In terms of the electromagnetic and weak forces, $\phi^{\mu}(x)$ and $W_{\mu}(x)$, this symmetry implies that there is not an em-type interaction for the massless particle described by $\psi(x)$. This is not the case for a massive particle because the mass term breaks the symmetry of the theory. So, the absence of em forces for a massless particle is a consequence of the invariance of the theory under the mapping (11).

IV. CONCLUSION

We have succeeded, in a very simple model, in showing that electromagnetic- and weak-type forces appear as a consequence of the self-interaction of the γ field as given by Eqs. (4) and (5). Furthermore, the symmetries of the equation for the spinor field give a natural explanation of the absence of the electric charge of the neutrino. These results suggest to us this question: Is it possible to assign values to the fundamental set $\{q_1, \ldots, q_5\}$ of the coupling constants? We intend to consider this problem elsewhere.

ACKNOWLEDGMENT

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[1]J. A. Wheeler, in *Battelle Rencontres,* edited by C. DeWitt and J. A. Wheeler (Benjamin, New York, 1968).

[2]M. Novello, J. Math. Phys. 12, 1039 (1971) and references therein.

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Massless Electrodynamics on the 5-Dimensional Unit Hypersphere: An Amplitude-Integral Formulation

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We further develop the O(5)-covariant formulation of massless quantum electrodynamics which was introduced in an earlier paper. We discuss the group theory of the free photon and electron propagators, and develop a simple amplitude-integral form for the interlacing operator which applies radiative corrections to closed fermion loops. Instead of involving path integrals defined by a limiting process, the amplitude integral involves an infinite product of *individually well-defined* ordinary integrals over coefficients appearing in the hyperspherical harmonic expansion of an external electromagnetic field A_a. We use the amplitude integral to study the analyticity properties in coupling constant α of single fermion loops, in a modified quantum electrodynamics in which the short-distance singularity of the photon propagator is cut off. In this model we find that $\alpha = 0$ is not a regular point, and that the single fermion loops cannot develop an infinite-order zero as α approaches a positive α_0 from below.

I. INTRODUCTION

The conformal-invariant (i.e., massless-fermion) limit of quantum electrodynamics is of great interest because of its connection with the eigenvalue problem for the coupling constant α imposed by requiring all of the renormalization constants of electrodynamics to be finite.[1] In a recent paper[2] we have studied reformulations of massless, Euclidean quantum electrodynamics made possible by invariance under the conformal group. Our principal result was that the Feynman rules for vacuum-polarization calculations and the equations of motion in this theory can be simply rewritten in terms of equivalent rules and equations of motion on the surface of the unit hypersphere embedded in 5-dimensional Euclidean space. The 5-dimensional rules are summarized in Table I; as in Ref. 2, $\eta_1, \eta_2, \ldots$ denote 5-dimensional unit vectors, $\int d\Omega_\eta$ denotes integration over the surface of the hypersphere, and α_a are a set of five 8×8 matrices satisfying the Clifford algebra $\{\alpha_a, \alpha_b\} = 2\delta_{ab}$ and all anticommuting with a sixth matrix α_6. The salient feature of these rules is that they involve a bounded, rather than an unbounded space, and, correspondingly, eigenfunction expansions involve summation over a discrete index, rather than the integration over a continuum index which

TABLE I. Five-dimensional and Euclidean Feynman rules.

	5-dimensional	Euclidean
electron propagator	$-\frac{1}{\pi^2}\frac{\frac{1}{2}(\alpha\cdot\eta_1-1)\frac{1}{2}(\alpha\cdot\eta_2+1)}{(\eta_1-\eta_2)^4}$	$-\frac{1}{2\pi^2}\frac{\gamma\cdot(x_1-x_2)}{(x_1-x_2)^4}$
photon propagator	$\frac{1}{4\pi^2}\frac{\delta_{ab}}{(\eta_1-\eta_2)^2}$	$\frac{1}{4\pi^2}\frac{\delta_{\mu\nu}}{(x_1-x_2)^2}$ +gauge terms
electron-photon vertex [a]	$ie\alpha_a\equiv\frac{1}{2}ie[\alpha\cdot\eta_1,\ \alpha_a]$	$ie\gamma_\mu$
each closed fermion loop	$-\mathrm{tr}_8$	$-\mathrm{tr}_4$
each virtual coordinate integration	$\int d\Omega_\eta$	$\int d^4x$

[a] The two forms of the electron-photon vertex are equivalent when sandwiched between electron propagators.

appears in the usual x-space formulation of electrodynamics.

In the present paper we continue to pursue the 5-dimensional formalism, with the particular aim of developing a 5-dimensional analog of the path-integral formulation of electrodynamics and applying it to the study of the eigenvalue problem for α. In Sec. II we describe some simple representation theory of the group $O(5)$. We discuss the group-theoretic properties of the vector spherical harmonics and the spinor harmonics which are the elementary eigensolutions of the free photon and electron wave equations, and obtain the eigenfunction expansions of the free photon and electron propagators. In Sec. III we consider the interlacing operator which applies virtual-photon radiative corrections to single closed fermion loops. In x space this operator can be represented by a path integral which, because of the limiting process involved in its definition, does not readily lend itself to approximations. In the 5-dimensional formalism the situation is quite different. Because of the discrete-index nature of eigenfunction expansions, we find that the interlacing operator on the hypersphere can be represented as an infinite product of individually well-defined ordinary integrals over the coefficients appearing in the vector-spherical-harmonic expansion of an external electromagnetic potential A. Inside this amplitude integral there stands the single-fermion-loop diagram in question, evaluated in the presence of the potential A. A particularly interesting approximation which can now be made consists of keeping only a finite number of eigenmodes in the vector-spherical-harmonic expansion of the photon propagator, or, equivalently, keeping only a finite number of amplitude integrals out of the infinite product appearing in the interlacing operator. (This approximation corresponds to cutting off the short-distance singularity of the photon propagator, while leaving the long-distance behavior qualitatively unchanged.) In Sec. IV we examine, in this truncated model, the analyticity properties of single-fermion-loop diagrams as a function of coupling constant α. As a preliminary step we summarize some properties of the external-field vacuum loop which appears inside the amplitude integral. Then we use these to show that the radiative-corrected single fermion loops in our model are not analytic at $\alpha=0$ and cannot develop an infinite-order zero as α approaches a positive α_0 from below. Other possible conclusions about the coupling-constant analyticity of single fermion loops are shown to depend on the detailed analyticity and asymptotic properties of the external-field problem, as a function of external-field amplitude. We formulate a simple one-mode problem, the study of which may help resolve some of the unanswered questions about the finite-photon-mode-number model. Finally, we discuss possible connections between our results for the truncated model and the behavior of the exact eigenvalue problem in which all photon modes are retained.

II. GROUP THEORY OF THE FREE PHOTON AND ELECTRON PROPAGATORS

The group O(5) is a simple Lie group of rank 2. Its irreducible representations are characterized[3] by two numbers, $\mu_1 \geq \mu_2 \geq 0$, which must be both integers or both half-integers; the half-integer representations are spinor representations. The dimension of the general representation (μ_1, μ_2) is given by

$$\dim(\mu_1, \mu_2) = \tfrac{1}{6}(\mu_1 - \mu_2 + 1)(\mu_1 + \mu_2 + 2) \times (2\mu_1 + 3)(2\mu_2 + 1) . \qquad (1)$$

The fundamental representations are $(\tfrac{1}{2}, \tfrac{1}{2})$, a spinor representation of dimension 4, and $(1,0)$, a vector representation of dimension 5; these play an important role in constructing the eigenfunction expansions of the free electron and the free photon propagators, respectively. The usual hyperspherical harmonics $Y_{nm}(\eta)$ transform according to the representation $(n, 0)$, with dimension

$$\dim(n, 0) = \tfrac{1}{6}(n+1)(n+2)(2n+3) . \qquad (2)$$

Let us first consider the free photon propagator, which according to Ref. 2 has the hyperspherical harmonic expansion

$$D_{ab}^{(0)}(\eta_1, \eta_2) \equiv \frac{\delta_{ab}}{4\pi^2(\eta_1 - \eta_2)^2} = \sum_{n=0}^{\infty} \sum_{m=1}^{\dim(n,0)} \frac{Y_{nm}(\eta_1) Y_{nm}(\eta_2)^* \delta_{ab}}{(n+1)(n+2)} . \qquad (3)$$

We wish to rewrite the right-hand side of Eq. (3) in the form

$$\sum_j c_j Y_{ja}(\eta_1) Y_{jb}(\eta_2)^* , \qquad (4)$$

where the $Y_{ja}(\eta)$ are vector spherical harmonics transforming according to definite irreducible representations of O(5). To do this, we write

$$\delta_{ab} = \sum_{l=1}^{5} (v_l)_a (v_l)_b , \qquad (5)$$

with $(v_l)_a$ the unit vector with components δ_{la}. This transforms the right-hand side of Eq. (3) to

$$\sum_{n=0}^{\infty} \sum_{m=1}^{\dim(n,0)} \sum_{l=1}^{5} \frac{[(v_l)_a Y_{nm}(\eta_1)][(v_l)_b Y_{nm}(\eta_2)]^*}{(n+1)(n+2)} , \qquad (6)$$

which is of course not yet in the desired form because the harmonics $(v_l)_a Y_{nm}(\eta)$ transform according to the product representation $(1,0)\otimes(n,0)$, which is reducible. We can readily infer the irreducible representation content of this product from the dimension formula in Eq. (1); we find

$$(1,0)\otimes(n,0) = (n+1,0)\oplus(n-1,0)\oplus(n,1) , \qquad (7a)$$

with

$$\dim(n+1, 0) = \tfrac{1}{6}(n+2)(n+3)(2n+5) ,$$
$$\dim(n-1, 0) = \tfrac{1}{6}n(n+1)(2n+1) , \qquad (7b)$$
$$\dim(n, 1) = \tfrac{1}{2}n(n+3)(2n+3) ,$$

$$\dim(1,0)\times\dim(n,0) = \tfrac{5}{6}(n+1)(n+2)(2n+3) = \dim(n+1,0) + \dim(n-1,0) + \dim(n,1) . \qquad (7c)$$

It is instructive to discuss the quantum numbers which characterize the three irreducible representations which appear on the right-hand side of Eq. (7a). For this purpose we introduce the (anti-Hermitian) angular momentum operator

$$L_{ab} = \eta_a \frac{\partial}{\partial \eta_b} - \eta_b \frac{\partial}{\partial \eta_a} , \qquad (8a)$$

and denote its square $L_{ab}L_{ab}$ by L^2. Since

$$L^2[(v_l)_a Y_{nm}(\eta)] = (v_l)_a L^2 Y_{nm}(\eta) = -2n(n+3)[(v_l)_a Y_{nm}(\eta)] \qquad (8b)$$

the three irreducible representations into which $(v_l)_a Y_{nm}(\eta)$ decomposes are all eigenstates of L^2 with the common eigenvalue $-2n(n+3)$. To find an eigenvalue which distinguishes the three irreducible representations, we consider the eigenvalue equation

$$L_{ab} A_b(\eta) = \lambda A_a(\eta) . \qquad (9)$$

It is easy to see that this equation is rotationally invariant: We consider the infinitesimal rotation $\eta_a \to \eta_a - \Omega_{ab}\eta_b$ and introduce the rotated vector field

$$A_a^R(\eta) = A_a(\eta - \Omega\cdot\eta) + \Omega_{ab}A_b(\eta) = A_a(\eta) + \tfrac{1}{2}\Omega_{cd} L_{cd} A_a(\eta) + \Omega_{ab} A_b(\eta) . \qquad (10)$$

Then, from the angular momentum commutation relations, it easily follows that if $A_a(\eta)$ satisfies Eq. (9) then $A_a^R(\eta)$ satisfies the identical equation

$$L_{ab} A_b^R(\eta) = \lambda A_a^R(\eta) . \qquad (11)$$

Consequently, if one basis function in an irreducible representation satisfies Eq. (9), then all the other basis functions do also; hence the eigenvalue λ is an additional quantum number which can be used to characterize vector representations of O(5). To determine the allowed values of λ, we use the fact that the operator

$$P_{ca} = 2L_{cb}L_{ba} - 6L_{ca} + L^2\delta_{ca} \qquad (12)$$

satisfies

$$L_{bc}P_{ca} = P_{ba} . \qquad (13)$$

Hence for given eigenvalue $-2n(n+3)$ associated with L^2, the eigenvalue λ must satisfy the equation

$$\lambda[2\lambda^2 - 6\lambda - 2n(n+3)] = 2\lambda^2 - 6\lambda - 2n(n+3) , \qquad (14)$$

with the three roots

$$\lambda = 1, -n, n+3 . \qquad (15)$$

Since the three eigenvalues are distinct, we see that the λ eigenvalue suffices to completely specify the representations appearing on the right-hand side of Eq. (7a). The correspondence between λ eigenvalues and representations can be found by explicitly constructing functions transforming

TABLE II. Vector spherical harmonics (not orthonormalized) for $n=1$.

$Y^{(-1)}_{1ma}$:	$(v_k)_a\eta\cdot v_l+(v_l)_a\eta\cdot v_k$ $\eta\cdot v_l(v_l)_a-\frac{1}{5}\eta_a$	$1\le k<l\le 5\Rightarrow$ 10 harmonics $m=1,\ldots,10$ $1\le l\le 4\quad\Rightarrow$ 4 harmonics $m=11,\ldots,14$	dim $(2,0)=14$
$Y^{(4)}_{1ma}$:	η_a	1 harmonic	dim $(0,0)=1$
$Y^{(1)}_{1ma}$:	$(v_k)_a\eta\cdot v_l-(v_l)_a\eta\cdot v_k$	$1\le k<l\le 5\Rightarrow$ 10 harmonics $m=1,\ldots,10$	dim $(1,1)=10$

according to the various representations, as follows.

