PHYSICAL REVIEW D VOLUME 8, NUMBER 8 15 OCTOBER 1973

Naked Singularities, Thin Shells, and the Reissner-Nordström Metric

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(Received 7 June 1973)

The time development of a thin charged shell is studied and it is shown that it can collapse to form a naked singularity if, and only if, the matter energy density of the shell is negative.

I. INTRODUCTION

The metric on the inside and outside of a thin shell of charged dust can be solved explicitly; the result, in Schwarschild coordinates, is just the Reissner-Nordström metric of total mass M and charge Q on the outside of the shell and the flat-(Minkowski) space metric on the inside. The metric can then be related to the stress-energy tensor of the shell by the discontinuity equations first derived by Israel[1] and Kuchař.[2] The stress-energy can be expressed in terms of the proper mass of the shell—the mass of each particle times the number of particles. The relation between the total mass and the proper mass may be written as the sum of kinetic- and potential-energy terms; this relation provides a means of solving for the motion of the shell in Schwarzchild coordinates. These results are summarized in Sec. II.

In order to interpret the results, the complete extension of the Reissner-Nordström metric must be used because, in the course of the time development, the world line of the shell will, in general, not remain within the region of space-time covered by the original coordinate patch. Section III is devoted to presenting the extension due to Carter.[3,4]

There are several possible motions of the shell for various proper mass, charge, and starting conditions; these are discussed in detail in Sec. IV. As long as the proper mass is positive and an asymptotically flat region of space-time exists, the total mass is always positive and no naked singularity can develop as seen by an observer in the asymptotically flat space. However, if the proper mass is taken to be negative (the matter stress-energy tensor has a negative energy density) then solutions exist in which the total mass is positive, there is an asymptotically flat space, and the shell collapses to a naked singularity.

II. THIN-SHELL EQUATIONS

The basic equations describing the time development of a thin shell have been derived by Israel[1] and Kuchař,[2] who have shown how to relate the matter stress-energy tensor to the discontinuity of the extrinsic curvature of the three-dimensional timelike hypersurface swept out by the shell in the course of time. This work is related to those of Arnowitt, Deser, and Misner,[5] who solved anologous equations for the initial-value problem, and Siegel,[5] who has studied the time development of the system in coordinates in which r^* measures the distance from the center of the shell.

It is convenient to use Schwarzschild coordinates in which the Reissner-Nordström metric takes the form

$$ds^2 = -[1-2\Phi(r)]\,dt^2 + dr^2[1-2\Phi(r)]^{-1} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,, \tag{1}$$

where, in units with $G=1=c$,

$\Phi(r) \equiv M/r - Q^2/2r^2$ outside the shell

and

$\Phi(r) \equiv 0$ inside the shell.

The total mass of the shell is M and its total charge is Q. If the shell collapses to the origin, an event horizon is well known to develop provided that $M>|Q|$; in that case $1-2\Phi(r)$ possesses zeros at

$$R_\pm = M \pm (M^2-Q^2)^{1/2} \tag{2}$$

and there are two pseudosingularities. The outer zero is the event horizon as seen from the exterior region.

The parameters M, Q, and $R(\tau)$, the radius of the shell at its proper time, τ, must now be related to the matter energy density. The shell is at $r=R(\tau)$ at the time $t=T(\tau)$, hence its four-velocity must be

$$u^\mu = \frac{dx^\mu}{d\tau} = (u^0, \dot{R}, 0, 0) \tag{3}$$

and

$$u^2 = -1 = -[1-2\Phi(R)](u^0)^2 + \dot{R}^2/[1-2\Phi(R)]\,. \tag{4}$$

The metric (Φ) is discontinuous across the shell; thus, the four-velocity will be different as seen from the two sides. However, in Schwarzschild coordinates, R is the same, being a measure of the (invariant) area of the shell. Thus,

$$u^0 = \frac{\pm 1}{1-2\Phi(R)}[1-2\Phi(R)+\dot{R}^2]^{1/2} . \quad (5)$$

The normal to the hypersurface, n_μ, must be orthogonal to u^μ and e_μ, the spatial vectors tangent to the surface; hence

$$n_\mu = \pm(-\dot{R}, u^0) . \quad (6)$$

As has been shown by Israel[1] and Kuchař,[2] the discontinuity of the extrinsic curvature is related to the surface stress-energy tensor of the shell. The extrinsic curvature, K_{ab}, is given by $K_{ab} = e^\mu_a K_{\mu\nu} e^\nu_b$, where

