p,m)$ has, for dimensional reasons, the form $\Gamma^{(2)}(p,m)=p^2\psi(p/m)$, the zero of $\Gamma^{(2)}$, which gives the inverse of the correlation length, is proportional to m. Now we know that $m^2=Z^{-1}\overline{m}^2$, Z is known from Appendix A, and we end up with

$$\begin{aligned} \xi^{-1}&=\text{const}\times m\\ &=\text{const}\times\overline{m}^{2/(2-\eta)}\\ &=\text{const}\times t^{\gamma/(2-\eta)}, \end{aligned} \quad \text{(B11)}$$

which establishes the scaling law

$$\nu=\frac{\gamma}{(2-\eta)}. \quad \text{(B12)}$$

To be complete, let us derive the scaling law which relates the specific heat exponent α to γ and η. The specific heat is related to the energy fluctuations, which diverge near T_c. They are dominated by the ϕ^2 terms which are those with the smaller dimension. Thus, one considers the quantity $\int d^dX\langle\phi_0{}^2(X)\phi_0{}^2(0)\rangle$, where $\phi_0{}^2$ is the bare field. This bare correlation function is related to the renormalized one through[21] a multiplicative and an additive renormalization. This additive renormalization necessitates the introduction of a new function, $K(u)$, finite in 4 dimensions, defined by

$$m\frac{\partial}{\partial m}\int d^dX\langle\phi_0{}^2(X)\phi_0{}^2(0)\rangle\bigg|_{g_0}=\left(\frac{Z_3}{Z_4}\right)^2m^{-\epsilon}K(u). \quad \text{(B13)}$$

Assuming that the function $K(u)$ has a finite limit when u approaches u_∞, which is true order by order in ϵ, since the behavior in m/μ of Z_3 and Z_4 is known [Eq. (B6)], for the energy fluctuations we end up with

$$\int d^dX\langle\phi_0{}^2(X)\phi_0{}^2(0)\rangle=\text{const}\times m^{2[\eta-\gamma_4(u_\infty)]-\epsilon} \quad \text{(B14)}$$

or, relating m to t,

$$\int d^dX\langle\phi_0{}^2(X)\phi_0{}^2(0)\rangle=\text{const}\times t^{\nu[2\eta-2\gamma_4(u_\infty)-\epsilon]}.$$

Therefore

$$\alpha=-\nu\{2[\eta-\gamma_4(u_\infty)]-\epsilon\}, \quad \text{(B15)}$$

and, taking into account (B10)–(B12), it is straightforward to write, instead of (B15),

$$\alpha=2-\nu d, \quad \text{(B16)}$$

which is one of Kadanoff's scaling laws.

APPENDIX C: FIRST-ORDER CALCULATION OF THE EQUATION OF STATE

In the calculations of Refs. 5 and 6 the Lagrangian was expressed in terms of a new field $\phi_i'(X)$ which has zero expectation value. In the present work, the theory in presence of an external field is expressed directly in terms of the symmetric theory in zero field since

$$\Gamma(M;m)=\sum_2^\infty\frac{M^n}{n!}\,\Gamma^{(n)}(p_i=0;m,u_\infty). \quad \text{(C1)}$$

The lowest-order contribution to $\Gamma(M;m)$ is given by the tree diagrams, and the first order in ϵ will be a result of the summation of the one-loop diagrams. An easy method to obtain the loop-wise expansion is to use the steepest-descent method on the Feynman path integral. At the one-loop order, the result for the generating functional $\Gamma(M)$ is

$$\Gamma(M;m) = -\mathfrak{L}(\phi) - \tfrac{1}{2}\,\mathrm{Tr}_{(\alpha\beta;\,xy)} \ln \frac{\delta^2 \mathfrak{L}}{\delta\phi_\alpha(x)\delta\phi_\beta(y)}\bigg|_{\phi=M}. \tag{C2}$$

In fact, for the equation of state, it is sufficient to know $\partial\Gamma/\partial M$, and from (C2) we obtain in the one-loop approximation

$$\frac{\partial\Gamma}{\partial M_\alpha} = m^2 M_\alpha + \tfrac{1}{6} m^\epsilon u M^2 M_\alpha + \tfrac{1}{6} m^\epsilon u (N-1) M_\alpha \int \frac{d^d p}{S_d}\left[\frac{1}{p^2+m^2+m^\epsilon u M^2/6} - \frac{1}{p^2+m^2} + \frac{m^\epsilon u M^2/6}{(p^2+m^2)^2}\right]$$
$$+ \tfrac{1}{2} m^\epsilon u M_\alpha \int \frac{d^d p}{S_d}\left[\frac{1}{p^2+m^2+m^\epsilon u M^2/2} - \frac{1}{p^2+m^2} + \frac{m^\epsilon u M^2/2}{(p^2+m^2)^2}\right]\bigg|_{u=u_\infty}$$
$$= m^2 M_\alpha\{1 + \tfrac{1}{6}y^2 + \tfrac{1}{12}(N-1)u_\infty[(1+\tfrac{1}{6}y^2)\ln(1+\tfrac{1}{6}y^2) - \tfrac{1}{6}y^2] + \tfrac{1}{4}u_\infty[(1+\tfrac{1}{2}y^2)\ln(1+\tfrac{1}{2}y^2) - \tfrac{1}{2}y^2]\}, \tag{C3}$$

where $y^2 \equiv u_\infty M^2 m^{\epsilon-2}$.