Representation $(n,1)$:

$$\lambda=1,\quad A_a\propto P_{ab}(v_l)_bY_{nm}(\eta)\,.$$

Representation $(n+1,0)$:

$$\lambda=-n\,,\quad A_a\propto[(n+1)\eta_a+\eta_bL_{ba}]Y_{n+1\ m}(\eta)\,. \tag{16}$$

Representation $(n-1,0)$:

$$\lambda=n+3\,,\quad A_a\propto[-(n+2)\eta_a+\eta_bL_{ba}]Y_{n-1\ m}(\eta)\,.$$

The number of functions exhibited for the representation $(n,1)$ is

$$5\dim(n,0)>\dim(n,1)\,, \tag{17}$$

so this set is obviously overcomplete. The basis functions given for the other two representations contain no redundancies. In Table II we give expressions for the $\lambda=1,-1,4$ basis functions for $n=1$.

Since, by the usual argument, eigenfunctions corresponding to different values of n or λ are orthogonal, we define orthonormalized vector spherical harmonics

$$Y^{(\lambda)}_{nm\,a}(\eta)\begin{cases} m=1,\ldots,\dim(n,1) & \text{for } \lambda=1\\ m=1,\ldots,\dim(n+1,0) & \text{for } \lambda=-n\\ m=1,\ldots,\dim(n-1,0) & \text{for } \lambda=n+3\end{cases} \tag{18}$$

which satisfy

$$\int d\Omega_\eta Y^{(\lambda)}_{nma}(\eta)Y^{(\lambda')}_{n'm'a}(\eta)^*=\delta_{nn'}\delta_{\lambda\lambda'}\delta_{mm'}\,. \tag{19}$$

Because the harmonics form a complete set, we have the completeness relation

$$\delta_S(\eta_1-\eta_2)\delta_{ab}=\sum_{n,\lambda,m}Y^{(\lambda)}_{nma}(\eta_1)Y^{(\lambda)}_{nmb}(\eta_2)^*\,, \tag{20}$$

which allows us, finally, to express the photon propagator in the desired form

$$D^{(0)}_{ab}(\eta_1,\eta_2)=\frac{1}{2-\frac{1}{2}L_1{}^2}\delta_S(\eta_1-\eta_2)\delta_{ab}=\sum_{n,\lambda,m}\frac{Y^{(\lambda)}_{nma}(\eta_1)Y^{(\lambda)}_{n\,mb}(\eta_2)^*}{(n+1)(n+2)}\,. \tag{21}$$

Let us now consider the effect of propagating a virtual-photon radiative correction from point η_1 to point η_2 within a single-closed-fermion-loop diagram. This is described by

$$\int d\Omega_{\eta_1}d\Omega_{\eta_2}J_{a_1}(\eta_1)J_{a_2}(\eta_2)\frac{1}{4\pi^2}\frac{\delta_{a_1a_2}}{(\eta_1-\eta_2)^2}\,, \tag{22}$$

where the currents J are, of course, just a convenient shorthand for describing the amplitude from which the photon is emitted and absorbed, with all variables other than those referring to the virtual photon in question suppressed. Because the amplitude for photon emission from a closed fermion loop is gauge-invariant, the currents in Eq. (22) satisfy the current conservation condition

$$L_{ab}J_b(\eta)=J_a(\eta)\,. \tag{23}$$

As a result, their expansions in terms of vector spherical harmonics contain only harmonics with $\lambda=1$, and hence, by orthonormality, only the terms in Eq. (21) with $\lambda=1$ actually contribute to the propagation of a virtual photon. In other words, as far as propagation of a virtual photon in closed loops is concerned, the photon propagator can be replaced by the effective propagator

$$D^{(0)}_{ab}(\eta_1,\eta_2)_{\text{eff}}=\sum_{nm}\frac{Y^{(1)}_{nma}(\eta_1)Y^{(1)}_{nmb}(\eta_2)^*}{(n+1)(n+2)}\,. \tag{24}$$

This fact will be of great utility in the next section, where it will allow us to immediately eliminate gauge degrees of freedom from the amplitude integral expression for the interlacing operator. (An explicit formula for the effective propagator is derived in the Appendix.)

We turn our attention next to the free electron propagator,

$$S^{(0)}(\eta_1,\eta_2)=-\frac{1}{\pi^2}\frac{\frac{1}{2}(\alpha\cdot\eta_1-1)\frac{1}{2}(\alpha\cdot\eta_2+1)}{(\eta_1-\eta_2)^4}\,, \tag{25}$$

which obeys the wave equation

$$h^{(0)}(\eta_1)S^{(0)}(\eta_1,\eta_2)=\delta_S(\eta_1-\eta_2)\tfrac{1}{2}(\alpha\cdot\eta_2+1)\,. \tag{26}$$

The electron Hamiltonian which appears here is

$$h^{(0)}(\eta)=2-L\cdot S\,, \tag{27a}$$

with the electron spin S [which satisfies the same O(5) commutation relations as does L] given by

$$S_{ab} = \tfrac{1}{4}[\alpha_a, \alpha_b]. \tag{27b}$$

Our aim is to find an expansion of the free electron propagator in terms of spinor eigenfunctions of the Hamiltonian in Eq. (27).

Let us begin by finding the irreducible representations of O(5) to which the spinor eigenfunctions of $h^{(0)}$ belong. Since it is obvious that

$$[L^2, h^{(0)}] = 0 \tag{28}$$

the spinor eigenfunctions may be characterized by a given eigenvalue $-2n(n+3)$ of L^2. For given n they will be linear combinations of terms of the form

$$Y_{nm}(\eta)\chi_s, \tag{29}$$

with χ_s a spinor with numerical (η-independent) components. Since we are using 8×8 α matrices, the spinor χ_s is an 8-component spinor and hence transforms under O(5) as the direct sum of *two* fundamental 4-dimensional spinor representations of type $(\frac{1}{2}, \frac{1}{2})$. The O(5) transformation behavior of the product wave function in Eq. (29) is therefore described by

$$(\tfrac{1}{2}, \tfrac{1}{2})\otimes(n,0)\oplus(\tfrac{1}{2}, \tfrac{1}{2})\otimes(n,0). \tag{30}$$

The irreducible representation content of each of the products in Eq. (30) may again be inferred from the dimension formula of Eq. (1), giving

$$(\tfrac{1}{2}, \tfrac{1}{2})\otimes(n,0) = (n+\tfrac{1}{2}, \tfrac{1}{2})\oplus(n-\tfrac{1}{2}, \tfrac{1}{2}) \tag{31a}$$

with

$$\begin{aligned}\dim(n+\tfrac{1}{2}, \tfrac{1}{2}) &= \tfrac{2}{3}(n+1)(n+2)(n+3),\\ \dim(n-\tfrac{1}{2}, \tfrac{1}{2}) &= \tfrac{2}{3}(n+1)(n+2)n,\end{aligned} \tag{31b}$$

$$\begin{aligned}\dim(\tfrac{1}{2}, \tfrac{1}{2})\times\dim(n,0) &= \tfrac{2}{3}(n+1)(n+2)(2n+3)\\ &= \dim(n+\tfrac{1}{2}, \tfrac{1}{2})+\dim(n-\tfrac{1}{2}, \tfrac{1}{2}).\end{aligned} \tag{31c}$$

Let us next find the eigenvalue spectrum of the electron Hamiltonian $h^{(0)}(\eta)$ in the subspace with a given eigenvalue $-2n(n+3)$ of L^2. This is easily done by noting that $L\cdot S$ satisfies the equation

$$\begin{aligned}(L\cdot S)^2 &= 3L\cdot S - \tfrac{1}{2}L^2\\ &= 3L\cdot S + n(n+3)\end{aligned} \tag{32}$$

with roots

$$L\cdot S = -n,\ n+3. \tag{33a}$$

The corresponding Hamiltonian eigenvalues are clearly given by

$$\begin{aligned}h^{(0)} &= 2 - L\cdot S\\ &= n+2,\ -(n+1).\end{aligned} \tag{33b}$$

Using Eq. (32) it is easy to construct projection operators on the subspaces with definite eigenvalues of $h^{(0)}$. We find

$$P_{-(n+1)} = \frac{n+L\cdot S}{2n+3}, \quad P_{n+2} = \frac{n+3-L\cdot S}{2n+3}; \tag{34a}$$

$$\begin{aligned}P_{-(n+1)}{}^2 &= P_{-(n+1)},\\ P_{n+2}{}^2 &= P_{n+2},\\ P_{-(n+1)} + P_{n+2} &= 1;\end{aligned} \tag{34b}$$

$$\begin{aligned}h^{(0)}P_{-(n+1)} &= -(n+1)P_{-(n+1)},\\ h^{(0)}P_{n+2} &= (n+2)P_{n+2}.\end{aligned} \tag{34c}$$

We can now proceed to construct a set of spinor eigenfunctions of $h^{(0)}$ by acting with the projection operators on the product wave functions of Eq. (29), obtaining[4]

$$\begin{aligned}\psi^{nms}_{-(n+1)} &= P_{-(n+1)}Y_{nm}\chi_s,\\ \psi^{nms}_{n+2} &= P_{n+2}Y_{nm}\chi_s.\end{aligned} \tag{35}$$

Finally, noting that

$$\frac{\mathrm{tr}_8 P_{-(n+1)}}{\mathrm{tr}_8 P_{n+2}} = \frac{n}{n+3} = \frac{\dim(n-\frac{1}{2}, \frac{1}{2})}{\dim(n+\frac{1}{2}, \frac{1}{2})}, \tag{36}$$

we infer that the eigenfunctions with $h^{(0)}$ eigenvalues of $-(n+1)$ and $n+2$ transform, respectively, according to the $(n-\frac{1}{2}, \frac{1}{2})$ and the $(n+\frac{1}{2}, \frac{1}{2})$ representations of O(5). We will find a direct confirmation of this when we discuss the completeness relation below.

Before proceeding, let us note a useful symmetry of the eigenvalue problem for $h^{(0)}$, which, as we will find later, will carry over to the case where an external electromagnetic field is present. The symmetry follows immediately from the relation

$$h^{(0)}\alpha\cdot\eta = -\alpha\cdot\eta h^{(0)}, \tag{37}$$

which may be verified by direct calculation. Using Eq. (37), we see that if $\psi_{\mu^{(0)}}$ is an eigenfunction of $h^{(0)}$ with eigenvalue $\mu^{(0)}$,

$$h^{(0)}\psi_{\mu^{(0)}} = \mu^{(0)}\psi_{\mu^{(0)}}, \tag{38a}$$

then

$$\psi_{-\mu^{(0)}} = \alpha\cdot\eta\psi_{\mu^{(0)}} \tag{38b}$$

is an eigenfunction with eigenvalue $-\mu^{(0)}$,

$$\begin{aligned}h^{(0)}\psi_{-\mu^{(0)}} &= -\alpha\cdot\eta h^{(0)}\psi_{\mu^{(0)}}\\ &= -\mu^{(0)}\psi_{-\mu^{(0)}}.\end{aligned} \tag{38c}$$

In other words, the eigenvalues of $h^{(0)}$ occur in pairs $\pm\mu^{(0)}$. We can see this explicitly from our construction of the spinor eigenfunctions given above. We note that when $n=0$ the eigenfunction $\psi^{nms}_{-(n+1)}$ vanishes,

$$\psi^{0ms}_{-1} = P_{-1}Y_{00}\chi_s = \tfrac{1}{3}L\cdot SY_{00}\chi_s = 0, \tag{39}$$

since Y_{00} is η-independent and hence vanishes when acted on by the η derivatives in L. Thus, the spectrum of $h^{(0)}$ contains

$$2\dim(n-\tfrac{1}{2},\tfrac{1}{2}) \text{ eigenvalues } -(n+1),\quad n=1,2,3,\ldots$$
$$2\dim(n+\tfrac{1}{2},\tfrac{1}{2}) \text{ eigenvalues } n+2,\quad n=0,1,2,\ldots, \tag{40}$$

which explicitly verifies the reflection symmetry.

In terms of the eigenfunctions of Eq. (35), we can write down the completeness relation for the electron eigenvalue problem,

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\dim(n,0)}\sum_{s=1}^{8}[\psi^{nms}_{-(n+1)}(\eta_1)\psi^{nms}_{-(n+1)}(\eta_2)^\dagger + \psi^{nms}_{n+2}(\eta_1)\psi^{nms}_{n+2}(\eta_2)^\dagger] = \delta_S(\eta_1-\eta_2)1_8, \tag{41}$$

with 1_8 the unit 8×8 matrix.

