This is the conserved vector[14] associated with the Brans-Dicke theory. Note that it reduces to Komar's expression if ϕ is taken to be constant. This is consistent with the reduction of the Brans-Dicke Lagrangian to the Lagrangian of general relativity for constant ϕ.

The physical consequences of (8) for the Brans-Dicke analog of the Schwarzschild solution[15] are under investigation at the present time. A general consideration of invariants, their compatibility identities, and Komar-type expressions, will be discussed in a forthcoming paper.

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[6]L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, (Addison-Wesley, Reading, Mass., 1962), Sec. 100.

[7]The arbitrary vector field arises through the formulation of the identity using the Lie derivative (in the direction of an *arbitrary* vector field).

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[10]Reference 4, pages 183–186, and Ref. 8 give the basic approach to such identities.

[11]$R \equiv g^{ab}g^{cd}R_{abcd} \equiv g^{ab}g^{cd}([ac,d]_{,b} - [ab,d]_{,c} + \{{}_{a}{}^{i}{}_{b}\}[dc,i] - \{{}_{a}{}^{i}{}_{c}\}[db,i])$ and a vertical stroke denotes covariant differentiation.

[12]Reference 4, p. 183.

[13]$Z^{a[bc]i} \equiv \frac{1}{2}(Z^{abci} - Z^{acbi}) \equiv \frac{1}{2}(Z^{baic} - Z^{biac})$.

[14]$(\phi\xi_{a|b} - 2\phi_{|b}\xi_a)Z^{a[ci]b} \equiv T^{ci}$, where $T^{ci} \equiv -T^{ic}$. Thus $T^{ci}{}_{|ci} \equiv 0$.

[15]Reference 9, Eqs. (31)–(34).

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Time-Asymmetric Two-Body Problem in Special Relativity

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We treat a Lorentz-covariant two-body problem due to Fokker: One electric charge experiences the retarded field of a second, while the second experiences the advanced field of the first; this is pure action at a distance, with no self-action; conservation principles exist. We show that this (apparently generally soluble) time-asymmetric problem is exactly soluble for straight-line motion and admits solutions in which the charges move in circles about a common center. We briefly consider nonelectrodynamic time-asymmetric interactions and aspects of quantizing the motions.

We begin a study here of the following Lorentz-covariant two-body problem, originally posed by Fokker[1]: One electric charge is acted upon by the retarded electromagnetic field of a second charge, while the second charge is acted upon by the advanced field of the first; the interaction is pure "action at a distance," with the fields treated merely as convenient tools for describing the interaction; there is no self-action. This problem is time-*a*symmetric, in contrast to the time-symmetric two-body problem in the Wheeler-Feynman[2] formulation of electrodynamics, in which each charge is acted upon by half the advanced plus half the retarded field of the other.

The symmetric problem has so far defied general analysis, since its solution apparently requires knowledge of an infinite set of position-velocity data for each charge. Attempts[3,4] have been made to reduce the Wheeler-Feynman two-body problem to Newtonian-type equations requiring only an ordinary set of initial conditions, but various mathematical questions arise,[5] e.g., whether the series expansions used converge. Only one class of rigorous solutions of the time-symmetric problem exists: Schild's motion,[6] in which the charges move in circles about a common center.[7]

The asymmetric two-body problem, on the other hand, is exactly soluble in the case of straight-line motion (as we show here), and appears to be soluble in general (we will investigate the general case in a later paper). It is hoped that a study of this problem and its solutions will lead to a better understanding of the general Lorentz-covariant two-body problem.

In Sec. I the dynamics of the problem is given: The Fokker action principle, the equations of motion, and the conserved quantities are written down. Section II deals with straight-line motion; a set of "generalized" coordinates for this case reduces the problem to an ordinary pair of coupled differential equations; all the tools of classical mechanics, e.g., Hamiltonian methods, become available; the generalized coordinates offer a bonus in that the simple form of a Lorentz transformation (along the line of the straight-line motion) for these coordinates facilitates the construction of other types of Lorentz-covariant interactions. In Sec. III we show that periodic motions are possible for the time-asymmetric interaction: A class of solutions (the analog of Schild's motion) is given in which the two charges move in circles about a common center. In Sec. IV we briefly discuss some of the nonelectrodynamic Lorentz-covariant interactions that can be constructed for the case of one space dimension; some of these can be readily generalized to three space dimensions, some to the time-symmetric case. In Sec. V we consider some aspects of the problem of quantizing the motions.