To obtain the equation of state, it is now sufficient to use the relations

$$\overline{\Gamma}(M) = \Gamma(M\sqrt{Z}), \qquad H = \frac{\partial\overline{\Gamma}}{\partial M}. \tag{C4}$$

Using Eq. (C3) and the large μ/m behavior of Z, we obtain in terms of $x = t/M^{1/\beta}$,

$$\frac{H}{M^\delta} = x^\gamma + \tfrac{1}{6}x^{\epsilon/2} + u_\infty \tfrac{1}{12}(N-1)\left[\left(x+\frac{1}{6}\right)\ln\left(1+\frac{1}{6x}\right) - \frac{1}{6}\right] + \frac{1}{4}u_\infty\left[\left(x+\frac{1}{2}\right)\ln\left(1+\frac{1}{2x}\right) - \frac{1}{2}\right]. \tag{C5}$$

If we now take into account the value of u_∞, which at order ϵ is

$$u_\infty = \frac{6}{N+8}\epsilon + O(\epsilon^2),$$

and the ϵ expansion of γ:

$$\gamma = 1 + \frac{(N+2)}{2(N+8)}\epsilon + O(\epsilon^2),$$

one obtains the cancellation of the $\ln x$ terms of Eq. (C4), which was expected since we derived in Sec. IV that the equation of state was regular, in the variable x around $x=0$. The result

$$\frac{H}{M^\delta} = x + \frac{1}{6} + \frac{1}{2}\frac{(N-1)}{(N+8)}\epsilon[(x+\tfrac{1}{6})\ln(x+\tfrac{1}{6}) - \tfrac{1}{6}] + \frac{3}{2(N+8)}\epsilon[(x+\tfrac{1}{2})\ln(x+\tfrac{1}{2}) - \tfrac{1}{2}]. \tag{C6}$$

If the normalizations of fields and temperature are chosen so that

$$H/M^\delta = 1 \quad \text{at } x=0$$

and

$$H=0 \quad \text{for } x=-1,$$

Eq. (C6) takes the form

$$\frac{H}{M^\delta} = 1 + x + \epsilon\left\{\frac{N-1}{2(N+8)}(x+1)\ln(x+1) + \frac{3}{2(N+8)}[(x+3)\ln(x+3) - 3\ln 3 + x(2\ln 2 - 3\ln 3)]\right\}, \tag{C7}$$

which coincides with the result of Ref. 6 at order ϵ.

APPENDIX D: LOGARITHMIC CORRECTIONS TO SCALING LAWS IN FOUR DIMENSIONS

The fact that logarithmic deviations from free-field theory are present in four dimensions has been discussed by Larkin and Khmel'nitskii[22] using the parquet approximation. Riedel and Wegner[23] have established the same results with the renormalization group techniques applied to the cutoff field theory. For the sake of completeness we want to show that the same results follow naturally from the methods developed in this article. The four dimensional theory will be studied within the same limit in which $\overline{u}$ is fixed and the subtraction mass μ is large.

As an example we shall discuss in some details the behavior on the critical isotherm. In the massless theory, i.e., $T=T_c$, the free field result is simply $H=0$. The term that we want to calculate will appear as a correction to this trivial limit. Let us start with the renormalization-group equations obtained when we vary the subtraction mass μ:

$$\left[\mu\frac{\partial}{\partial\mu} + \overline{\beta}(\overline{u})\frac{\partial}{\partial\overline{u}} - \frac{n}{2}\overline{\gamma}_3(\overline{u})\right]\overline{\Gamma}^{(n)}(p_i;\overline{m}=0,\overline{u},\mu) = 0, \tag{D1}$$

and introduce the generating functional

$$\overline{\Gamma}(M;\mu,\overline{u})=\sum_{n=0}^{\infty}\frac{1}{n!}\int\overline{\Gamma}^{(n)}(x_1\cdots x_n;\overline{m}=0,\overline{u},\mu)\times\phi(x_1)\cdots\phi(x_n)dx_1\cdots dx_n\big|_{\phi(x)=M}. \tag{D2}$$