Equation (41) is easily verified by noting that

$$\sum_{s=1}^{8}\chi_s\chi_s^\dagger = 1_8,$$
$$\sum_{m=1}^{\dim(n,0)} Y_{nm}(\eta_1)Y_{nm}(\eta_2)^* = \frac{2n+3}{8\pi^2}C_n^{3/2}(\eta_1\cdot\eta_2), \tag{42}$$

with $C_n^{3/2}$ a Gegenbauer polynomial, as in Ref. 2. Substituting Eqs. (35) and (42) into Eq. (41) and using the Hermiticity of the projection operators and Eq. (34b), we find that the sum in Eq. (41) becomes

$$\sum_{n=0}^{\infty}\frac{2n+3}{8\pi^2}C_n^{3/2}(\eta_1\cdot\eta_2)1_8 = \delta_S(\eta_1-\eta_2)1_8, \tag{43}$$

as required. From Eq. (41) we can verify our identification of the eigenvalues $-(n+1)$ and $n+2$ with the respective O(5) representations $(n-\frac{1}{2},\frac{1}{2})$ and $(n+\frac{1}{2},\frac{1}{2})$. Letting Tr denote the complete trace

$$\mathrm{Tr}\,\mathcal{O} = \int d\Omega_\eta \mathrm{tr}_8\langle\eta|\mathcal{O}|\eta\rangle, \tag{44}$$

and letting

$$1^n_{-(n+1)},\ 1^n_{n+2} \tag{45}$$

denote the unit matrices in the subspaces with L^2 eigenvalue $-2n(n+3)$ and $h^{(0)}$ eigenvalues $-(n+1)$, $n+2$, respectively, we find

$$\begin{aligned}\mathrm{Tr}1^n_{-(n+1)} &= \mathrm{tr}_8\int d\Omega_\eta\sum_{m=1}^{\dim(n,0)}\sum_{s=1}^{8}\psi^{nms}_{-(n+1)}(\eta)^\dagger\psi^{nms}_{-(n+1)}(\eta)\\ &= \mathrm{tr}_8\frac{n+L\cdot S}{2n+3}\frac{8\pi^2}{3}\frac{2n+3}{8\pi^2}C_n^{3/2}(1)\\ &= 2\dim(n-\tfrac{1}{2},\tfrac{1}{2}),\\ \mathrm{Tr}1^n_{n+2} &= \mathrm{tr}_8\int d\Omega_\eta\sum_{m=1}^{\dim(n,0)}\sum_{s=1}^{8}\psi^{nms}_{n+2}(\eta)^\dagger\psi^{nms}_{n+2}(\eta)\\ &= \mathrm{tr}_8\frac{n+3-L\cdot S}{2n+3}\frac{8\pi^2}{3}\frac{2n+3}{8\pi^2}C_n^{3/2}(1)\\ &= 2\dim(n+\tfrac{1}{2},\tfrac{1}{2}).\end{aligned} \tag{46}$$

[In the final step we have used the fact that $C_n^{3/2}(1) = \frac{1}{2}(n+1)(n+2)$.] Equation (46) agrees with the eigenvalue counting summarized in Eq. (40).

We now have all the group-theoretic apparatus required for writing down the eigenfunction expansion of the free electron propagator. Let us begin by defining two auxiliary electron propagators $\tilde{S}^{(0)}(\eta_1,\eta_2)$ and $S_T^{(0)}(\eta_1,\eta_2)$,

$$\begin{aligned}\tilde{S}^{(0)}(\eta_1,\eta_2) &= -\frac{1}{\pi^2}\frac{\frac{1}{2}(\alpha\cdot\eta_1+1)\frac{1}{2}(\alpha\cdot\eta_2-1)}{(\eta_1-\eta_2)^4}\\ &= S^{(0)}(-\eta_1,-\eta_2),\\ S_T^{(0)}(\eta_1,\eta_2) &= S^{(0)}(\eta_1,\eta_2)+\tilde{S}^{(0)}(\eta_1,\eta_2)\\ &= \frac{1}{2\pi^2}\frac{1-\alpha\cdot\eta_1\alpha\cdot\eta_2}{(\eta_1-\eta_2)^4},\end{aligned} \tag{47}$$

which obey the wave equations

$$\begin{aligned}h^{(0)}(\eta_1)\tilde{S}^{(0)}(\eta_1,\eta_2) &= \delta_S(\eta_1-\eta_2)\tfrac{1}{2}(1-\alpha\cdot\eta_2),\\ h^{(0)}(\eta_1)S_T^{(0)}(\eta_1,\eta_2) &= \delta_S(\eta_1-\eta_2)1_8.\end{aligned} \tag{48}$$

Since $S^{(0)}$ and $\tilde{S}^{(0)}$ can be obtained from $S_T^{(0)}$ by projection,

$$\begin{aligned}S^{(0)}(\eta_1,\eta_2) &= \tfrac{1}{2}(1-\alpha\cdot\eta_1)S_T^{(0)}(\eta_1,\eta_2)\tfrac{1}{2}(1+\alpha\cdot\eta_2),\\ \tilde{S}^{(0)}(\eta_1,\eta_2) &= \tfrac{1}{2}(1+\alpha\cdot\eta_1)S_T^{(0)}(\eta_1,\eta_2)\tfrac{1}{2}(1-\alpha\cdot\eta_2),\end{aligned} \tag{49}$$

it suffices to find the eigenfunction expansion of $S_T^{(0)}$. This, however, can be immediately obtained from the completeness relation,[4]

$$\begin{aligned}S_T^{(0)}(\eta_1,\eta_2) &= (h^{(0)})^{-1}\delta_S(\eta_1-\eta_2)1_8\\ &= \sum_{n=0}^{\infty}\sum_{m=1}^{\dim(n,0)}\sum_{s=1}^{8}\left[\frac{\psi^{nms}_{-(n+1)}(\eta_1)\psi^{nms}_{-(n+1)}(\eta_2)^\dagger}{-(n+1)}+\frac{\psi^{nms}_{n+2}(\eta_1)\psi^{nms}_{n+2}(\eta_2)^\dagger}{n+2}\right].\end{aligned} \tag{50}$$

As our final consistency check we explicitly evaluate the sum in Eq. (50). Summing over m and s, we get

$$S_T^{(0)}(\eta_1,\eta_2) = \sum_{n=0}^{\infty}\frac{2n+3}{8\pi^2}\left[\frac{n+L_1\cdot S}{2n+3}\frac{C_n^{3/2}(\eta_1\cdot\eta_2)}{-(n+1)}+\frac{n+3-L_1\cdot S}{2n+3}\frac{C_n^{3/2}(\eta_1\cdot\eta_2)}{n+2}\right]. \tag{51}$$

Substituting the relation

$$L_1 \cdot S C_n^{3/2}(\eta_1\cdot\eta_2) = (\alpha\cdot\eta_1\alpha\cdot\eta_2 - z)\frac{d}{dz}C_n^{3/2}(z)\,,\qquad z=\eta_1\cdot\eta_2\,, \tag{52}$$

Eq. (51) can be reduced to the form

$$\begin{aligned}\frac{1}{8\pi^2}\left[1+(z-\alpha\cdot\eta_1\alpha\cdot\eta_2)\frac{d}{dz}\right]\sum_{n=0}^{\infty}\frac{2n+3}{(n+1)(n+2)}C_n^{3/2}(z) &= \frac{1}{8\pi^2}\left[1+(z-\alpha\cdot\eta_1\alpha\cdot\eta_2)\frac{d}{dz}\right]\frac{1}{1-z}\\ &= \frac{1}{8\pi^2}\frac{1-\alpha\cdot\eta_1\alpha\cdot\eta_2}{(1-\eta_1\cdot\eta_2)^2}\\ &= S_T^{(0)}(\eta_1,\eta_2)\,, \end{aligned} \tag{53}$$

which completes the check.

III. AMPLITUDE INTEGRAL FOR THE INTERLACING OPERATOR

With the group-theoretic preliminaries of the preceding section in mind, we proceed to develop an amplitude-integral formula for the interlacing operator which applies virtual-photon radiative corrections to fermion loops. We restrict ourselves at the outset to diagrams containing a single closed fermion loop, coupling to an arbitrary number of external photons. A convenient generating functional for this entire class of diagrams is obtained by calculating the vacuum amplitude W in the presence of an external electromagnetic potential A. We let $W^{(0)}[A]$ denote the single-fermion-loop vacuum amplitude with *no* internal virtual photons [Fig. 1(a)] and $W[A]$ denote the single-fermion-loop vacuum amplitude with all internal virtual-photon radiative corrections included [Fig. 1(b)]. The interlacing operator I is then defined as the linear operator which converts the functional $W^{(0)}[A]$ to $W[A]$, that is,

$$W[A]=I\,W^{(0)}[A]\,. \tag{54}$$

Our starting point in deriving an expression for I is the usual x-space path-integral formula for the interlacing operator,[5] which, in terms of the electromagnetic potential $a_\mu(x)$ in Euclidean x space, states that

$$W[a']=C\int[da]\exp\left[\int d^4x\,\mathcal{L}(x)\right]W^{(0)}[a+a']\,, \tag{55a}$$

where C is a normalizing constant, $\int[da]$ is the path integral

$$\int[da]=\prod_{\mu,x}\int da_\mu(x)\,, \tag{55b}$$

and $\mathcal{L}(x)$ is the electromagnetic kinetic Lagrangian

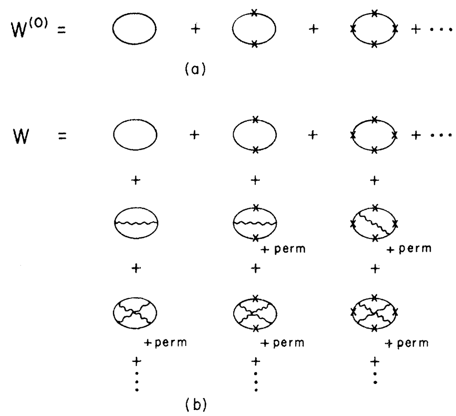

FIG. 1. (a) Diagrams contributing to the single-fermion-loop vacuum amplitude with no internal virtual photons. Each × denotes a coupling to the external potential A. (b) Diagrams contributing to the single-fermion-loop vacuum amplitude with all internal-virtual-photon radiative corrections included. "Perm" indicates that we must include all distinct permutations of the internal virtual photons and the external potential couplings ×.

$$\mathcal{L}(x)=-\tfrac{1}{4}f_{\mu\nu}f^{\mu\nu}\,,\quad f_{\mu\nu}=\partial_\mu a_\nu-\partial_\nu a_\mu\,. \tag{55c}$$

Equation (55) can be transcribed directly to the hyperspherical formalism, giving

$$W[A]=C'\int[dA]\exp\left[\int d\Omega_\eta\mathcal{L}(\eta)\right]W^{(0)}[A+A']\,, \tag{56a}$$

with $\int[dA]$ the path integral

$$\int[dA]=\prod_{a,\eta}\int dA_a(\eta) \tag{56b}$$

and with the kinetic Lagrangian given by

$$\mathcal{L}(\eta)=-\tfrac{1}{12}(F_{abc})^2\,,\qquad F_{abc}=L_{ab}A_c+L_{bc}A_a+L_{ca}A_b\,. \tag{56c}$$

The argument of the exponential in Eq. (56a) can be rewritten in a more convenient form by an integration by parts, giving

$$\begin{aligned}-\tfrac{1}{12}\int d\Omega_\eta(F_{abc})^2 &= -\tfrac{1}{4}\int d\Omega_\eta L_{ab}A_cF_{abc}\\ &=\tfrac{1}{4}\int d\Omega_\eta A_cL_{ab}F_{abc}\\ &=\tfrac{1}{4}\int d\Omega_\eta A_cP_{ca}A_a\,,\end{aligned} \tag{57}$$

with P_{ca} the wave operator given in Eq. (12). We next expand the potential A in terms of the orthonormalized vector spherical harmonics of Eq. (18),

$$A_a=\sum_{n,\lambda,m}c_{nm}^{(\lambda)}Y_{nma}^{(\lambda)}(\eta)\,, \tag{58}$$

thereby introducing a set of expansion amplitudes

$c_{nm}^{(\lambda)}$. Our aim is to reexpress the path integral in terms of an infinite product of ordinary integrals over the expansion amplitudes. We begin by substituting Eq. (58) into the expression in Eq. (57) for the argument of the exponential. Using the orthonormality of the vector spherical harmonics and the fact that

$$P_{ca}Y_{nma}^{(\lambda)}=0,\quad \lambda=-n,n+3$$
$$P_{ca}Y_{nma}^{(1)}=(L^2-4)Y_{nmc}^{(1)}=-2(n+1)(n+2)Y_{nmc}^{(1)},\tag{59}$$

we readily find that

$$\int d\Omega_\eta \mathcal{L}(\eta)=-\tfrac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\dim(n,1)}(n+1)(n+2)[c_{nm}^{(1)}]^2.\tag{60}$$

We see that the exponent has no dependence on the coefficient of those harmonics which fail to satisfy the hyperspherical Lorentz condition $L_{ab}Y_b=Y_a$. The same is true of the vacuum amplitude $W^{(0)}[A+A']$, by virtue of the argument which follows Eq. (23), and hence the entire integrand in Eq. (56a) depends only on the set of coefficients $c_{nm}^{(1)}$. We next must express the path integral $C'\int d[A]$ in terms of an integral over the amplitudes $c_{nm}^{(\lambda)}$. Proceeding heuristically, we write

$$\int[dA]=\left(\prod_{n,\lambda,m}\int_{-\infty}^{\infty}dc_{nm}^{(\lambda)}\right)J,\tag{61}$$

with J the Jacobian

$$J=\det\left[\frac{\partial A_a(\eta)}{\partial c_{nm}^{(\lambda)}}\right].\tag{62}$$