$$K_{\mu\nu} \equiv n_{\mu;\nu} \quad (7)$$

and the three e^μ_a are an orthonormal triad lying in the surface,

$$n_\mu e^\mu_a = 0,$$

$$e^\mu_a g_{\mu\nu} e^\nu_b = {}^3g_{ab} , \quad (8)$$

and

$$^3g^{ab} e^\mu_a e^\nu_b = [g^{\mu\nu} - n^\mu n^\nu / n^2] ,$$

with $^3g_{ab}$ being the metric in the timelike hypersurface. Rather than the extrinsic curvature, it is convenient to use

$$\Pi^a_b \equiv e^{a\mu} e^\nu_b K_{\mu\nu} - \delta^a_b K^c_c , \quad (9)$$

where

$$e^{a\mu} = {}^3g^{ab} e^\mu_b .$$

The discontinuity of π^a_b is then given by the surface matter stress-energy density,[1,2] S^a_b:

$$[\Pi^a_b] \equiv \Pi^a_b(R+) - \Pi^a_b(R-) = -4\pi S^a_b = -4\pi\sigma u^a u_b . \quad (10)$$

In particular, if $e^\mu_a = u^\mu = e^\mu_b$, Eq. (10) reads, using Eqs. (9) and (7),

$$[u^\mu n_{\mu;\nu} u^\nu + n^\mu{}_{;\mu} - n^\mu n^\nu n_{\mu;\nu}] = -4\pi\sigma , \quad (11)$$

where σ is the surface energy density of the shell in the comoving frame; the total proper mass of the shell, $\mathfrak{M}$, is the surface density times the area of the shell, $4\pi R^2$ or $4\pi\sigma = \mathfrak{M}/R^2$, and, since S^{ab} is covariantly conserved in the hypersurface, $\mathfrak{M}$ is a constant. Using the metric, Eq. (11), the identity

$$\delta^\lambda_\sigma = \delta^\lambda_2 \delta^2_\sigma + \delta^\lambda_3 \delta^3_\sigma + n^\lambda n_\sigma - u^\lambda u_\sigma , \quad (12)$$

and the equations for the affinity

$$\Gamma^2_{\alpha 2} = \Gamma^3_{\alpha 3} = \delta^1_\alpha / r \quad (13)$$

it follows that

$$-\mathfrak{M}/R^2 = (n^1(R+) - n^1(R-))/R$$

or

$$\mathfrak{M} = R(n^1(R-) - n^1(R+)) . \quad (14)$$

The normal on the inside is just $(1+\dot{R}^2)^{1/2}$, corresponding to the flat space inside the shell and the fact that the invariant radius decreases in the direction of the center of the shell.

The outward normal may have either sign; this is discussed in more detail below. Hence

$$n^1(R+) = \pm[1-2\phi(R)+\dot{R}^2]^{1/2} \equiv n \quad (15)$$

and Eq. (14) may be solved for $n^1(R+)$, squared, and rewritten as

$$M = \mathfrak{M}(1+\dot{R}^2)^{1/2} - (\mathfrak{M}^2 - Q^2)/2R . \quad (16)$$

Here, the total mass (energy) has been written as the kinetic plus rest-mass energy of the shell, $\mathfrak{M}(1+R^2)^{1/2}$, plus the potential-energy term, which is exactly the same as the classical potential energy. The factor of 2 reflects the fact that the shell responds to the average field, not the field on either side, and that this holds both for the gravitational and electric fields.

This result for $(1+\dot{R}^2)^{1/2} = n^1(R-)$ as a function of R may be reinserted in Eq. (14) to give an explicit result for $n^1(R+)$:

$$n^1(R+) = M/\mathfrak{M} - (\mathfrak{M}^2 + Q^2)/2R\mathfrak{M}. \quad (17)$$

It is now trivial to show that the shell cannot collapse to a naked singularity if $\mathfrak{M}$ is positive. If the shell is in an asymptotically flat region, $n^1(R+)$ must be positive, because the Schwarzschild radius increases as one moves away from the shell. Then the sign of the normal must remain positive until the shell passes an event horizon since

$$n = [1-2\Phi(R+) + \dot{R}^2]^{1/2} \quad (18)$$

and the argument of the square root is positive as long as $1-2\Phi$ is positive. But, by Eq. (17), R cannot go to zero unless either $\mathfrak{M}<0$ or $n^1(R+)$ changes sign. Thus, for $\mathfrak{M}>0$, the shell can collapse to a point only if an event horizon forms. However, if $\mathfrak{M}<0$, then collapse to $R=0$ may occur without the formation of an event horizon; an example of appropriate starting conditions is given below.

There is a geometrical reason why n^1 can only change sign if there is an event horizon: The normal to the surface of the shell must be a spacelike vector, hence

$$n^\mu n_\mu = (n^1)^2/[1-2\Phi(R+)] - (n^0)^2[1-2\Phi(R+)] = 1 . \quad (19)$$

If there is no event horizon, $(1-2\Phi)$ is positive and, at the radius where n^1 changes sign, the normal must be timelike. This contradiction implies that n^1 can change sign only where $(1-2\Phi)<0$, i.e., inside the event horizon.