From dimensional considerations one can write

$$\overline{\Gamma}(M,\mu,\overline{u})=\frac{M^4}{4!}f(\overline{u},M/\mu), \tag{D3}$$

and from (D1) we derive for the function f the equation

$$\left[-x\frac{\partial}{\partial x}+\tilde{\beta}(\overline{u})-2\tilde{\gamma}(\overline{u})\right]f(\overline{u},x)=0, \tag{D4}$$

with

$$\tilde{\beta}=\frac{\overline{\beta}}{1+\frac{1}{2}\overline{\gamma}_3},\quad \tilde{\gamma}=\frac{\overline{\gamma}_3}{1+\frac{1}{2}\overline{\gamma}_3}.$$

Integration of (D4) gives

$$f(\overline{u},\lambda x)=f(\overline{u}(\lambda),x)\exp\left(\int_{\overline{u}(\lambda)}^{\overline{u}}2\frac{\tilde{\gamma}(u')}{\tilde{\beta}(u')}du'\right), \tag{D5}$$

where $\overline{u}(\lambda)$ is defined by

$$\int_{\overline{u}}^{\overline{u}(\lambda)}\frac{du'}{\tilde{\beta}(u')}=\ln\lambda. \tag{D6}$$

The large μ limit is obtained by letting λ go to zero, and if we write

$$\begin{aligned}&\tilde{\beta}(u)=bu^2+O(u^3),\quad b=\tfrac{1}{6}(n+8)S^{-1}\\&\tilde{\gamma}(u)=c_3u^2+O(u^3),\quad c_3=\tfrac{1}{72}(n+2)S^{-2}\\&S^{-1}\equiv 8\pi^2,\end{aligned} \tag{D7}$$

the solution of (D6) for λ small is

$$\overline{u}(\lambda)=-\frac{1}{b\ln\lambda}+O\left(\frac{1}{(\ln\lambda)^2}\right). \tag{D8}$$

Since $\overline{u}(\lambda)$ is small we can study $f(\overline{u},\lambda x)$ by calculating $f(\overline{u}(\lambda),x)$ in perturbation theory. Here, we are in a situation similar to the one studied recently by Symanzik[24] for a ϕ^4 theory with a negative coupling constant.

At lowest order in $\overline{u}$ (tree approximation) one has

$$\overline{\Gamma}(M,\overline{u},\mu)=\overline{u}\frac{M^4}{4!}, \tag{D9}$$

and therefore

$$f(\overline{u}(\lambda),x)=\overline{u}(\lambda)=-\frac{1}{b\ln\lambda}+O\left(\frac{1}{(\ln\lambda)^2}\right). \tag{D10}$$

From (D3)–(D5) we end up with

$$\Gamma(M,\overline{u},\mu)\underset{M\ll\mu}{\sim}\frac{M^4}{|\ln(M/\mu)|}, \tag{D11}$$

which yields for $H=\partial\overline{\Gamma}/\partial M$

$$H\sim\frac{M^3}{|\ln(M/\mu)|}. \tag{D12}$$

Corrections to scaling laws which involve the temperature are obtained in a similar fashion. The central remark is that the coupling constant u of the massive theory renormalized at zero momentum vanishes like

$$u=-\frac{1}{b\ln(m/\mu)}+O\left(\frac{1}{\ln^2(m/\mu)}\right). \tag{D13}$$

The equation (D13) follows directly from Eqs. (58)–(59). The mass-temperature relation becomes

$$\frac{\partial t}{\partial\chi^{-1}}\sim u^{c_4/b}, \tag{D14}$$

where χ is the magnetic susceptibility, which is here proportional to m^{-2}, and c_4 is the first term in the expansion of γ_4:

$$\gamma_4(u)=c_4u+O(u^2),\quad c_4=-\frac{n+2}{6}\frac{1}{S}. \tag{D15}$$

Taking into account the relation (D13) we obtain

$$\frac{\partial\chi^{-1}}{\partial t}\sim|\ln\chi^{-1}|^{c_4/b},$$

which, after integration, gives

$$\chi\sim\frac{1}{t}|\ln|t||^{(n+2)/(n+8)}. \tag{D16}$$

Finally let us note another consequence of (D13) for the equation of state. In a region where $\ln(m/M)$ remains finite, perturbation theory in u gives

$$\begin{aligned}H&=m^2M+\tfrac{1}{6}uM^3\\&=m^2M-\frac{1}{6b}\frac{M^3}{\ln(m/\mu)}.\end{aligned} \tag{D17}$$

For the spontaneous magnetization this leads to $M\sim m(|\ln m|)^{1/2}$, and from (D16) and $m\sim\chi^{-1/2}$ we obtain

$$M\sim(-t)^{1/2}|\ln(-t)|^{3/(n+8)}. \tag{D18}$$

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