To calculate J, we note that

$$J^2=\det\left[\int d\Omega_\eta\frac{\partial A_a(\eta)}{\partial c_{nm}^{(\lambda)}}\frac{\partial A_a(\eta)}{\partial c_{n'm'}^{(\lambda')}}\right]$$
$$=\det\left[\int d\Omega_\eta Y_{nma}^{(\lambda)}(\eta)Y_{n'm'a}^{(\lambda')}(\eta)\right]$$
$$=\det[\delta_{nn'}\delta_{\lambda\lambda'}\delta_{mm'}]=1,\tag{63}$$

where we have used Eq. (19) and *have assumed that the orthonormalized vector spherical harmonics have been chosen to be real*. Hence we have $J=1$, and the Jacobian of the transformation is trivial. Finally we must choose the normalizing constant C'. Since the integrand has no dependence on $c_{nm}^{(-n)}$ and $c_{nm}^{(n+3)}$, the product

$$\prod_{n,m}\int_{-\infty}^{\infty}dc_{nm}^{(-n)}\int_{-\infty}^{\infty}dc_{nm}^{(n+3)}\tag{64}$$

gives simply an infinite constant factor which can be absorbed into C'. We can then determine C' by noting that Eq. (56a) must reduce to the trivial identity

$$W^{(0)}[0]=C'\left(\prod_{n=1}^{\infty}\prod_{m=1}^{\dim(n,1)}\int_{-\infty}^{\infty}dc_{nm}^{(1)}\right)\times\exp\left[\int d\Omega_\eta\mathcal{L}(\eta)\right]W^{(0)}[0]\tag{65}$$

when the electromagnetic coupling e is set equal to zero. We thus find that

$$C'=\prod_{n=1}^{\infty}\prod_{m=1}^{\dim(n,1)}\left[\frac{(n+1)(n+2)}{2\pi}\right]^{1/2},\tag{66}$$

giving as our final formula for the interlacing operator

$$W[A']=\left\{\prod_{n=1}^{\infty}\prod_{m=1}^{\dim(n,1)}\int_{-\infty}^{\infty}dc_{nm}^{(1)}\left[\frac{(n+1)(n+2)}{2\pi}\right]^{1/2}\right\}\exp\left\{-\tfrac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\dim(n,1)}(n+1)(n+2)[c_{nm}^{(1)}]^2\right\}W^{(0)}[A+A'],\tag{67a}$$

with

$$A_a(\eta)=\sum_{n=1}^{\infty}\sum_{m=1}^{\dim(n,1)}c_{nm}^{(1)}Y_{nma}^{(1)}(\eta).\tag{67b}$$

Two features of Eq. (67) merit special emphasis. First, the formula no longer involves path integrals and the troublesome limiting processes implicit in their definition, but rather involves only an infinite product of individually well-defined ordinary integrals. Second, gauge degrees of freedom, represented by the vector spherical harmonics with $\lambda=-n,n+3$, have been completely and explicitly eliminated.

Since our derivation of Eq. (67) has been somewhat heuristic, we will check the formula explicitly by expanding $W^{(0)}$ in a power series in A and showing that the interlacing operator does correctly reassemble the virtual-photon propagators. We write

$$W^{(0)}[A+A']=W^{(0)}[A']+\int d\Omega_{\eta_1}d\Omega_{\eta_2}A_{a_1}(\eta_1)A_{a_2}(\eta_2)W^{(0)}_{a_1a_2}[\eta_1\eta_2;A']$$
$$+\int d\Omega_{\eta_1}d\Omega_{\eta_2}d\Omega_{\eta_3}d\Omega_{\eta_4}A_{a_1}(\eta_1)A_{a_2}(\eta_2)A_{a_3}(\eta_3)A_{a_4}(\eta_4)W^{(0)}_{a_1a_2a_3a_4}[\eta_1\eta_2\eta_3\eta_4;A']$$
$$+\cdots+\int d\Omega_{\eta_1}\cdots d\Omega_{\eta_{2n}}A_{a_1}(\eta_1)\cdots A_{a_{2n}}(\eta_{2n})W^{(0)}_{a_1\cdots a_{2n}}[\eta_1\cdots\eta_{2n};A']+\cdots+\text{terms odd in }A,\tag{68}$$

where we have not indicated explicitly the terms odd in A because they vanish when substituted into the amplitude integral. The diagrammatic content of $W^{(0)}[A']$, $W^{(0)}_{a_1a_2}[\eta_1\eta_2;A'],\ldots$ is indicated in Fig. 2; when the free photon ends are linked with virtual photon propagators these become equal respectively to the first, second,... lines in Fig. 1(b). So to verify Eq. (67) we must check that the amplitude integral in Eq. (67a) converts the products of potentials $A_{a_1}(\eta_1)A_{a_2}(\eta_2)$, $A_{a_1}(\eta_1)\cdots A_{a_4}(\eta_4),\ldots$ appearing in Eq. (68) into the appropriate photon propagator factors. Beginning with the term containing two factors of A, and denoting as above

$$I=\left\{\prod_{n=1}^{\infty}\prod_{m=1}^{\dim(n,1)}\int_{-\infty}^{\infty}dc^{(1)}_{nm}\left[\frac{(n+1)(n+2)}{2\pi}\right]^{1/2}\right\}\times\exp\left\{-\tfrac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\dim(n,1)}(n+1)(n+2)[c^{(1)}_{nm}]^2\right\},\tag{69}$$

we find

$$IA_{a_1}(\eta_1)A_{a_2}(\eta_2)=\sum_{n_1m_1}\sum_{n_2m_2}Y^{(1)}_{n_1m_1a_1}(\eta_1)Y^{(1)}_{n_2m_2a_2}(\eta_2)Ic^{(1)}_{n_1m_1}c^{(1)}_{n_2m_2}.\tag{70}$$

Defining

$$J_l(n)=\int_{-\infty}^{\infty}dc\left[\frac{(n+1)(n+2)}{2\pi}\right]^{1/2}\exp[-\tfrac{1}{2}(n+1)(n+2)c^2]\,c^{2l}=\frac{(2l-1)!!}{[(n+1)(n+2)]^l},\tag{71}$$

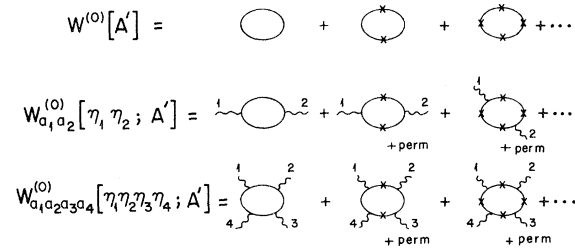

FIG. 2. Diagrammatic content of the amplitudes which appear in the expansion of $W^{(0)}$ in powers of A. "Perm" indicates that we must include all distinct permutations of the external potential couplings × with respect to the external photon vertices (external photon labels are *not* permuted).

we easily see that

$$Ic^{(1)}_{n_1m_1}c^{(1)}_{n_2m_2}=\delta_{n_1n_2}\delta_{m_1m_2}J_1(n_1)=\frac{\delta_{n_1n_2}\delta_{m_1m_2}}{(n_1+1)(n_1+2)}.\tag{72a}$$

Hence the interlacing operator acting on the product of two A's gives

$$IA_{a_1}(\eta_1)A_{a_2}(\eta_2)=\sum_{n_1m_1}\frac{Y^{(1)}_{n_1m_1a_1}(\eta_1)Y^{(1)}_{n_1m_1a_2}(\eta_2)}{(n_1+1)(n_1+2)}=D^{(0)}_{a_1a_2}(\eta_1,\eta_2)_{\rm eff},\tag{72b}$$

which is just the effective photon propagator discussed in Sec. II. We next consider the term in Eq. (68) containing four factors of A. Noting that

$$J_2(n)=\frac{3}{[(n+1)(n+2)]^2},\tag{73a}$$

we find that

$$\begin{aligned}IA_{a_1}(\eta_1)A_{a_2}(\eta_2)A_{a_3}(\eta_3)A_{a_4}(\eta_4)=&\sum_{n_1m_1\neq n_2m_2}\frac{1}{(n_1+1)(n_1+2)(n_2+1)(n_2+2)}[Y^{(1)}_{n_1m_1a_1}(\eta_1)Y^{(1)}_{n_1m_1a_2}(\eta_2)Y^{(1)}_{n_2m_2a_3}(\eta_3)Y^{(1)}_{n_2m_2a_4}(\eta_4)\\&+Y^{(1)}_{n_1m_1a_1}(\eta_1)Y^{(1)}_{n_1m_1a_3}(\eta_3)Y^{(1)}_{n_2m_2a_2}(\eta_2)Y^{(1)}_{n_2m_2a_4}(\eta_4)\\&+Y^{(1)}_{n_1m_1a_1}(\eta_1)Y^{(1)}_{n_1m_1a_4}(\eta_4)Y^{(1)}_{n_2m_2a_2}(\eta_2)Y^{(1)}_{n_2m_2a_3}(\eta_3)]\\&+3\sum_{nm}\frac{1}{(n+1)^2(n+2)^2}Y^{(1)}_{nma_1}(\eta_1)Y^{(1)}_{nma_2}(\eta_2)Y^{(1)}_{nma_3}(\eta_3)Y^{(1)}_{nma_4}(\eta_4)\\=&D^{(0)}_{a_1a_2}(\eta_1,\eta_2)_{\rm eff}D^{(0)}_{a_3a_4}(\eta_3,\eta_4)_{\rm eff}\\&+D^{(0)}_{a_1a_3}(\eta_1,\eta_3)_{\rm eff}D^{(0)}_{a_2a_4}(\eta_2,\eta_4)_{\rm eff}+D^{(0)}_{a_1a_4}(\eta_1,\eta_4)_{\rm eff}D^{(0)}_{a_2a_3}(\eta_2,\eta_3)_{\rm eff},\end{aligned}\tag{73b}$$

which is again the required string of photon propagators. The argument continues in the same fashion for the higher terms in Eq. (68). In particular, wherever an overlap of $2l$ modes with the same indices n,m is encountered, the amplitude integral in Eq. (71) supplies a combinatoric factor $(2l-1)!!$, which is just the number of ways of dividing $2l$ objects into l groups of 2. But this is just equal to the number of terms in the propagator subchain

$$D^{(0)}_{a_{i(1)}a_{i(2)}}(\eta_{i(1)},\eta_{i(2)})_{\rm eff}\cdots D^{(0)}_{a_{i(2l-1)}a_{i(2l)}}(\eta_{i(2l-1)},\eta_{i(2l)})_{\rm eff}+[(2l-1)!!-1]\text{ other orderings},\tag{74}$$

each term of which contains one l-fold overlap involving the mode n,m. We conclude that for the general term in Eq. (68) we have

$$IA_{a_1}(\eta_1)\cdots A_{a_{2n}}(\eta_{2n})=D^{(0)}_{a_1a_2}(\eta_1,\eta_2)_{\rm eff}\cdots D^{(0)}_{a_{2n-1}a_{2n}}(\eta_{2n-1},\eta_{2n})_{\rm eff}+[(2n-1)!!-1]\text{ other orderings}, \tag{75}$$

which by Wick's theorem is just the required virtual-photon propagator string.

Having concluded our discussion of the interlacing operator, we consider next the problem of constructing the non-radiative-corrected vacuum functional $W^{(0)}[A]$. According to the usual perturbation-theory rules, we find

$$W^{(0)}[A]-W^{(0)}[0]=-\sum_{n=1}^{\infty}\frac{\mathrm{Tr}[S^{(0)}ie\,\alpha\cdot A]^n}{n}, \tag{76}$$

where we have introduced an operator notation

$$\begin{aligned}\langle\eta_1|S^{(0)}|\eta_2\rangle&=S^{(0)}(\eta_1,\eta_2),\\ \langle\eta_1|A_a|\eta_2\rangle&=\delta_S(\eta_1-\eta_2)A_a(\eta_1),\end{aligned} \tag{77}$$

and where the factor n^{-1} in Eq. (76) is a combinatoric factor characteristic of vacuum amplitudes. [The terms in Eq. (76) with n odd will of course vanish, by Furry's theorem.] Noting that

$$S^{(0)}=(2-L\cdot S)^{-1}\tfrac{1}{2}(1+\alpha\cdot\eta) \tag{78}$$

and summing the power series in Eq. (76), we get

$$W^{(0)}[A]-W^{(0)}[0]=\mathrm{Tr}\ln[1-(2-L\cdot S)^{-1}\tfrac{1}{2}(1+\alpha\cdot\eta)ie\,\alpha\cdot A]. \tag{79}$$

Hence, up to an A-independent constant (which is of no interest in our subsequent discussion) we can write

$$W^{(0)}[A]=\mathrm{Tr}\ln[h], \tag{80}$$

where

$$h=2-L\cdot S-\tfrac{1}{2}(1+\alpha\cdot\eta)ie\,\alpha\cdot A.$$

Equation (80), while simple in appearance, has an important defect: Although the free part of h is a Hermitian operator, the interaction term $\frac{1}{2}(1+\alpha\cdot\eta)ie\,\alpha\cdot A$ is not Hermitian, and therefore we cannot express the problem of evaluating Eq. (80) in terms of a Hermitian eigenvalue problem.