I. DYNAMICS

The Minkowski line element in the usual coordinates x^0, x^1, x^2, x^3, with $x^0 = ct$, is written as $(cds)^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$. The scalar product of two four vectors A^μ, B^μ will be denoted by (AB):

$$(AB) \equiv (A^0B^0)^2 - (A^1B^1)^2 - (A^2B^2)^2 - (A^3B^3)^2 .$$

The two charges will be labeled a, b in such a way (see Fig. 1) that a is acted upon by the retarded potential of b, while b is acted upon by the advanced potential of a. Quantities referring to the two charges will be labeled by the appropriate subscript, e.g., the proper time for a is taken as $ds_a = (dx_a^0/c)(1 - z_a{}^2)^{1/2}$, where z_a is the magnitude of the three vector $\vec{z}_a$ with components $(dx_a^1/dx_a^0$, dx_a^2/dx_a^0, $dx_a^3/dx_a^0)$. The masses and charges will be denoted by m_a, m_b and e_a, e_b, respectively.

Dots will be used with the above set of coordinates to indicate differentiation with respect to the proper time, e.g., $\dot{x}_b^1 = dx_b^1/ds_b$. (Another usage of the dot notation, which should not cause confusion, is given in Sec. II.)

When the two charges are on the same light cone, the scalar product of the vector $R^\mu \equiv x_a^\mu - x_b^\mu$ with itself is zero:

$$(RR) = (x_a^0 - x_b^0)^2 - r^2 = 0 , \tag{1}$$

where $r = |\vec{r}|$, $\vec{r} = \vec{r}_a - \vec{r}_b$, with, e.g., $\vec{r}_a = (x_a^1, x_a^2, x_a^3)$. The two charges interact only when a is on the forward light cone of b (we shall refer to this portion of b's light cone as the "interaction light cone;" see Fig. 1); for this case we have

$$x_a^0 - x_b^0 = r . \tag{2}$$

If a and b lie on an interaction light cone, then for infinitesimal displacements dx_a^μ, dx_b^μ such that they again lie on an interaction light cone, Eq. (1) gives $(Rdx_a) = (Rdx_b)$, and so $ds_a/ds_b = (R\dot{x}_b)/(R\dot{x}_a)$.

The Fokker action for this problem can now be written as (using Gaussian units)[8]

$$S = -m_a c\int ds_a (\dot{x}_a\dot{x}_a)^{1/2} - m_b c\int ds_b (\dot{x}_b\dot{x}_b)^{1/2} - \lambda\int ds_a (\dot{x}_a\dot{x}_b)/(R\dot{x}_b),$$

$$\lambda \equiv e_a e_b/c , \tag{3}$$

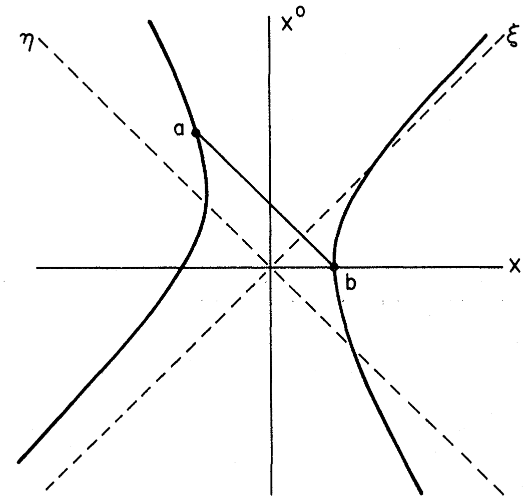

FIG. 1. The two charges on an "interaction light cone" (a on the forward light cone of b). The ξ, η are the "generalized" coordinates used for straight-line motion (Sec. II).

together with the subsidiary condition (2). The form of S in (3) is convenient when obtaining the equations of motion of a; since $(R dx_a) = (R dx_b)$, the integrand in the last term of (3) may be replaced by $ds_b(\dot{x}_a \dot{x}_b)/(R\dot{x}_a)$ when obtaining the equations of motion of b. References 1 and 2 give the usual procedure[9] for obtaining the equations of motion and conserved quantities from an action principle of the above type; the results below can, of course, also be obtained by starting with the expressions for the retarded fields of b and the advanced fields of a and using the Lorentz force $\vec{F} = e(\vec{E} + \vec{z} \times \vec{B})$.