III. KRUSKAL DIAGRAM

In order to understand the changing sign of the outward normal of the shell it is necessary to look at a diagram of the maximal analytic extension of the Reissner-Nordström metric. This extension has been given by Graves and Brill and by Carter.[4] For $M>|Q|$, the maximal extension by Carter is shown in Fig. 1. The light cones are 45° lines on the diagram. The 45° lines shown correspond to spatial infinity, R_+ or R_- as indicated, and to the coordinate t equal to plus or minus infinity. The dashed lines indicate lines of constant r and the arrowheads indicate the direction of increasing t. For $M=|Q|$, the maximal extension is shown in Fig. 2, where a similar notation is used.[4] For $M<|Q|$, there are no event horizons and the metric does not need to be extended—the space $0<r<\infty$ and $-\infty<t<\infty$ is geodesically complete.

The world line of the shell may now be plotted on the appropriate diagram (two possible lines are shown in Fig. 1). This must be a timelike line (slope greater than 45°). The I_+ regions are taken to be the normal regions outside the shell,

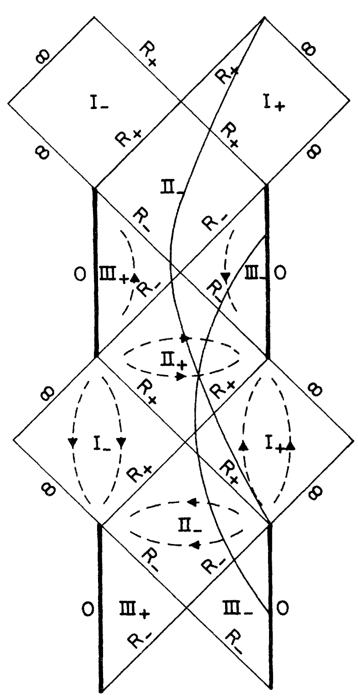

FIG. 1. Maximal analytic extension of the Reissner-Nordström metric for $M>|Q|$.

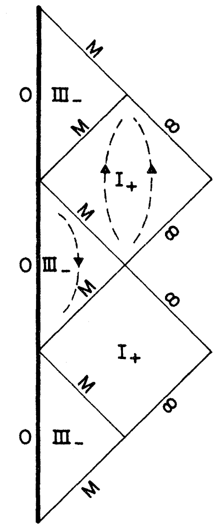

FIG. 2. Maximal analytic extension of the Reissner-Nordström metric for $M=|Q|$.

or (if that is appropriate) the event horizon shielding the shell. The region to the left of the world line of the shell is the interior of shell—a flat-space region of radius R. If the world line of the shell passes through regions I_+ and III_+ the outward normal to the shell is positive; it points toward larger values of r. In regions I_- and III_- the outward normal is negative, pointing toward smaller values of r. In regions $II_\pm$, the radial component of the outward normal may have either sign; however, n^0 must be positive (negative) in II_+ (II_-). The points on the diagram where regions I, II, and III meet are singular points of the coordinates; they represent temporal infinity for all timelike geodesics in region I, spatial infinity for all spacelike geodesics in region II, and temporal infinity $R<R_-$ for region III. The three infinite regions are disjoint and cannot be identified.

In the next section, the world line of the shell will be plotted on the appropriate Kruskal diagram for each of the possible starting conditions for a shell of proper mass $\mathfrak{M}$ and charge Q. It is then trivial to read off the properties of the solutions. A discussion of some of the properties of thin charged shells is given by de la Cruz and Israel,[6] who explain many of the properties in more detail, and by Bekenstein[6] and Chase.[6]

IV. BEHAVIOR OF THE SHELL

The energy equation, Eq. (16), and the equation for $n^1(R+)$, Eq. (17), combined with the Kruskal diagrams, provide the basis for a discussion of the time development of the shell.

The equation for $R(\tau)$, Eq. (17), can be integrated directly, yielding R as a function of τ as long as R is greater than zero. For $M>\mathfrak{M}$

$$[(R-r_+)(R-r_-)]^{1/2}+\frac{(r_+ +r_-)\frac{1}{2}\cosh^{-1}(2R-r_+-r_-)}{r_+-r_-}=[(M/\mathfrak{M})^2-1]^{1/2}|\tau-\tau_0|,$$

where

$$r_\pm\equiv-(\mathfrak{M}^2-Q^2)/2(M\mp\mathfrak{M})$$

and

$$R>r_+ .$$

For $M<\mathfrak{M}$ and $\mathfrak{M}^2<Q^2$

$$[(r_+ -R)(R-r_-)]^{1/2}+\frac{(r_+ +r_-)\frac{1}{2}\cos^{-1}(2R-r_+-r_-)}{r_+ -r_-}=[1-(M/\mathfrak{M})^2]^{1/2}|\tau-\tau_0|,$$

with

$$r_+>R.$$

These equations cease to be applicable when the shell reaches the origin, $R=0$.

