Conserved Vector of the Komar Type for Brans-Dicke Theory*

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The analog of Komar's identically conserved vector density in general relativity is found for the Brans-Dicke theory. The relation of such *unique* expressions to given Lagrangians is described, thereby exhibiting the reason for the physical consequences of the identities.

Some years ago Komar[1] found that conservation laws for general relativity may be obtained from the tensor identity for covariant divergences,

$$(\xi^{i|j}-\xi^{j|i})_{|ij}\equiv 0\ , \tag{1}$$

where ξ^i is an arbitrary vector field. For suitable choices of this vector the expression reduces to the various suggested conservation laws.[2] For instance, Moss[3] has recently shown that if one chooses the vector to be a Killing vector field, then (1) becomes Trautman's conservation law.[4] Also, for an asymptotically Schwarzschild universe Komar showed[1] that the energy content of the manifold given by (1) is mc^2.

Since identity (1) is a mathematical triviality it seems strange that physical consequences should follow from it. However, it is not generally realized that every invariant has associated with it an identically conserved vector (density),[5] or, to be more specific, a skew-symmetric second-rank tensor which is a functional of the field variables as well as an arbitrary vector field.[6,7] Since the double covariant divergence of any skew-symmetric second-rank tensor vanishes identically by Ricci's lemma, this accounts for the conserved quantity above. In the case of the scalar curvature the associated skew-symmetric tensor is found in (1). This property of the scalar curvature was first pointed out by Davis and Moss[8] in a different context.

In view of the current interest in the Brans-Dicke theory it is useful to present the counterpart of (1) associated with this theory.[9] The term of interest in their Lagrangian is $\mathcal{L}\equiv\phi R(g)^{1/2}$ and one considers the Lie derivative[10] of $\mathcal{L}$ in the direction of an arbitrary vector field ξ^i. One can show that the scalar ϕR satisfies the identity[11]

$$(\phi R\xi^i)_{|i}\equiv 2(E^{ij}\xi_j+\phi Z^{abci}\xi_{a|bc}-\phi_{|c}Z^{abci}\xi_{a|b})_{|i}+(\mathcal{E}\phi_{|i}-2E^j_{i|j})\xi^i\ , \tag{2}$$

where

$$E^{ij}\equiv\frac{1}{\sqrt{g}}\frac{\delta\mathcal{L}}{\delta g_{ij}}\equiv\phi(\tfrac{1}{2}g^{ij}R-R^{ij})+Z^{ijkl}\phi_{|kl}\ ,$$
$$\mathcal{E}\equiv\frac{1}{\sqrt{g}}\frac{\delta\mathcal{L}}{\delta\phi}\equiv R\ , \tag{3}$$
$$Z^{ijkl}\equiv\frac{\partial R}{\partial g_{ij,kl}}\ .$$

Using standard techniques[12] one then argues that the coefficient of ξ^i in the final term of (2) vanishes identically. This is the "compatibility identity" involving the two variational derivatives of $\mathcal{L}$ (which reduces to the identical vanishing of the divergence of the Einstein tensor if ϕ is constant). For this reason (2) yields the conserved vector

$$V^i\equiv-\tfrac{1}{2}\phi R\xi^i+E^{ij}\xi_j+\phi Z^{abci}\xi_{a|bc}-\phi_{|c}Z^{abci}\xi_{a|b}\ . \tag{4}$$

We shall now rewrite this in the form of Komar's expression. It is easily seen[11] that

$$Z^{abcd}\equiv g^{ab}g^{cd}-\tfrac{1}{2}g^{ac}g^{bd}-\tfrac{1}{2}g^{ad}g^{bc}$$

and hence that Z^{abcd} is cyclic on any three indices and possesses the additional symmetries $Z^{abcd}\equiv Z^{abdc}\equiv Z^{badc}\equiv Z^{cdab}$. We may then write[13]

$$Z^{abci}\equiv\tfrac{4}{3}Z^{a[bc]i}+\tfrac{2}{3}Z^{c[ab]i} \tag{5}$$

to find

$$Z^{abci}\xi_{a|bc}\equiv R^{ij}\xi_j+\tfrac{2}{3}Z^{c[ab]i}\xi_{a|bc} \tag{6}$$

which follows from the commutation relations. This reduces (4) to

$$V^i\equiv Z^{abci}\phi_{|ab}\xi_c+\tfrac{2}{3}\phi Z^{c[ab]i}\xi_{a|bc}-\phi_{|c}Z^{abci}\xi_{a|b}\ . \tag{7}$$