In order to remedy this defect, we use the fact that the projection operator P_{ca} satisfies

$$\eta_c P_{ca}=0; \tag{81a}$$

hence the individual harmonics $Y^{(1)}_{nma}(\eta)$ entering into the amplitude-integral formula, as well as the potential $A_a(\eta)$ itself, satisfy the constraint[6]

$$\eta_a Y^{(1)}_{nma}(\eta)=0, \tag{81b}$$

$$\eta_a A_a(\eta)=0. \tag{81c}$$

We will now show that for the class of potentials satisfying Eq. (81c), Eq. (76) can be rewritten in the alternative form

$$W^{(0)}[A]-W^{(0)}[0]=-\tfrac{1}{2}\sum_{n=1}^{\infty}\frac{\mathrm{Tr}[S_T^{(0)}ie\,\alpha\cdot\eta\,\alpha\cdot A]^n}{n}, \tag{82}$$

where $S_T^{(0)}$ is the auxiliary propagator defined in Eq. (47). To see this, we consider a portion of the propagator-vertex string appearing in the nth-order term in Eq. (82):

$$\begin{aligned}\int d\Omega_{\eta_2}\cdots S_T^{(0)}(\eta_1,\eta_2)ie\,\alpha\cdot\eta_2\alpha\cdot A(\eta_2)S_T^{(0)}(\eta_2,\eta_3)\cdots\\ =\int d\Omega_{\eta_2}\cdots[S^{(0)}(\eta_1,\eta_2)ie\,\alpha\cdot\eta_2\alpha\cdot A(\eta_2)S^{(0)}(\eta_2,\eta_3)+\tilde S^{(0)}(\eta_1,\eta_2)ie\,\alpha\cdot\eta_2\alpha\cdot A(\eta_2)S^{(0)}(\eta_2,\eta_3)\\ +S^{(0)}(\eta_1,\eta_2)ie\,\alpha\cdot\eta_2\alpha\cdot A(\eta_2)\tilde S^{(0)}(\eta_2,\eta_3)+\tilde S^{(0)}(\eta_1,\eta_2)ie\,\alpha\cdot\eta_2\alpha\cdot A(\eta_2)\tilde S^{(0)}(\eta_2,\eta_3)]\cdots.\end{aligned} \tag{83}$$

The terms containing one factor $S^{(0)}$ and one factor $\tilde S^{(0)}$ vanish, since

$$(\alpha\cdot\eta_2+1)\,\alpha\cdot\eta_2\alpha\cdot A(\eta_2)(\alpha\cdot\eta_2+1)=(\alpha\cdot\eta_2-1)\,\alpha\cdot\eta_2\alpha\cdot A(\eta_2)(\alpha\cdot\eta_2-1)=0$$

by virtue of the constraint, Eq. (81c). The terms containing two factors $S^{(0)}$ or two factors $\tilde S^{(0)}$ can be simplified by using

$$(\alpha\cdot\eta_2\pm1)\alpha\cdot\eta_2=(\alpha\cdot\eta_2\pm1)(\pm1), \tag{84}$$

reducing Eq. (83) to

$$\int d\Omega_{\eta_2}\cdots\{S^{(0)}(\eta_1,\eta_2)ie\,\alpha\cdot A(\eta_2)S^{(0)}(\eta_2,\eta_3)+\tilde S^{(0)}(\eta_1,\eta_2)[-ie\,\alpha\cdot A(\eta_2)]\tilde S^{(0)}(\eta_2,\eta_3)\}\cdots. \tag{85}$$

Continuing in this fashion entirely around the nth-order propagator-vertex string, we find that Eq. (82) reduces to

$$W^{(0)}[A]-W^{(0)}[0]=-\tfrac{1}{2}\sum_{n=1}^{\infty}\frac{\mathrm{Tr}[S^{(0)}ie\,\alpha\cdot A]^n+\mathrm{Tr}[\tilde S^{(0)}ie\,\alpha\cdot A]^n(-1)^n}{n}. \tag{86}$$

But on reverse-ordering the α-matrix factors and relabeling the coordinates in the second term, one sees that

$$\mathrm{Tr}[\tilde{S}^{(0)}ie\,\alpha\cdot A]^n = \mathrm{Tr}[S^{(0)}ie\,\alpha\cdot A]^n\,, \tag{87}$$

which gives finally

$$W^{(0)}[A] - W^{(0)}[0] = -\sum_{n\ \mathrm{even}}^{\infty} \frac{\mathrm{Tr}[S^{(0)}ie\,\alpha\cdot A]^n}{n}\,, \tag{88}$$

which by Furry's theorem is identical to Eq. (76). Having verified the correctness of Eq. (82), we substitute

$$S_T^{(0)} = (2 - L\cdot S)^{-1} \tag{89}$$

and sum the series into a logarithm, giving

$$W^{(0)}[A] - W^{(0)}[0] = \tfrac{1}{2}\,\mathrm{Tr}\ln[1-(2-L\cdot S)^{-1}\alpha\cdot\eta ie\,\alpha\cdot A]\,. \tag{90}$$

Again, up to an A-independent constant, this is equivalent to

$$W^{(0)}[A] = \tfrac{1}{2}\,\mathrm{Tr}\ln h_T\,,$$
$$h_T = 2 - L\cdot S - ie\,\alpha\cdot\eta\,\alpha\cdot A\,. \tag{91a}$$

The Hamiltonian appearing in Eq. (91) has the desired Hermiticity property, since by virtue of Eq. (81c) we have

$$[ie\,\alpha\cdot\eta\,\alpha\cdot A]^\dagger = -ie\,\alpha\cdot A\,\alpha\cdot\eta$$
$$= ie\,\alpha\cdot\eta\,\alpha\cdot A\,. \tag{91b}$$

Having now a Hermitian Hamiltonian, we consider the eigenvalue problem

$$h_T\psi_\mu = \mu\psi_\mu\,, \tag{92}$$

analogous to Eq. (38a) of the noninteracting case. Because

$$h_T\,\alpha\cdot\eta = -\alpha\cdot\eta\, h_T\,, \tag{93}$$

the argument of Eq. (38) tells us that the eigenvalues occur in pairs $\pm\mu$. Since the eigenstates ψ_μ of h_T form a complete set, we can evaluate the trace appearing in Eq. (91) in the basis in which h_T is diagonal, giving for the vacuum amplitude (again modulo a constant)

$$W^{(0)}[A] = \sum_{\mu>0} \ln\mu\,. \tag{94}$$

Thus, the study of the behavior of $W^{(0)}[A]$ is equivalent to the study of the eigenvalue problem posed in Eqs. (91) and (92). Up to this point we have neglected renormalizations, which are necessary because the two smallest vacuum diagrams, illustrated in Fig. 3(a), are divergent. To make Eq. (94) well defined, we must subtract off the first two terms in its power-series expansion in α, obtaining the renormalized expression

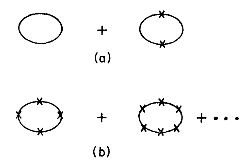

FIG. 3. (a) Diagrams contributing to $W^{(0)}$ which are divergent and therefore require renormalization subtractions; (b) diagrams contributing to $W^{(0)}$ which are convergent.

$$\tilde{W}^{(0)}[A] = W^{(0)}[A] - W^{(0)}[0] - \alpha\left\{\frac{d}{d\alpha}W^{(0)}[A]\right\}_{\alpha=0}\,. \tag{95a}$$

Substituting Eq. (94), Eq. (95a) reads

$$\tilde{W}^{(0)}[A] = \ln\Big[\prod_{\mu>0}\mu\Big/\prod_{\mu^{(0)}>0}\mu^{(0)}\Big] - \alpha\left\{\frac{d}{d\alpha}\ln\Big[\prod_{\mu>0}\mu\Big/\prod_{\mu^{(0)}>0}\mu^{(0)}\Big]\right\}_{\alpha=0}\,. \tag{95b}$$

Equations (95a) and (95b) receive contributions only from the larger loop diagrams of Fig. (3b), and therefore are finite.

IV. FINITE-MODE-NUMBER APPROXIMATION

We now wish to apply our amplitude-integral formalism to study the analyticity properties in coupling constant $\alpha = e^2/4\pi$ of single-fermion-loop diagrams, and in particular to examine whether they can develop an infinite-order zero (i.e., an essential singularity) when α approaches some positive value α_0. We distinguish at the outset two alternative ways in which an essential singularity could appear. (i) Any finite subproduct of the infinite product of ordinary integrals in Eq. (67a) exists, but a divergence appears when the limit of an infinite product is taken. The analysis of this case is difficult, and will not be considered further, apart from a brief mention in the concluding paragraph. We confine our attention henceforth to a more interesting possibility: (ii) An essential singularity occurs even when Eq. (67a) is truncated down to a finite product. In this case, instead of studying the full $W[A']$ we study the approximant $W_k[A']$ defined by

$$W_k[A'] = \left\{\prod_{n=1}^{k}\prod_{m=1}^{\dim(n,1)}\int_{-\infty}^{\infty} dc_{nm}^{(1)}\left[\frac{(n+1)(n+2)}{2\pi}\right]^{1/2}\right\} \times\exp\left\{-\tfrac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{\dim(n,1)}(n+1)(n+2)[c_{nm}^{(1)}]^2\right\} \times W^{(0)}[A+A']\,,$$
$$A_a(\eta) = \sum_{n=1}^{k}\sum_{m=1}^{\dim(n,1)} c_{nm}^{(1)}\,Y_{nma}^{(1)}(\eta)\,. \tag{96}$$

Following the argument of Eqs. (68)–(75), we see that Eq. (96) corresponds to the approximation of replacing the full effective photon propagator $D^{(0)}_{a_1a_2}(\eta_1,\eta_2)_{\text{eff}}$ by the truncated effective propagator

$$D^{(0)}_{ka_1a_2}(\eta_1,\eta_2)_{\text{eff}}=\sum_{n=1}^{k}\sum_{m=1}^{\dim(n,1)}\frac{Y^{(1)}_{nma_1}(\eta_1)Y^{(1)}_{nma_2}(\eta_2)}{(n+1)(n+2)}\,, \tag{97}$$

and then using this truncated propagator to calculate virtual-photon radiative corrections to all orders. For distances $(\eta_1-\eta_2)^2$ of order unity, Eq. (97) is qualitatively similar to the full propagator; substantial differences appear only in the short-distance limit $\eta_1\to\eta_2$, where the full propagator becomes singular, while Eq. (97) remains finite. Thus, the study of case (ii), on which we now embark, is essentially an examination of whether an essential singularity in coupling constant can appear in a model of massless electrodynamics in which fermions are treated exactly, but in which the short-distance singularity of the photon propagator is cut off.

To proceed, we note that $W^{(0)}[A+A']$ depends on the electric charge e only through the combination $e(A+A')$, so by writing

$$W^{(0)}[A+A']=\overline{W}^{(0)}[e(A+A')] \tag{98a}$$

and then making the rescalings

$$A\to e^{-1}A,\quad A'\to e^{-1}A' \tag{98b}$$

we can make the coupling-constant dependence of the right-hand side of Eq. (96) completely explicit. Introducing the abbreviated notation

$$\begin{aligned}&\prod_{n=1}^{k}\prod_{m=1}^{\dim(n,1)}\equiv\prod_{1}^{N_k},\\ &\sum_{n=1}^{k}\sum_{m=1}^{\dim(n,1)}=\sum_{1}^{N_k},\\ &N_k=\sum_{n=1}^{k}\dim(n,1)\\ &\quad=\tfrac{1}{2}k(k+1)[\tfrac{1}{2}k(k+1)+3(k+2)]\,,\end{aligned} \tag{99}$$

we find

$$\begin{aligned}W_k[e^{-1}A']=&\left\{\prod_1^{N_k}\int_{-\infty}^{\infty}dc^{(1)}_{nm}\left[\frac{(n+1)(n+2)}{2\pi e^2}\right]^{1/2}\right\}\\ &\times\exp\left\{-\frac{1}{2e^2}\sum_1^{N_k}(n+1)(n+2)[c^{(1)}_{nm}]^2\right\}\\ &\times\overline{W}^{(0)}[A+A']\,.\end{aligned} \tag{100}$$

Finally, transforming to polar coordinates in the N_k-dimensional space of the $c^{(1)}_{nm}$ by introducing the new integration variables

$$\begin{aligned}&a^2=\sum_1^{N_k}(n+1)(n+2)[c^{(1)}_{nm}]^2\,,\\ &a^{(1)}_{nm}=[(n+1)(n+2)]^{1/2}c^{(1)}_{nm}/a\,,\end{aligned} \tag{101a}$$

and using the formula

$$\prod_1^{N_k}\int_{-\infty}^{\infty}\frac{da^{(1)}_{nm}}{(2\pi)^{1/2}}\,\delta\left(1-\sum_1^{N_k}[a^{(1)}_{nm}]^2\right)=[2^{N_k/2}\Gamma(\tfrac{1}{2}N_k)]^{-1}\,, \tag{101b}$$

we arrive at the expression

$$\begin{aligned}W_k[e^{-1}A']=&\frac{1}{(2e^2)^{N_k/2}\Gamma(\frac{1}{2}N_k)}\\ &\times\int_0^{\infty}d(a^2)a^{N_k-2}\exp\left(-\frac{a^2}{2e^2}\right)w_k(a^2,A')\,.\end{aligned} \tag{102a}$$

Here

$$w_k(a^2,A')=\frac{\left(\prod_1^{N_k}\int_{-\infty}^{\infty}da^{(1)}_{nm}\right)\delta\left(1-\sum_1^{N_k}[a^{(1)}_{nm}]^2\right)\overline{W}^{(0)}[A+A']}{\left(\prod_1^{N_k}\int_{-\infty}^{\infty}da^{(1)}_{nm}\right)\delta\left(1-\sum_1^{N_k}[a^{(1)}_{nm}]^2\right)} \tag{102b}$$

is the angular average of $\overline{W}^{(0)}[A+A']$ in the space of the coefficients $c^{(1)}_{nm}$ for fixed "radius" a.

It is immediately evident from Eq. (102) that the study of the analyticity in coupling constant of W_k depends on the properties of the external-field vacuum amplitude $\overline{W}^{(0)}[A+A']$.[7] Hence we begin our discussion by summarizing some facts about the behavior of the external-field problem.