The equations of motion for the two charges are similar in form, and we shall write them as one equation by using subscripts i, j and introducing factors $G_{i,j} = \pm 1$; the equations of motion for a can be obtained from (4) below by substituting $i = a$, $j = b$; the substitution $i = b$, $j = a$ yields the equations of motion for b. *Equations (4)–(8) hold only under the condition (2)*, e.g., (4) gives the four-accelerations of a and b when they lie on an interaction light cone:

$$m_i \ddot{x}_i^\mu/\lambda = -\dddot{x}_j^\mu \rho_i/\rho_j^{\,2} + \dot{x}_j^\mu (R\ddot{x}_j)\rho_i/\rho_j^{\,3} + R^\mu[(\dot{x}_i \ddot{x}_j)/\rho_j^{\,2} - (\dot{x}_i \dot{x}_j)(R\ddot{x}_j)/\rho_j^{\,3}] + G_i(c^2/\rho_j^3)[-\dot{x}_j^\mu \rho_i + R^\mu(\dot{x}_i \dot{x}_j)]\,; \qquad (4)$$

$$\rho_a \equiv (R\dot{x}_a); \quad \rho_b \equiv (R\dot{x}_b); \quad G_a \equiv +1; \quad G_b \equiv -1.$$

The conserved quantities energy-momentum (P^μ) and angular momentum ($L^{\mu\nu}$) are given by

$$P^\mu = m_a \dot{x}_a^\mu + m_b \dot{x}_b^\mu + \lambda[\dot{x}_a^\mu/\rho_a + \dot{x}_b^\mu/\rho_b - R^\mu(\dot{x}_a \dot{x}_b)/\rho_a \rho_b]\,, \qquad (5)$$

$$L^{\mu\nu} = m_a(x_a^\mu \dot{x}_a^\nu - x_a^\nu \dot{x}_a^\mu) + m_b(x_b^\mu \dot{x}_b^\nu - x_b^\nu \dot{x}_b^\mu) + \lambda[(x_a^\mu \dot{x}_b^\nu - x_a^\nu \dot{x}_b^\mu)/\rho_b + (x_b^\mu \dot{x}_a^\nu - x_b^\nu \dot{x}_a^\mu)/\rho_a + (\dot{x}_a \dot{x}_b)(x_a^\mu x_b^\nu - x_a^\nu x_b^\mu)/\rho_a \rho_b]\,. \qquad (6)$$

Equations (4) can be simplified by noting that they yield

$$(R\ddot{x}_i) = -G_i \lambda c^2 \rho_i/m_i \rho_j^{\,2}\,. \qquad (7)$$

Using (7) and substituting $\ddot{x}_a^\mu$ from the equations of motion of a into the equations of motion of b and vice versa, and using $(\dot{x}_i \ddot{x}_i) = 0$, we find

$$\ddot{x}_i^{\,\mu}(m_i - \lambda^2/m_j \rho_i \rho_j) = G_i c^2\{(\dot{x}_i^{\,\mu} - R^\mu c^2/\rho_i)(\lambda^2/m_j \rho_i \rho_j)(\lambda/m_i \rho_j^{\,2} - 1/\rho_i) + [\dot{x}_j^\mu \rho_i - R^\mu(\dot{x}_i \dot{x}_j)](\lambda/\rho_j^{\,2})(\lambda/m_j \rho_i^{\,2} - 1/\rho_j)\}\,. \qquad (8)$$

In addition to the expected singularity for $r = 0$, there is the interesting possibility of a singularity when the factor $m_i - \lambda^2/m_j \rho_i \rho_j$ is zero. In the case of straight-line motion, such a singularity does not occur, since then the same factor occurs on the right-hand side of the applicable equations of (8); for the circular motions considered in Sec. III it is impossible for the factor multiplying $\ddot{x}_i^\mu$ in (8) to be zero. Yet we can pick values of the coordinates and velocities that make this factor zero, but for which the right-hand side of (8) is not in general zero; e.g., b is instantaneously at rest at the origin, $x_a^1 = x_a^2 = 2^{-1/2} r$, $x_a^3 = 0$, $x_a^0 = r = (\lambda/c)(m_a m_b)^{-1/2}$, $\dot{x}_a^1 = 2^{3/2} c$, $\dot{x}_a^2 = \dot{x}_a^3 = 0$. For this set of conditions the right-hand side of (8) for $\ddot{x}_a^1$ is not zero unless $m_a = m_b$. The role, if any, that this singularity plays in the three-space-dimensional solutions will be investigated in later work.