From here, with a direct but rather long calculations, we find

$$V^i\equiv\tfrac{2}{3}[(\phi\xi_{a|b}-2\phi_{|b}\xi_a)Z^{a[ci]b}]_{|c}\ . \tag{8}$$

This is the conserved vector[14] associated with the Brans-Dicke theory. Note that it reduces to Komar's expression if ϕ is taken to be constant. This is consistent with the reduction of the Brans-Dicke Lagrangian to the Lagrangian of general relativity for constant ϕ.

The physical consequences of (8) for the Brans-Dicke analog of the Schwarzschild solution[15] are under investigation at the present time. A general consideration of invariants, their compatibility identities, and Komar-type expressions, will be discussed in a forthcoming paper.

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[1] A. Komar, Phys. Rev. 113, 934 (1959).

[2] J. L. Anderson, *Principles of Relativity Physics* (Academic, New York, 1967), Sec. 13-1.

[3] M. K. Moss, Nuovo Cimento Lett. 5, 543 (1972).

[4] A. Trautman, *Lectures on General Relativity* (Prentice-Hall, New York, 1965), p. 185.

[5] For the most simple invariants the conserved vector may itself vanish identically.

[6] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, (Addison-Wesley, Reading, Mass., 1962), Sec. 100.

[7] The arbitrary vector field arises through the formulation of the identity using the Lie derivative (in the direction of an *arbitrary* vector field).

[8] W. R. Davis and M. K. Moss, J. Math. Phys. 7, 975 (1966).

[9] C. Brans and R. H. Dicke, Phys. Rev. 124, 925 (1961), Eq. (6).

[10] Reference 4, pages 183–186, and Ref. 8 give the basic approach to such identities.

[11] $R \equiv g^{ab} g^{cd} R_{abcd} \equiv g^{ab} g^{cd} ([ac,d]_{,b} - [ab,d]_{,c} + \{{}_a{}^i{}_b\}[dc,i] - \{{}_a{}^i{}_c\}[db,i])$ and a vertical stroke denotes covariant differentiation.

[12] Reference 4, p. 183.

[13] $Z^{a[bc]i} \equiv \frac{1}{2}(Z^{abci} - Z^{acbi}) \equiv \frac{1}{2}(Z^{baic} - Z^{biac})$.

[14] $(\phi \xi_{a|b} - 2\phi_{|b} \xi_a) Z^{a[ci]b} \equiv T^{ci}$, where $T^{ci} \equiv -T^{ic}$. Thus $T^{ci}{}_{|ci} \equiv 0$.

[15] Reference 9, Eqs. (31)–(34).

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Time-Asymmetric Two-Body Problem in Special Relativity

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We treat a Lorentz-covariant two-body problem due to Fokker: One electric charge experiences the retarded field of a second, while the second experiences the advanced field of the first; this is pure action at a distance, with no self-action; conservation principles exist. We show that this (apparently generally soluble) time-asymmetric problem is exactly soluble for straight-line motion and admits solutions in which the charges move in circles about a common center. We briefly consider nonelectrodynamic time-asymmetric interactions and aspects of quantizing the motions.

We begin a study here of the following Lorentz-covariant two-body problem, originally posed by Fokker[1]: One electric charge is acted upon by the retarded electromagnetic field of a second charge, while the second charge is acted upon by the advanced field of the first; the interaction is pure "action at a distance," with the fields treated merely as convenient tools for describing the interaction; there is no self-action. This problem is time-*a*symmetric, in contrast to the time-symmetric two-body problem in the Wheeler-Feynman[2] formulation of electrodynamics, in which each charge is acted upon by half the advanced plus half the retarded field of the other.

The symmetric problem has so far defied general analysis, since its solution apparently requires knowledge of an infinite set of position-velocity data for each charge. Attempts[3,4] have been made to reduce the Wheeler-Feynman two-body problem to Newtonian-type equations requiring only an ordinary set of initial conditions, but various mathematical questions arise,[5] e.g., whether the series expansions used converge. Only one class of rigorous solutions of the time-