(i) For small enough a, the perturbation series for the renormalized vacuum functional $\tilde{W}^{(0)}[A]$ is convergent. This is easily proved by majorization of the power-series expansion for $\tilde{W}^{(0)}[A]$. To begin, we use the Schwarz inequality to bound the absolute magnitude of the vector potential $A_b(\eta)$ on the hypersphere in terms of a. Writing

$$A_b(\eta)=a\sum_1^{N_k}\frac{a^{(1)}_{nm}Y^{(1)}_{nmb}(\eta)}{[(n+1)(n+2)]^{1/2}}\,, \tag{103a}$$

we have

$$|A_b(\eta)|^2\le|a|^2\left(\sum_1^{N_k}[a^{(1)}_{nm}]^2\right)\left[\sum_{n=1}^{k}\frac{1}{(n+1)(n+2)}\sum_{m,a}|Y^{(1)}_{nma}(\eta)|^2\right]. \tag{103b}$$

But

$$\sum_{m,a} |Y^{(1)}_{nma}(\eta)|^2 \le \sum_{m,a} [|Y^{(1)}_{nma}(\eta)|^2 + |Y^{(-n)}_{nma}(\eta)|^2 + |Y^{(n+3)}_{nma}(\eta)|^2]$$
$$= 5\,\frac{2n+3}{8\pi^2}\, C_n^{3/2}(1)$$
$$= \frac{5}{16\pi^2}(n+1)(n+2)(2n+3)\,, \tag{104a}$$

giving the bound

$$|A_b(\eta)| \le |a|\left[\frac{5}{16\pi^2}k(k+4)\right]^{1/2} \equiv A_{\max}. \tag{104b}$$

We turn now to the power series for $\tilde{W}^{(0)}[A]$ which, since we have subtracted off the terms of order e^0 and e^2, takes the form

$$\tilde{W}^{(0)}[A] = \sum_{l=2}^{\infty} e^{2l}\, \tilde{W}^{(0)}_{2l}[A]\,. \tag{105}$$

Consulting the Feynman rules in Table I, we find that the general term in Eq. (105) is given by

$$\tilde{W}^{(0)}_{2l}[A] = \left(\prod_{j=1}^{2l} \int d\Omega_{\eta_j}\right)(-)\,\mathrm{tr}_8[\alpha\cdot A(\eta_1) S^{(0)}(\eta_1,\eta_2)\,\alpha\cdot A(\eta_2) S^{(0)}(\eta_2,\eta_3)\cdots\alpha\cdot A(\eta_{2l}) S^{(0)}(\eta_{2l},\eta_1)][(-1)^l/2l]\,. \tag{106}$$

To bound the trace in Eq. (106), we note that for a product $M_1\cdots M_p$ of 8×8 matrices we have

$$|\mathrm{tr}_8(M_1\cdots M_p)| \le 8 \max_{1\le i,j\le 8} |(M_1\cdots M_p)_{ij}|\,. \tag{107}$$

But for any i and j we have

$$|(M_1\cdots M_p)_{ij}|^2 \le \sum_i\sum_j |(M_1\cdots M_p)_{ij}|^2 = \sum_i\sum_j\Big|\sum_k M_{1ik}(M_2\cdots M_p)_{kj}\Big|^2$$
$$\le \sum_i\sum_k |M_{1ik}|^2 \sum_k\sum_j |(M_2\cdots M_p)_{kj}|^2 = \mathrm{tr}_8(M_1M_1^\dagger)\sum_k\sum_j |(M_2\cdots M_p)_{kj}|^2$$
$$\le \cdots \le \mathrm{tr}_8(M_1M_1^\dagger)\cdots\mathrm{tr}_8(M_pM_p^\dagger) \tag{108a}$$

so we learn that

$$|\mathrm{tr}_8(M_1\cdots M_p)| \le 8[\mathrm{tr}_8(M_1M_1^\dagger) \times \mathrm{tr}_8(M_2M_2^\dagger)\cdots\mathrm{tr}_8(M_pM_p^\dagger)]^{1/2}. \tag{108b}$$

To apply this inequality to Eq. (106), we note that

$$\mathrm{tr}_8(\alpha\cdot A(\eta_1)[\alpha\cdot A(\eta_1)]^\dagger) \le 8 A_{\max}^2\,,$$
$$\mathrm{tr}_8[S^{(0)}(\eta_1,\eta_2) S^{(0)}(\eta_1,\eta_2)^\dagger] = \frac{1}{8\pi^4(1-\eta_1\cdot\eta_2)^3}\,, \tag{109}$$

giving

$$|\tilde{W}^{(0)}_{2l}[A]| \le \frac{8 A_{\max}{}^{2l}}{2l\pi^{4l}}\, t_{2l}\,,$$
$$t_{2l} = \left(\prod_{j=1}^{2l}\int d\Omega_{\eta_j}\right)(1-\eta_1\cdot\eta_2)^{-3/2}(1-\eta_2\cdot\eta_3)^{-3/2} \times\cdots(1-\eta_{2l-1}\cdot\eta_{2l})^{-3/2}(1-\eta_{2l}\cdot\eta_1)^{-3/2}. \tag{110}$$

To evaluate t_{2l}, we substitute the hyperspherical harmonic expansion

$$(1-\eta_1\cdot\eta_2)^{-3/2} = \tfrac{1}{3}(2^{3/2}8\pi^2) \times\sum_{nm}\frac{3}{2n+3}\,Y_{nm}(\eta_1)Y_{nm}(\eta_2)^*\,, \tag{111}$$

obtaining

$$t_{2l} = (\tfrac{1}{3}2^{3/2}8\pi^2)^{2l}\sum_{n=0}^{\infty}\left(\frac{3}{2n+3}\right)^{2l}\tfrac{1}{6}(n+1)(n+2)(2n+3)$$
$$= (\tfrac{1}{3}2^{3/2}8\pi^2)^{2l}[1+5\times(\tfrac{3}{5})^{2l}+\cdots]\,. \tag{112}$$

Thus, we conclude that

$$e^{2l}|\tilde{W}^{(0)}_{2l}[A]| \le (\tfrac{8}{3}2^{3/2}eA_{\max})^{2l}\frac{4}{l}[1+5\times(\tfrac{3}{5})^{2l}+\cdots], \quad l\ge 3 \tag{113}$$

indicating that $\tilde{W}^{(0)}[A]$ has a convergent power-series expansion[8] for small external fields, extending at least to a radius of convergence $eA_{\max} = (\frac{8}{3}2^{3/2})^{-1}$. The fact that there is a nonzero radius of convergence is important, since it means that the quantity $\tilde{W}^{(0)}[A]$ appearing in the amplitude integral formula is uniquely specified by perturbation theory for small A.

(ii) The renormalized vacuum functional $\tilde{W}^{(0)}[A]$ *satisfies the inequality* $\mathrm{Re}\tilde{W}^{(0)}[A] \le P_{12}(|a|)$, *with* P_{12} *a twelth-degree polynomial with finite coefficients.*[9] To show this we start from Eq. (90), which, we recall, states that

$$W^{(0)}[A] - W^{(0)}[0] = \tfrac{1}{2}\,\mathrm{Tr}\ln[1 - S_T^{(0)}\alpha\cdot\eta ie\,\alpha\cdot A]\,. \tag{114}$$

Since $W^{(0)}[A]$ is even in A, we can average Eq. (114) for A and $-A$, obtaining

$$W^{(0)}[A]-W^{(0)}[0]=\tfrac{1}{4}\{\mathrm{Tr}\ln[1-S_T^{(0)}\alpha\cdot\eta ie\alpha\cdot A]+\mathrm{Tr}\ln[1+S_T^{(0)}\alpha\cdot\eta ie\alpha\cdot A]\}=\tfrac{1}{4}\mathrm{Tr}\ln(1+K)\,, \qquad (115a)$$

with

$$K=-(S_T^{(0)}\alpha\cdot\eta ie\alpha\cdot A)^2\,. \qquad (115b)$$

Noting that $\frac{1}{4}\mathrm{Tr}K$ is just the second-order diagram in Fig. 3(a), we see that the renormalized vacuum functional is given by

$$\tilde{W}^{(0)}[A]=\tfrac{1}{4}\mathrm{Tr}\,[\ln(1+K)-K]\,. \qquad (116)$$

Hence

$$\mathrm{Re}\tilde{W}^{(0)}[A]=\tfrac{1}{8}\mathrm{Tr}[\ln(1+K)-K+\ln(1+K^\dagger)-K^\dagger]=\tfrac{1}{8}\mathrm{Tr}\{[\ln(1+B)-B+\tfrac{1}{2}B^2-\tfrac{1}{3}B^3]+B-\tfrac{1}{2}B^2+\tfrac{1}{3}B^3-(K+K^\dagger)\}\,, \qquad (117a)$$

with

$$B=K+K^\dagger+KK^\dagger=(1+K)(1+K)^\dagger-1\,. \qquad (117b)$$

Since $(1+K)(1+K)^\dagger$ is a positive-semidefinite operator, B satisfies the operator inequality $B\geq -1$. Thus, using the fact that

$$\ln(1+B')-B'+\tfrac{1}{2}(B')^2-\tfrac{1}{3}(B')^3\leq 0\,,\quad B'\geq -1 \qquad (118a)$$

we can conclude that

$$\mathrm{Tr}[\ln(1+B)-B+\tfrac{1}{2}B^2-\tfrac{1}{3}B^3]\leq 0\,, \qquad (118b)$$

which gives immediately an upper bound for $\mathrm{Re}\tilde{W}^{(0)}[A]$,

$$\mathrm{Re}\tilde{W}^{(0)}[A]\leq\tfrac{1}{8}\mathrm{Tr}[B-\tfrac{1}{2}B^2+\tfrac{1}{3}B^3-(K+K^\dagger)]\,. \qquad (119)$$

The right-hand side of Eq. (119) is a twelfth-degree polynomial in the external potential A; to see that it has finite coefficients we substitute Eq. (117b) for B to get

$$\mathrm{Tr}[B-\tfrac{1}{2}B^2+\tfrac{1}{3}B^3-(K+K^\dagger)]=-\mathrm{Re}\,\mathrm{Tr}[K^2]+\mathrm{Tr}[O(K^3,K^\dagger K^2,(K^\dagger)^2K)+\cdots]\,. \qquad (120)$$

But $\mathrm{Tr}[K^2]$ is just proportional to the fourth-order diagram in Fig. 3(b), and hence is finite (although only conditionally convergent), while the remaining terms are easily seen to be absolutely convergent by the bounding argument of Eqs. (105)–(113).

(iii) The complete (multi-fermion-loop) vacuum amplitude $\Delta(a)=\exp\{\tilde{W}^{(0)}[A]\}$ *is an entire function of the complex variable a, for arbitrary direction cosines* $a_{nm}^{(1)}$. This result, first derived by Matthews and Salam[9] and Schwinger,[9] follows heuristically from the fact that exponentiation of Eq. (79) gives

$$\exp\{W^{(0)}[A]\}\propto\exp\mathrm{Tr}\ln[1+S^{(0)}\tfrac{1}{2}(1+\alpha\cdot\eta)ie\alpha\cdot A]=\det[1+S^{(0)}\tfrac{1}{2}(1+\alpha\cdot\eta)ie\alpha\cdot A]\,, \qquad (121)$$

which is a Fredholm determinant and therefore is an entire function of a. The need for renormalization subtractions does not alter the conclusion, since the factor needed to convert $\exp\{W^{(0)}[A]\}$ to $\exp\{\tilde{W}^{(0)}[A]\}$ is the exponential of a quadratic polynomial in a, and is therefore entire. Note that if instead of using Eq. (79) we exponentiate Eq. (90), we get

$$\exp\{W^{(0)}[A]\}\propto\{\det[1-S_T^{(0)}\alpha\cdot\eta ie\alpha\cdot A]\}^{1/2}\,. \qquad (122)$$

The presence of the square root in this version produces no singularities, because the $\mu\leftrightarrow-\mu$ reflection symmetry of the spectrum of h_T implies that any zero of the Fredholm determinant in Eq. (122) must necessarily be a double zero.

(iv) For arbitrary (fixed) direction cosines $a_{nm}^{(1)}$, *the function* $\Delta(a)=\exp\{\tilde{W}^{(0)}[A]\}$ *has at least one zero in the complex a plane, and hence* $\tilde{W}^{(0)}[A]$ *has at least one logarithmic-type branch point*. An obvious corollary is that *the perturbation expansion for* $\tilde{W}^{(0)}[A]$ *(which, as we have seen, exists for small enough a) has a finite radius of convergence*. To prove this statement, let us suppose that $\Delta(a)$ is an entire function with no zeros. Then $E(a)\equiv\tilde{W}^{(0)}[A]$ is also an entire function, which according to (ii) satisfies the bound $\mathrm{Re}E(a)\leq P_{12}(|a|)$ in the whole complex a plane. But applying Caratheadory's inequality[10]

$$\max_\theta|E(re^{i\theta})|\leq[\max_\theta\mathrm{Re}E(Re^{i\theta})-\mathrm{Re}E(0)]\frac{2r}{R-r}+|E(0)|\,,\quad r<R \qquad (123a)$$

on a circle of radius $r=\frac{1}{2}R$ and noting that $E(0)=0$, we have

$$\max_\theta|E(\tfrac{1}{2}Re^{i\theta})|\leq 2\max_\theta\mathrm{Re}E(Re^{i\theta})\leq 2P_{12}(R)\,, \qquad (123b)$$

indicating that the modulus of E is polynomial-bounded. Hence E is a polynomial, which is false, since the perturbation expansion for $\tilde{W}^{(0)}[A]$ does not terminate in finite order. Thus $\Delta(a)$ must have at least one zero. We actually expect there to be an infinite number of zeros of $\Delta(a)$. These zeros of course correspond to the values of a which produce vanishing eigenvalues μ in the eigenvalue problem $h_T\psi=\mu\psi$.