As the equations of motion stand in (8), it appears possible to construct solutions given a set of initial conditions which satisfy the constraint (2) and for which $\ddot{x}_a^\mu$ and $\ddot{x}_b^\mu$ are finite and well defined. Using the subscript zero to indicate initial values, we suppose that the initial position and velocity of b are specified at $x_{b\,0}^0 = 0$; then the initial position and velocity of a are specified at $x_{a\,0}^0 = r_0$, according to (2); the three accelerations can be found using (8), and the positions and velocities of a and b found short times Δ_a and Δ_b later, i.e., at $x_b^0 = \Delta_b$ and $x_a^0 = r_0 + \Delta_a$. To continue this step-by-step integration, we have to pick Δ_a and Δ_b so that (2) still holds, i.e., $r_0 + \Delta_a - \Delta_b = r(r_0 + \Delta_a, \Delta_b)$. The constraint introduces a complication, but does not seem to prohibit this method of solution.

From the point of view of the above method of solution, there are only six degrees of freedom, i.e., if we take x_b^0 as the parameter for the motion, we can pick the spatial coordinates of a and b arbitrarily (away from possible singularities), and then x_a^0 is fixed by (2). It seems reasonable, then, to attempt to find a set of generalized coordinates for the problem which would take the constraint into account automatically (as in Lagrangian descriptions of ordinary mechanics problems), and at the same time try to find a convenient parameter which would play the role of the time variable in ordinary mechanics.

It is possible to do this in the case of straight-line motion, as shown in Sec. II; preliminary work indicates no essential difficulty in carrying this out for the full four-dimensional case.

II. STRAIGHT-LINE MOTION

If the initial conditions are such that $\dot{x}_a^2$, $\dot{x}_a^3$, $\dot{x}_b^2$, $\dot{x}_b^3$, R^2, R^3 are all zero, we have straight-line motion parallel to the x^1 axis. We shall simplify the notation here: Let $x \equiv x^1$, $\tau \equiv x^0$. The equations of motion for this case are most readily obtained from (7); they are [using the i, j notation as for Eq. (4)]

$$\frac{m_i d^2 x_i / d\tau_i^2}{(1 - z_i^2)^{3/2}} = \mp G_i \frac{\lambda/c}{(x_i - x_j)^2} \frac{1 \mp z_j}{1 \pm z_j}, \qquad (9)$$

with the constraint

$$\tau_a - \tau_b = \pm(- x_a + x_b), \qquad (10)$$

where the upper (lower) sign holds if a is to the left (right) of b, where, e.g., "a left of b" means $x_a < x_b$. *From now on we shall presume that a is initially to the left of b* (if the charges repel, a will remain to the left of b, while if they attract a singularity occurs at $r=0$); we shall, however, indicate how the procedure below is modified if a is initially to the right of b.

We introduce the coordinates ξ, η (see Fig. 1) by[10]

$$\xi = 2^{-1/2}(x + \tau); \quad x = 2^{-1/2}(\xi - \eta);$$
$$\eta = 2^{-1/2}(-x + \tau); \quad \tau = 2^{-1/2}(\xi + \eta).$$

We have, then, for example, $dx_a/d\tau_a = (1 - d\eta_a/d\xi_a)/(1 + d\eta_a/d\xi_a)$. The constraint (10) becomes simply $\xi_a = \xi_b$. We can thus drop the subscript on the ξ's and write $\dot\eta_a = d\eta_a/d\xi$, $\dot\eta_b = d\eta_b/d\xi$, i.e., ξ can be taken as a convenient parameter for describing the motion. As indicated, dots will be used with the ξ, η coordinates to indicate differentiation with respect to ξ. The values 0, 1, ∞ of the generalized velocity $\dot\eta$ correspond to $dx/dt = c, 0, -c$, respectively.

The equations of motion (9) (using the upper signs) can now be written as two ordinary coupled differential equations[11]:

$$m_i \ddot\eta_i / \dot\eta_i^{3/2} = G_i k \dot\eta_j / (\eta_i - \eta_j)^2,$$
$$k \equiv 2^{5/2}\lambda/c. \qquad (11)$$

For a Lorentz transformation to a frame moving along the line of motion of the charges (we shall hereafter refer to such a transformation as a *one-space Lorentz transformation*), we have $x' = \gamma(x - \beta\tau)$, $\tau' = \gamma(\tau - \beta x)$; here $\gamma = (1-\beta^2)^{-1/2}$, and βc is the velocity of the frame (x', τ') relative to the frame (x, τ). Defining $\xi' = 2^{-1/2}(x' + \tau')$, $\eta' = 2^{-1/2}(-x' + \tau')$, we have

$$\xi' = \sigma\xi, \quad \eta' = \eta/\sigma; \quad \sigma \equiv [(1-\beta)/(1+\beta)]^{1/2}, \qquad (12)$$

i.e., the one-space Lorentz transformation corresponds merely to a change in scale of the ξ, η; we have also $\dot\eta = \sigma^2 \dot\eta'$, $\ddot\eta = \sigma^3 \ddot\eta'$, so it is clear that (11) is invariant in form under this one-space Lorentz transformation (it also becomes clear how to construct other types of covariant interactions for the case of one space dimension—see Sec. IV).

The action (3) becomes here $\int \mathfrak{L}\, d\xi$, with

$$\mathfrak{L} = 4m_a \dot\eta_a^{1/2} + 4m_b \dot\eta_b^{1/2} + \tfrac{1}{2}k(\dot\eta_a + \dot\eta_b)/(\eta_a - \eta_b). \qquad (13)$$

The three conserved quantities (denoted by p_v, H, l) obtained using this Lagrangian are (with p_a, p_b the momenta conjugate to η_a, η_b; $u \equiv \eta_a - \eta_b$)

$$p_v = p_a + p_b = 2m_a/\dot\eta_a^{1/2} + 2m_b/\dot\eta_b^{1/2} + k/(\eta_a - \eta_b); \qquad (14)$$

$$H = -2m_a \dot\eta_a^{1/2} - 2m_b \dot\eta_b^{1/2} = -4m_a^2/(p_a - k/2u) - 4m_b^2/(p_b - k/2u); \qquad (15)$$

$$l = p_a \eta_a + p_b \eta_b + H\xi; \qquad (16)$$

here H is the Hamiltonian[12]; (16) follows from the invariance of form of $\int \mathfrak{L} d\xi$ under a one-space Lorentz transformation. These quantities are related to the P^0, P^1, L^{01} of (5) and (6) by

$$P^0 = \tfrac{1}{4}c(p_v - H); \quad P^1 = \tfrac{1}{4}c(p_v + H);$$
$$L^{01} = cl/2^{3/2}. \qquad (17)$$

The Hamiltonian for this problem is thus not equal to the energy of the system, although in the center-of-momentum frame (frame in which $P^1 = 0$), we have $H = -2P^0/c$. In the nonrelativistic limit ($c \to \infty$), the center-of-momentum frame becomes the center-of-mass (c.m.) frame [despite the fact that P^1 as given by (17) is not simply the sum of the free-particle momenta], while the energy $E_{\text{c. mom.}}$ of the center-of-momentum frame becomes in this limit the rest-mass energy plus the internal or c.m. energy (as opposed to the translational energy). We have, then, for the internal energy E_{int} the invariant:

$$E_{\text{int}} = E_{\text{c. mom.}} = \tfrac{1}{2}c^2(-p_v H)^{1/2} = c[(P^0)^2 - (P^1)^2]^{1/2}. \qquad (18)$$

If we introduce coordinates u, v by $u = \eta_a - \eta_b$, $v = \frac{1}{2}(\eta_a + \eta_b)$, then the p_v of (14) is the momentum conjugate to v, and $p_u = \frac{1}{2}(p_a - p_b)$; v is then an ignorable coordinate (but $dv/d\xi$ is not a constant). This transformation, or a more general one with $v = a_1\eta_a + a_2\eta_b$, does not greatly simplify the problem as does the c.m. transformation for the nonrelativistic problem.

We shall not write down here the explicit solutions of the equations of motion (they can be obtained in a straightforward manner using, e.g., the Hamilton-Jacobi method) since, as in the classical problem, the general nature of the motion is more readily determined from the equations of motion and the conservation theorems.

In particular, if we plot Eqs. (14) and (15) as in Fig. 2 (with p_v and H as constants), then the motion of the two charges can be represented by the motion of a point along the straight line obtained from the plot of (15). The plot of (14) yields a family of curves corresponding to various possible values of u (only the right branch of each of the latter curves is admissible, since the left branch corresponds to $\dot{\eta}_a{}^{1/2} < 0$). Equations (11) show that $\dot{\eta}_a$ monotonically increases (decreases) for $k > 0$ (< 0), while $\dot{\eta}_b$ has just the opposite behavior. *For repulsion,* then, the system point moves along the straight line from I to III; point I corresponds to infinite separation with the particles approaching each other; point II corresponds to the "closest approach;" point III corresponds to infinite separation with the particles receding from each other. *For attraction,* the system point moves along the line directed from I to IV; point I corresponds to infinite separation, point IV to $r = 0$; as the system point approaches IV, the speed of charge a approaches the speed of light c, while the speed of b approaches a value less than c. As in the classical problem, point IV is singular.