With the above properties of the external field problem in mind, we turn to an examination of the behavior of the truncated amplitude-integral formulas of Eqs. (100) and (102). We divide our discussion into four parts. First, we show that the perturbation series in coupling constant α for the single-fermion-loop vacuum amplitude and $2n$-point functions have zero radius of convergence; hence they are only asymptotic series and do not uniquely specify the physical theory. We then argue that this nonuniqueness is associated with the freedom to make inequivalent choices of contour in the amplitude integral, and suggest that unitarity restricts the actual choice of contour to one (or possibly two) at most. Third, for the choices of contour suggested by unitarity we examine the conditions under which an infinite-order zero in coupling constant can occur. Finally, we formulate a simple one-mode problem, the study of which may help resolve some of the unanswered questions about the finite-mode-number case.

A. Zero Radius of Convergence of Perturbation Expansions

We consider first the radiative-corrected single-fermion-loop vacuum amplitude in zero external field, obtained by setting $A'=0$ in Eq. (102a). According to property (i) above, the weighting function $w_k(a^2,0)$ has a convergent power-series expansion in a, with a radius of convergence of at least

$$R_{\min}=\left[\frac{5}{16\pi^2}k(k+4)\right]^{-1/2}(\tfrac{8}{3}2^{3/2})^{-1}. \qquad (124)$$

However, according to (iv), w_k must develop a singularity for finite a, since for large enough complex values of a singularities of $\overline{W}^{(0)}[A]$ will be encountered at endpoints in the integration over direction cosines in Eq. (102b). Thus the actual radius of convergence R is finite, and if we write

$$w_k(a^2,0)=\sum_{n=0}^{\infty}w_k^{(n)}a^{2n}, \qquad (125a)$$

then we have

$$\lim_{n\to\infty}\sup|w_k^{(n)}|^{1/n}=\frac{1}{R}>0\,. \qquad (125b)$$

Now let us substitute the power-series expansion of Eq. (125a) into Eq. (102a) and integrate term by term, giving the perturbation series for the vacuum amplitude in powers of e^2,

$$W_k[0]=\sum_{n=0}^{\infty}w_k^{(n)}(2e^2)^n\frac{\Gamma(n+\frac{1}{2}N_k)}{\Gamma(\frac{1}{2}N_k)}\,. \qquad (126)$$

The inverse radius of convergence of this series is

$$\lim_{n\to\infty}\sup|w_k^{(n)}2^n\Gamma(n+\tfrac{1}{2}N_k)/\Gamma(\tfrac{1}{2}N_k)|^{1/n} =\frac{2}{R}\lim_{n\to\infty}\sup[\Gamma(n+\tfrac{1}{2}N_k)/\Gamma(\tfrac{1}{2}N_k)]^{1/n}=\infty\,, \qquad (127)$$

that is, Eq. (126) has zero radius of convergence.[11] An analagous argument applies to the radiative-corrected single-fermion-loop $2n$-point function in zero external field. An expression for this quantity is obtained by functionally differentiating Eq. (102) with respect to A' and then setting A' to zero:

$$T_k(\eta_1a_1,\ldots,\eta_{2n}a_{2n})=\frac{1}{(2e^2)^{N_k/2}\Gamma(\frac{1}{2}N_k)}\int_0^{\infty}d(a^2)a^{N_k-2}\exp\left(-\frac{a^2}{2e^2}\right)t_k(a^2;\eta_1a_1,\ldots,\eta_{2n}a_{2n})\,, \qquad (128a)$$

$$t_k(a^2;\eta_1a_1,\ldots,\eta_{2n}a_{2n})=\frac{\left(\prod_1^{N_k}\int_{-\infty}^{\infty}da_{nm}^{(1)}\right)\delta\left(1-\sum_1^{N_k}[a_{nm}^{(1)}]^2\right)T_k[A;\eta_1a_1,\ldots,\eta_{2n}a_{2n}]}{\left(\prod_1^{N_k}\int_{-\infty}^{\infty}da_{nm}^{(1)}\right)\delta\left(1-\sum_1^{N_k}[a_{nm}^{(1)}]^2\right)}\,.$$

Here

$$T_k[A;\eta_1a_1,\ldots,\eta_{2n}a_{2n}]=-\tfrac{1}{2}\mathrm{tr}_8[i\alpha\cdot\eta_1\alpha_{a_1}\langle\eta_1|h_T^{-1}|\eta_2\rangle i\alpha\cdot\eta_2\alpha_{a_2}\langle\eta_2|h_T^{-1}|\eta_3\rangle\ldots i\alpha\cdot\eta_{2n}\alpha_{a_{2n}}\langle\eta_{2n}|h_T^{-1}|\eta_1\rangle] +\text{permutations of }\eta_2a_2,\ldots,\eta_{2n}a_{2n}\,, \qquad (128b)$$

with

$$h_T=2-L\cdot S-i\alpha\cdot\eta\alpha\cdot A \qquad (128c)$$

the same expression as in Eq. (91a) except that the charge e has been scaled out. Again, properties (i) and (iv) tell us that $t_k(a^2;\eta_1a_1,\ldots,\eta_{2n}a_{2n})$ is power-series-expandable with a finite, nonzero radius of convergence R, determined by the complex values a which produce vanishing eigenvalues of h_T. The argument then proceeds as in Eqs. (125)–(127).

B. Choice of Contour in the Amplitude Integrals

The fact that the perturbation series for the radiative-corrected vacuum amplitude and $2n$-point functions have zero radius of convergence means that they are at best asymptotic series, which do not uniquely specify the physical theory for nonzero coupling. To better understand this nonuniqueness, let us return to Eq. (102) for the vacuum amplitude. Although we have assumed up to this point that the a-integration contour runs along the real axis, in fact the term-by-term integration to get the perturbation expansion in e^2 is convergent and yields the same answer for any integration contour which lies within the sector

$$-\tfrac{1}{2}\pi < \arg a^2 < \tfrac{1}{2}\pi \,. \tag{129}$$

Let us now consider two such contours, which pass on opposite sides of a branch cut of $w_k(a^2, 0)$ which originates from a branch point $\bar{a}$. They make contributions to Eq. (102) which differ by the amount

$$\frac{1}{(2e^2)^{N_k/2}\Gamma(\frac{1}{2}N_k)} \int_{\bar{a}^2}^{\infty} d(a^2) a^{N_k-2} \exp\left(-\frac{a^2}{2e^2}\right) \times \mathrm{disc}[w_k(a^2,0)] \tag{130}$$

where disc denotes the discontinuity of w_k across its branch cut. The nonvanishing difference in Eq. (130) shows that different amplitude-integration contours yield inequivalent physical theories. However, since $\mathrm{Re}(a^2) > 0$ on the contour in Eq. (130), the asymptotic expansion of this equation about $e^2 = 0$ vanishes, consistent with our statement above that all integration contours in the sector of Eq. (129) yield the same perturbation-expansion coefficients.

Clearly, we can uniquely specify the choice of physical theory by specifying the contour on which the amplitude integral is to be taken. *It seems a plausible conjecture that this choice of contour will be dictated by the requirement of unitarity.* In nonrelativistic potential theory a unitary or probability-conserving system has a real Lagrangian density. We speculate that the corresponding unitarity requirement, in defining Euclidean quantum electrodynamics by a photon amplitude integral, is that the effective photon Lagrangian $\mathcal{L}_{\mathrm{eff}}[A]$, obtained after integration over charged-particle coordinates, should be real. Since $\mathcal{L}_{\mathrm{eff}}$ is even in A, this suggests that we must choose our amplitude-integration contour to be purely real, or (a much more speculative possibility)[12] purely imaginary. The real contour corresponds to the choice which we have implicitly made in all of the above discussion. Amplitude integrals evaluated on a real contour will be unambiguously defined only *if h_T has no vanishing eigenvalues for a real; we conjecture this to be the case.* The imaginary contour will be of interest only if (i) h_T has no zero eigenvalues for a imaginary, and (ii) the external-field amplitude $w_k[a^2, 0]$ decreases in Gaussian fashion [$w_k \sim \exp(ca^2)$, $c > 0$] as $a \to i\infty$, so that the integral in Eq. (102) exists for e large enough.

C. Conditions for the Occurrence of an Infinite-Order Zero in Coupling Constant

Let us now examine the conditions for single-fermion-loop diagrams to develop an infinite-order zero in coupling constant, for the cases of real and imaginary amplitude-integral contours described above.

(i) Real contour. When Eq. (102) is integrated on a real contour, there are two possibilities of interest, depending on the asymptotic rate of growth of $w_k(a^2, A')$ for large a. If w_k grows asymptotically more weakly than a Gaussian [for example, as either a power of a or as an exponential $\exp(\lambda a)$, with any value of λ] then Eq. (102a) defines a function of e^2 analytic in the right half-plane, and an essential singularity obviously cannot occur. On the other hand, if the leading asymptotic behavior of w_k is Gaussian as $a \to \infty$,

$$w_k = \exp\left(\frac{a^2}{2e_k^2}\right) \times (\text{non-Gaussian factor}) \,, \tag{131}$$

then Eq. (102a) will converge for $0 < e < e_k$, but will develop a singularity at e_k. If the non-Gaussian factors oscillate an infinite number of times, as in the specific example

$$\left(\frac{2}{\pi}\right)^{1/2} \int_0^{\infty} \frac{da}{e} \exp\left(\frac{-a^2}{2e^2}\right) \exp\left(\frac{a^2}{2e_k^2}\right)(e^{ita} + e^{-ita}) = \left(\frac{e_k^2}{e_k^2 - e^2}\right)^{1/2} \exp\left(\tfrac{1}{2}t^2 \frac{e^2 e_k^2}{e^2 - e_k^2}\right), \quad \mathrm{Re}\,t^2 > 0 \tag{132}$$

then the singularity can take the form of an infinite-order zero as $e \to e_k$ from below. However, *such oscillatory Gaussian growth is ruled out by property (ii), which states that w_k is bounded from above by a polynomial in a.* Although the example of Eq. (132) is very special, the argument which we have just sketched can be made general and precise,[13] and indicates that *for real amplitude-integral contour Eq. (102a) cannot develop an infinite-order zero as e approaches a positive singularity from below.* The bound from above on the real part of w_k does not rule out nonoscillatory Gaussian growth of w_k, such as

$$w_k = -\exp\left(\frac{a^2}{2e_k^2}\right)(e^{ta} + e^{-ta}), \quad \mathrm{Re}\,t^2 > 0 \tag{133}$$

which would lead to singular behavior of Eq. (102a) as $e \to e_k$ from below. For w_k to behave as in Eq. (133), the complete vacuum amplitude $\Delta(a)$ would have to decrease in modulus as $\exp[-\exp(a^2/2e_k^2)]$ along the real axis. But this can happen only if there is a clustering of zeros of $\Delta(a)$ in the strip of the complex plane adjacent to the real axis,[14] with a linear density (when the zeros are projected onto the real axis) which increases in roughly Gaussian fashion as $\mathrm{Re}\,a \to \infty$. Such a high density of zeros seems very implausible. We conjecture that w_k in fact grows more weakly than a Gaussian, which, as noted, would imply that in the finite-mode-number approximation with a real contour, all single-fermion-loop diagrams are analytic functions of e^2 in the right half-plane.

(ii) Imaginary contour. If $\Delta(a)$ has no zeros for a imaginary, and *if* $w_k[a^2, A']$ decreases as rapidly as

$$w_k = \exp\left(\frac{a^2}{2e_k^2}\right) \times (\text{non-Gaussian factor}) \qquad (134)$$

along the imaginary axis, then Eq. (102a) will exist when integrated along the imaginary axis provided that $\mathrm{Re}\, e^2 > e_k^2$. Clearly, Gaussian decreasing behavior of w_k is consistent with our polynomial upper bound irrespective of the sign of the non-Gaussian factors; hence an infinite-order zero as e approaches a finite singularity from above may be possible in the finite-mode-number case.

D. One-Mode Model

Distinguishing among the various possibilities discussed above will require a knowledge of the distribution of zeros and the asymptotic behavior of the vacuum amplitude $\Delta(a)$. The simplest model which permits one to study these questions is the one-mode approximation on the hypersphere, in which only one of the ten modes belonging to the (1, 1) representation is kept. That is, for the potential $A_b(\eta)$ one takes the linear form

$$A_b(\eta) = a\left(\frac{15}{16\pi^2}\right)^{1/2}(v_{1b}\eta\cdot v_2 - v_{2b}\eta\cdot v_1)\,,$$
$$v_1^2 = v_2^2 = 1, \quad v_1\cdot v_2 = 0\,, \qquad (135)$$

which has unit normalization when $a = 1$. The eigenvalue problem $h_T\psi = \mu\psi$ for the potential of Eq. (135) can be reduced to a single ordinary second-order differential equation, which may permit the study of its properties by standard techniques. The details of this reduction will be fully described elsewhere.

Finally, let us close by briefly discussing possible connections between our results for the truncated model and the behavior of the exact eigenvalue problem in electrodynamics, the study of which is, of course, our ultimate goal. Because our methods are not powerful enough at this point to handle the infinite-mode-number case, we can make no definite statements, but instead pose some pertinent questions for future investigation.