III. CIRCULAR MOTION

We assume a and b travel at constant speeds cz_a, cz_b in circular orbits of radii A and B, respectively; we have, e.g., $A = cz_a/\omega$, where ω is the frequency of the motion. We further assume that a is an angle $\pi + \phi$ ahead of b at time t (see Fig. 3), so that the positions of a and b at the same time t are given by $x_a^3 = x_b^3 = 0$ and

$$x_a^1(t) = A\cos(\omega t + \pi + \phi); \quad x_b^1(t) = B\cos\omega t;$$

$$x_a^2(t) = A\sin(\omega t + \pi + \phi); \quad x_b^2(t) = B\sin\omega t.$$

The points a_t, b_t in Fig. 3 give the positions of a and b at time t; $a_{t+r/c}$, b_t are the positions of a and b when they lie on an interaction light cone. If a light signal leaves b at time t (leaves b_t), it arrives at a at time $t + r/c$, with r given by $r^2 = A^2 + B^2 + 2AB\cos\theta$; a travels through the angle $\theta - \phi$ during the time r/c: $\omega r/c = \theta - \phi$. When the two charges lie on an interaction light cone, we then have

$$\begin{aligned} x_a^1(t + r/c) &= -(cz_a/\omega)\cos(\omega_a s_a + \theta); \\ x_a^2(t + r/c) &= -(cz_a/\omega)\sin(\omega_a s_a + \theta); \\ x_b^1(t) &= (cz_b/\omega)\cos\omega_b s_b; \\ x_b^2(t) &= (cz_b/\omega)\sin\omega_b s_b; \\ s_i &= t(1 - z_i{}^2)^{1/2}; \quad \omega_i \equiv \omega(1 - z_i{}^2)^{-1/2}, \end{aligned} \tag{19}$$

where we have assumed for the proper times $s_a = s_b = 0$ when $t = 0$.

Substitution of (19) into the equations of motion (4) or (8) yields the conditions which the four quantities z_a, z_b, ω, θ must satisfy in order that (19) be a solution. We find

$$m_i z_i c^2/\omega(1 - z_i{}^2)^{1/2} = \frac{-\lambda(1 - z_j{}^2)}{(z_i + z_j\cos\theta)(z_i z_j \sin\theta + Z)}; \tag{20}$$

$$z_a{}^2\sin\theta + z_b{}^2\sin\theta + z_a z_b\sin\theta\cos\theta + (z_a z_b + \cos\theta)Z - \sin\theta = 0; \tag{21}$$

$$Z \equiv (z_a{}^2 + z_b{}^2 + 2z_a z_b\cos\theta)^{1/2}.$$

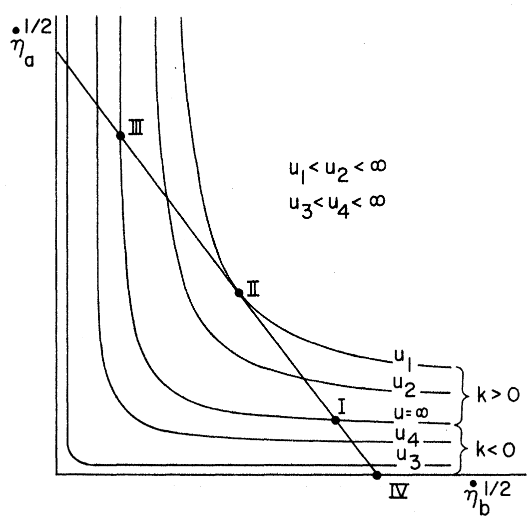

FIG. 2. The motion of the two charges can be represented by the motion of a point along the straight line from I to III for repulsion and from I to IV for attraction.

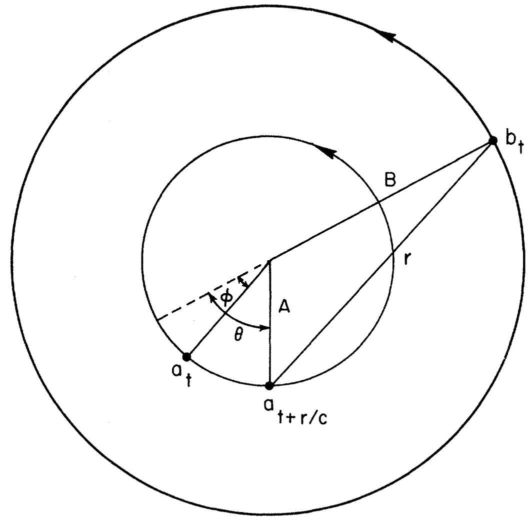

FIG. 3. At time t, a is at a_t and b is at b_t. When a is at $a_{t+r/c}$, it is on the forward ("interaction") light cone of b_t.

Equations (20) correspond to the radial equations of motion [they are obtained most readily from (7)]; (21) is just the requirement that the tangential component of the electric field of each charge vanish at the appropriate space-time position of the other charge (the existence of the circular-motion solutions depends on the fact that the same condition is obtained for each charge).

It is easy to show that solutions of (20) and (21) exist. For example, we can assume $\theta=\frac{1}{2}\pi$, $z_a/z_b=3^{1/2}$; then (21) gives $z_b\approx 0.4$. If we eliminate ω from Eqs. (20) and substitute these values into the resulting equation, we find $m_a/m_b\approx 0.5$, i.e., the above values for θ, z_a, z_b constitute a solution for this particular ratio of the masses. In general, the allowed range of θ depends on the comparative sizes of m_a and m_b, e.g., if $m_a=m_b$ then $z_a=z_b$ and $0<\theta<\pi$.

The only quantities P^μ and $L^{\mu\nu}$ which are nonzero for this circular motion can be written as

$$P^0=m_a c(1-z_a{}^2)^{1/2}+m_b c(1-z_b{}^2)^{1/2},$$

$$L^{12}=-\lambda(1+z_a z_b\cos\theta)/(z_a z_b\sin\theta+Z).$$

These are identical in form to the P^0, L^{12} of the Schild motion.[6] As for the Schild motion, the energy of the bound system is less than the free-particle energy. It is interesting to note that while the total momentum $\vec{P}=0$, the center of mass of the two charges moves with constant speed in a circle about the origin.

IV. NONELECTRODYNAMIC INTERACTIONS

The form (12) of the one-space Lorentz transformation using the ξ, η coordinates makes it easy to construct other types of covariant dynamics for the case of one space dimension; all we need do is see that on transformation the same power of σ occurs in all terms of the equations written down.

For example,[13] $m_i\ddot{\eta}_i/\dot{\eta}_i{}^{3/2}=k(\eta_j-\eta_i)/\dot{\eta}_i{}^{1/2}$ for a to the left of b and $m_i\ddot{\xi}_i/\dot{\xi}_i{}^{3/2}=k(\xi_j-\xi_i)/\dot{\xi}_i{}^{1/2}$ for a to the right[11] of b $(k>0)$ is one of several possible generalizations of the classical harmonic oscillator equations of motion. This particular dynamics (i) admits super-light speeds; however, if the initial speeds are less than c they remain so and the motion is oscillatory; (ii) can (for speeds less than c) be obtained from the Hamiltonian $H=m_a\exp(p_a/m_a)+m_b\exp(p_b/m_b)+\frac{1}{2}ku^2$; for this H, the one-space Lorentz transformation is not canonical[14]; (iii) can be readily generalized to three space dimensions—simply use (8) with the right-hand side multiplied by the invariant $\rho_i{}^2\rho_i$.

We can guarantee a Lagrangian-Hamiltonian dynamics for which the one-space Lorentz transformation is canonical by starting with (13) and varying only the last or "potential" term; for a relativistic oscillator, one possible potential[13] is $\frac{1}{2}k(\eta_a-\eta_b)^2/(\dot{\eta}_a+\dot{\eta}_b)^{1/2}$. The equations of motion for this case are considerably more involved than for the oscillator given above.

If the resulting equations of motion are of the form $m_i\ddot{\eta}_i/\dot{\eta}_i{}^{3/2}=f(\eta_i,\eta_j,\dot{\eta}_i,\dot{\eta}_j)$ for a left of b and $m_i\ddot{\xi}_i/\dot{\xi}_i{}^{3/2}=f(\xi_i,\xi_j,\dot{\xi}_i,\dot{\xi}_j)$ for a right of b, where f is arbitrary, they can be readily generalized to the time-symmetric case.

V. ASPECTS OF QUANTIZING THE MOTIONS

The one-space-dimension Klein-Gordon equation for a free particle in the ξ, η coordinates is $\partial^2\psi/\partial\xi\partial\eta=-\frac{1}{2}(mc/\hbar)^2\psi$, which can be obtained from the free-particle Hamiltonian $H=-4m^2/p$ if we make the replacements $H\to -i\,8^{1/2}(\hbar/c)\partial/\partial\xi$, $p\to i\,8^{1/2}(\hbar/c)\partial/\partial\eta$. Attempts to obtain a Schrödinger-type wave equation from the H of (15) lead to difficulties, e.g., the operators $(p_i-k/2u)^{-1}$ have to be defined. Since H is not in general the energy of the system, we might ask if it is the proper quantity to quantize.