(i) The first question is whether the result that $\alpha = 0$ is not a regular point is just an artifact of the truncation approximation, or in fact continues to be valid when all photon modes are kept. Extension of this result to the full theory would constitute a proof (for the class of single-fermion-loop diagrams) of Dyson's[15] old conjecture that perturbation expansions are asymptotic series at best. If the perturbation series for the eigenvalue function $F^{[1]}(\alpha)$ is divergent, then as discussed in Sec. IV B, the perturbation coefficients do not determine the function uniquely; to define the full theory additional information, perhaps from a specific choice of contour in an amplitude-integral formulation, will be needed.

(ii) *If* analysis of the finite-mode-number case shows that

$$\lim_{|a|\to\infty} \frac{\ln \ln w_k(a^2, 0)}{\ln|a|} < 2\,, \qquad (136)$$

then the real amplitude-integral contour discussed in Sec. IV C is likely to be the only correct one, and will give vacuum amplitudes which are analytic functions of e^2 in the right half-plane. It then becomes an important question to determine the convergence properties of the vacuum amplitudes as the photon mode number N_k becomes infinite. The appearance of an essential singularity from the limit of a sequence of analytic functions would require nonuniform convergence; on the other hand, if one could prove sufficiently uniform convergence, one could rule out the presence of an essential singularity in the full theory, and hence prove that a finite quantum electrodynamics is not possible. If nonuniform convergence turns out to be the case, the challenge, of course, will be to find alternative approximation schemes, not involving the restriction to finite-photon-mode number, for studying whether an essential singularity in e^2 is present.

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APPENDIX

We derive here an explicit formula for the effective photon propagator $D_{ab}^{(0)}(\eta_1, \eta_2)_{\text{eff}}$ defined in Eq. (24). It is convenient to remove normalization constants, so we define

$$d_{ab}^{(0)}(\eta_1,\eta_2)=8\pi^2 D_{ab}^{(0)}(\eta_1,\eta_2)$$
$$=\frac{\delta_{ab}}{1-z}\,,\quad z=\eta_1\cdot\eta_2\,,$$
$$d_{ab}^{(0)}(\eta_1,\eta_2)_{\rm eff}=8\pi^2 D_{ab}^{(0)}(\eta_1,\eta_2)_{\rm eff}$$
$$=d_{ab}^{(0)}+\Delta d_{ab}^{(0)}\,, \tag{A1}$$

with $\Delta d_{ab}^{(0)}$ the negative of the contribution to $d_{ab}^{(0)}$ of the vector spherical harmonics with $\lambda=-n$, $n+3$ (the gauge degrees of freedom). Since these harmonics are linear combinations of terms of the form

$$\eta_a Y(\eta)\,,$$
$$\eta_b L_{ba} Y(\eta)\,, \tag{A2}$$

and since $d_{ab}^{(0)}(\eta_1,\eta_2)_{\rm eff}$ is symmetric under interchange of the photon variables, the most general structure of $\Delta d_{ab}^{(0)}$ is

$$\Delta d_{ab}^{(0)}=\eta_{1a}\eta_{2b}\Phi_1(z)+(\eta_{1a}\eta_{2d}L_{2db}+\eta_{2b}\eta_{1c}L_{1ca})\Phi_2(z)$$
$$+\eta_{1c}L_{1ca}\eta_{2d}L_{2db}\Phi_3(z)$$
$$=\eta_{1a}\eta_{2b}\Phi_1(z)+[\eta_{1a}(\eta_{1b}-z\eta_{2b})+\eta_{2b}(\eta_{2a}-z\eta_{1a})]\Phi_2{}'(z)$$
$$+(\delta_{ab}-\eta_{1a}\eta_{1b}-\eta_{2a}\eta_{2b}+z\eta_{1a}\eta_{2b})\Phi_3{}'(z)$$
$$+(\eta_{2a}-z\eta_{1a})(\eta_{1b}-z\eta_{2b})\Phi_3{}''(z)\,. \tag{A3}$$

We can determine the unknown functions appearing in Eq. (A3) by requiring that Eq. (A1) satisfy the known properties of the effective propagator. Imposing the constraint

$$\eta_{1a}d_{ab}^{(0)}(\eta_1,\eta_2)_{\rm eff}=0 \tag{A4}$$

implies that

$$\Phi_1(z)=-\frac{z}{1-z}\,,\quad \Phi_2{}'(z)=-\frac{1}{1-z}\,. \tag{A5}$$

The second constraint.

$$L_{1ca}d_{ab}^{(0)}(\eta_1,\eta_2)_{\rm eff}=d_{cb}^{(0)}(\eta_1,\eta_2)_{\rm eff}\,, \tag{A6}$$

requires that $\Phi_3{}'(z)$ satisfy the inhomogeneous differential equation

$$\frac{d}{dz}[(1-z^2)\Phi_3{}''(z)-6z\Phi_3{}'(z)]=\frac{6-5z}{(1-z)^2}\,. \tag{A7}$$

The unique solution of this equation which is regular at $z=-1$ and no more singular than $(z-1)^{-1}$ at $z=1$ is

$$\Phi_3{}'(z)=-\frac{1}{4}\frac{1}{1-z}+\frac{1}{(z+1)^3}\left((z^2+3z+\tfrac{8}{3})\{\ln[\tfrac{1}{2}(1-z)]+\tfrac{47}{60}\}-(\tfrac{1}{5}z^2+\tfrac{17}{20}z+\tfrac{211}{180})\right)\,. \tag{A8}$$

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[1]For a review of work on finite quantum electrodynamics, see S. L. Adler, Phys. Rev. D 5, 3021 (1972); K. Johnson and M. Baker, Phys. Rev. D 8, 1110 (1973).

[2]S. L. Adler, Phys. Rev. D 6, 3445 (1972). We will repeatedly draw upon results contained in this paper.

[3]We follow the notation of M. Hamermesh, *Group Theory* (Addison-Wesley, Reading, Mass., 1962), p. 309. [Hamermesh does not discuss the spinor representations of O(5); these are dealt with in the review article of R. E. Behrends *et al.*, Rev. Mod. Phys. 34, 1 (1962).]

[4]Since $4\dim(n,0)>\dim(n\pm\frac{1}{2},\frac{1}{2})$, the eigenfunctions defined in Eq. (35) form an overcomplete, nonorthonormalized set. The definition is convenient, however, since it leads to the simple form for the completeness relation given in Eq. (41). If the redundancies are eliminated, to give orthonormalized eigenfunctions $\psi^{nk}_{-(n+1)}$ and ψ^{nk}_{n+2}, the completeness relation takes the form

$$\sum_{n=0}^{\infty}\left[\sum_{k=1}^{\dim(n-\frac{1}{2},\frac{1}{2})}\psi^{nk}_{-(n+1)}(\eta_1)\psi^{nk}_{-(n+1)}(\eta_2)^\dagger+\sum_{k=1}^{\dim(n+\frac{1}{2},\frac{1}{2})}\psi^{nk}_{n+2}(\eta_1)\psi^{nk}_{n+2}(\eta_2)^\dagger\right]=\delta_S(\eta_1-\eta_2)1_8\,.$$

The corresponding form for $S_T^{(0)}(\eta_1,\eta_2)$ can be obtained immediately from the relation $S_T^{(0)}(\eta_1,\eta_2)=(h_T^{(0)})^{-1}\times\delta_S(\eta_1-\eta_2)1_8$.

[5]For a review see B. W. Lee and J. Zinn-Justin, Phys. Rev. D 5, 3121 (1972), Sec. II and references contained therein.

[6]Referring to Eq. (16), we see that the harmonics

$$Y^{(-n+1)}_{n-1\;ma}(\eta)\propto[n\eta_a+\eta_b L_{ba}]\,Y_{nm}(\eta)\,,$$
$$Y^{(n+4)}_{n+1\;ma}(\eta)\propto[-(n+3)\eta_a+\eta_b L_{ba}]\,Y_{nm}(\eta)\,,$$

which represent gauge degrees of freedom, *do not* satisfy the analogous constraint

$$\eta_a Y^{(-n+1)}_{n-1\;ma}(\eta)=0,\quad \eta_a Y^{(n+4)}_{n+1\;ma}(\eta)=0\,.$$

Since Eq. (81c) for the potential must always be maintained, this means that the gauge harmonics must always occur correlated in the combination

$$(n+3)[n\eta_a+\eta_b L_{ba}]\,Y_{nm}(\eta)+n[-(n+3)\eta_a+\eta_b L_{ba}]\,Y_{nm}(\eta)$$
$$=(2n+3)\eta_b L_{ba}Y_{nm}(\eta)\,.$$

The general form for gauge transformations on the hypersphere (first pointed out to me by S.-J. Chang) is then

$$A_a(\eta)\to A_a(\eta)+\eta_b L_{ba}\Phi(\eta)\,.$$

[7]The importance of the behavior of the external-field problem was first pointed out to me in conversations with S. Coleman and D. Gross.

[8]An analogous statement is *not* true for bounded potentials in x space, because the x-space analog of Eq. (110),

$$\prod_{j=1}^{2l}\int d^4x_j|x_1-x_2|^{-3/2}|x_2-x_3|^{-3/2}\times\cdots|x_{2l-1}-x_{2l}|^{-3/2}|x_{2l}-x_1|^{-3/2},$$

is infrared-divergent. This dissimilarity between x space and the hyperspherical formalism is not surprising, since the transformations of Ref. 2 indicate that the general bound x-space potential $a_\mu(x)$ transforms into an unbounded hyperspherical potential $A_a(\eta)$. Note that although the upper bound of Eq. (112) diverges for $l = 2$, exact calculation for this case, taking gauge invariance properly into account, indicates that $\tilde{W}_4^{(0)}[A]$ is convergent.

[9]A. Salam and P. T. Matthews, Phys. Rev. 90, 690 (1953); J. Schwinger, *ibid.* 93, 615 (1954). These authors actually make fewer subtractions than we do in the argument of Eqs. (116)–(118), because they claim that $\mathrm{Tr}(KK^\dagger)$ is finite. However, although $\mathrm{Tr}(KK^\dagger)$ is formally a four-point function, it does not have the propagator and vertex factors arranged in the correct order to be the usual gauge-invariant four-point function. Since the four-point function is only conditionally and not absolutely convergent, this suggests that $\mathrm{Tr}(KK^\dagger)$ will be divergent, and that the Fredholm argument of Matthews and Salam and Schwinger will need additional subtractions to be made precise.

[10]B. Ja. Levin, *Distribution of Zeros of Entire Functions,* Translations of Mathematical Monographs Vol. 5 (American Mathematical Society, Providence, R. I., 1964), p. 17. I wish to thank A. S. Wightman for conversations which suggested the argument of property (iv).

[11]We note that our argument fails when the number of modes kept is infinite. As $N_k \to \infty$, the factor $\Gamma(n+\frac{1}{2}N_k)/\Gamma(\frac{1}{2}N_k)$ in Eq. (26) approaches unity, and so Eq. (126) becomes

$$w_\infty[0] = \sum_{n=1}^{\infty} w_\infty^{(n)}(2e^2)^n .$$

Although this equation no longer contains a combinatoric factor which grows as $n!$ for large n, we can draw no conclusion about its radius of convergence, because the estimate of Eq. (124) for an upper bound $R_{\min}{}^{-1}$ on

$$\lim_{n\to\infty} \sup |w_k^{(n)}|^{1/n}$$

diverges as $k \to \infty$. Nonanalyticity results related to ours are given by E. R. Caianiello, A. Campolattaro, and M. Marinaro [Nuovo Cimento 38, 1777 (1965)] and D. Kershaw (unpublished).

[12]The imaginary contour of course lies outside the sector of Eq. (129), and hence cannot be developed in an asymptotic expansion in e^2 by direct term-by-term integration. It may still have an asymptotic development agreeing with perturbation theory after an appropriate analytic continuation.

[13]S. Coleman and S. B. Treiman (unpublished). Their argument takes into account the presence of the asymptotically subdominant parts of w_k, which give rise to a "background" amplitude integral which is analytic at e_k.

[14]This clustering is implied by Thm. 11 on p. 21 of Levin, Ref. 10.

[15]F. J. Dyson, Phys. Rev. 85, 631 (1952).

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Approach to Scaling in Renormalized Perturbation Theory

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The consequences of the large magnitude of the bare coupling constant in Wilson's theory of critical phenomena are examined for renormalized field theory in $4-\epsilon$ dimensions. The scaling behavior of the correlation functions, the relations among critical exponents, and the existence of a scaling equation of state, regular in the temperature around T_c, are then obtained in this framework. Some corrections to the scaling laws are also discussed and are shown to be dependent on another exponent.

I. INTRODUCTION

In a previous work[1] we have studied the existence of asymptotic scaling forms for the correlation functions near the critical point within the framework of Wilson's theory[2] in $d = 4-\epsilon$ dimensions. The main tool was the use of renormalized perturbation theory and of the Callan-Symanzik equations.[3] All renormalizations were performed at zero momenta. It was shown then that the renormalized coupling constant was fixed at a nontrival solution of an eigenvalue condition simply because one lets the bare coupling constant go to infinity. This corresponds to the physical situations in which the bare coupling is measured in units of the inverse lattice spacing a^{-1}, whereas the masses and relevant momenta are proportional to the inverse correlation length ξ^{-1}, and to the fact that, in the vicinity of the critical point, ξ is much greater than a.

However, the scaling behavior of the correlation functions is not sufficient to obtain all the scaling laws. In addition there are the Widom-Kadanoff[4] relations among the critical exponents; there is also an equation of state, i.e., a relation between the applied field H, the magnetization M, and the temperature T, in scaling form:

$$\frac{H}{M^{\delta}} = f\left[\frac{(T-T_c)}{M^{1/\beta}}\right]. \qquad (1)$$