The Bohr-Sommerfeld quantum conditions (BSQC) suggest otherwise; they can be applied when the action-angle integrals exist (these exist for, e.g., versions of a relativistic oscillator). If the first, or "kinetic," part of the Lagrangian has the same form as in (13), the quantity $p_u du$ is invariant under a one-space Lorentz transformation, and thus so is the BSQC $\oint p_u du=$constant, where the constant is presumed to be the same in all frames. If as in the Hamilton-Jacobi method, p_u is expressed in terms of the constants of the motion p_v and H, the BSQC is of the form $f(p_v,H)=$invariant; this implies that f must be a function of the invariant p_vH. The BSQC thus specifies the allowed values of $E_{\rm int}$ as given by (18). This agrees with the classical quantum result: The internal energy is quantized, while the translational energy is not. These ideas do not apply if $p_u du$ is not an invariant, e.g., in the case of the first relativistic oscillator given in Sec. IV.

For the circular motions of Sec. III, we can apply naive Bohr quantization by setting $L^{12}=n\hbar$ [this is an invariant quantum condition since it corresponds in a general frame to $L^{\mu\nu}L_{\mu\nu}=(n\hbar)^2$]. This quantum condition yields, of course, the nonrelativistic result as a first approximation. For positronium,[15] for example, we find $cP_n^0=2mc^2(1-\alpha^2/8n^2+\Delta_n)$; here m is the mass of the electron, $\alpha=e^2/\hbar c$ is the fine-structure constant; $2mc^2\Delta_n$ is the deviation from the nonrelativistic result. For $n=1$, we have $\Delta_1 \approx -5\alpha^4/128$; this is smaller than the perturbation theory correction[16] ($\approx -\alpha^4/6$) for the singlet ground state of positronium. We find nothing here corresponding to spin.

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[2]J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. 21, 425 (1949).

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[7]The difficulties in solving the Wheeler-Feynman problem apply also to the van Dam-Wigner-Katz generalization of the time-symmetric problem [in this generalization, each particle (not necessarily charged) is affected by the portion of the other's trajectory for which $(RR) \leq 0$]: H. van Dam and E. P. Wigner, Phys. Rev. 138, B1576 (1965); A. Katz, J. Math. Phys. 10, 1929 (1969); 10, 2215 (1969). Degasperis has found the exact circular-motion solutions for this dynamics: A. Degasperis, Phys. Rev. D 3, 273 (1971).

[8]Strictly, s_a, s_b should be considered as quite arbitrary parameters, and not taken as the proper times, until *after* the variation of (3) is carried out. See, e.g., J. L. Anderson, *Principles of Relativity Physics* (Academic, N.Y., 1967), p. 193; F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, Mass., 1965), Chap. 6.

[9]See also, e.g., the references in Ref. 8.

[10]This is a simple example of Dirac's "front form" [P. A. M. Dirac, Rev. Mod. Phys. 21, 392 (1949)].

[11]If a is to the right of b, the constraint (10) becomes $\eta_a = \eta_b$, so we can use η as the motion parameter; Equations (9) become $m_i \ddot{\xi}_i / \dot{\xi}_i^{3/2} = G_i k \dot{\xi}_j / (\xi_i - \xi_j)^2$; here $\dot{\xi}_i = d\xi_i/d\eta$. The Lagrangian and Hamiltonian can be obtained from (13) and (15) by replacing the letter η by the letter ξ. The generalization of the equations of motion to the Wheeler-Feynman two-body problem is straightforward.

[12]The Currie-Jordan-Sudarshan "zero-interaction" theorem [D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, Rev. Mod. Phys. 35, 350 (1963)] does not apply here, as the position coordinates x_i^μ are not canonical (cf. Ref. 3).

[13]Since the paths of the particles cross, we have to decide what equations of motion to use when a is to the right of b; if the same Hamiltonian is used for both parts of the motion, then the particles interchange roles (i. e., on the second half of the motion $\tau_b \geq \tau_a$); if $\tau_a \geq \tau_b$ always, then two Hamiltonians are required, one for each half of the motion.

[14]It may be possible to construct another Hamiltonian for this problem, canonically inequivalent to the one given here, but for which the Lorentz transformation in canonical. See, e. g., R. N. Hill, J. Math. Phys. 8, 1756 (1967).

[15]The asymmetric problem has a sort of symmetry under time reversal plus charge conjugation if b is an electron (particle) and a is a positron (antiparticle), for then we can consider a as a Feynman electron traveling backward in time: The retarded potential of an electron becomes the advanced potential of a positron under time reversal.

[16]A. O. Barut and A. Baiquini, Phys. Rev. 184, 1342 (1969).