The behavior is just what one would expect from studying the turning points (points where $\dot{R}=0$) of Eq. (16). The qualitative nature of the solutions may, then, be read off from that equation, except that R does not completely specify the point; $R>R_+$ may imply that the surface of the shell lies in either region I_+ or I_- of Fig. 1. In order to complete the analysis the behavior of the outward normal $n^1(R+)$ must be followed also. This information is provided by Eq. (17) and, given M, $\mathfrak{M}$, and Q, n changes sign at

$$R_n=(\mathfrak{M}^2+Q^2)/2M. \tag{20}$$

By the argument given above, this point must lie between the two event horizons; it is a straightforward exercise to verify that, if the shell can in fact reach R_n, then R_n does lie between R_+ and R_-, i.e., in a type-$\mathrm{II}_\pm$ region where t is a spacelike coordinate.

The energy equation, Eq. (16), requires that $\dot{R}$ be equal to zero at no more than one radius, R_T, where

$$R_T=-(\mathfrak{M}^2-Q^2)/2(M-\mathfrak{M}); \tag{21}$$

hence $\dot{R}$ may change sign only if the right-hand side is greater than zero; if it is less than zero the shell must proceed inexorably from $R=\infty$ in a I_+ region since $n^1(\infty)>0$ to $R=0$ in a III_- region since $n^1(0)<0$.

The various possibilities are now spelled out in detail.

(1) $M>\mathfrak{M}>|Q|$. From the equation for $(1+\dot{R}^2)^{1/2}$, Eq. (17), it is apparent that $\dot{R}$ cannot be zero at any time; the shell either explodes or collapses to zero radius depending on the sign of $\dot{R}$. Since $M>Q$, an event horizon develops and, since the shell reaches zero radius, n must change sign; hence, the shell must pass through region III_-. Thus, the shell follows one of the paths in Fig. 3. Note that a spacelike surface passing through the world line of the shell in region III_- or III_+ is closed and that there is a singularity in the metric on the outside (non-flat-space side) of the shell. This is a charged singularity since the flux through an enclosing surface is $-Q$ (the total charge in any compact space must be zero). The 3-space topology has changed from that of a Euclidean 3-space to that of a closed universe. The interior of the sphere is a Euclidean sphere of radius R; the other side has a point charge $-Q$ at the origin of a sphere of radius R with the Reissner-Nordström geometry. There is no space, at this time, which is asymptotically flat. To the extent that the equations can be believed, as the singularity develops and the shell collapses to a point the universe does so also, and after $R(\tau)=0$ the universe consists of an uncharged point.

(2) $M>\mathfrak{M}$; $|Q|>\mathfrak{M}$. There is now a turning point at R_T, the minimum radius of the shell. It is convenient to characterize the shell by the turning radius rather than M,

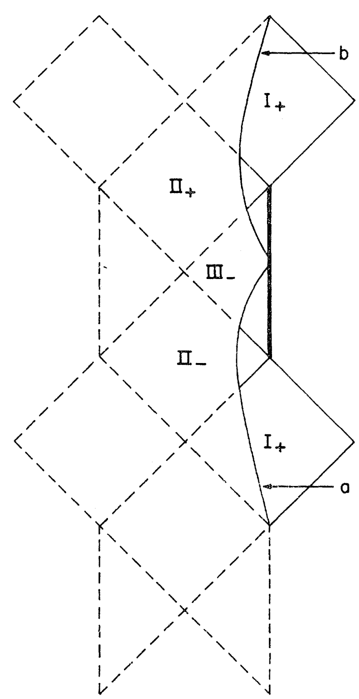

FIG. 3. (a) Space-time for a collapsing charged shell with $M>\mathfrak{M}>|Q|$; (b) space-time for an exploding charged shell with $M>\mathfrak{M}>|Q|$.

$$M=\mathfrak{M}+(Q^2-\mathfrak{M}^2)/2R_T>0\,. \qquad (22)$$

There are now three subcases:

(a) $R_T>\frac{1}{2}(\mathfrak{M}+|Q|)$; then, $M<|Q|$ and no event horizon can form in any case; the shell simply contracts to R_T, then reexpands to infinity. As R_T approaches $\frac{1}{2}(\mathfrak{M}+|Q|)$, the coordinate time required for the shell to bounce a finite distance from R_T diverges, providing physical continuity between this case and the next.

(b) $R_T=\frac{1}{2}(\mathfrak{M}+|Q|)$. Now $M=|Q|$, $R_T<R_+=M$, and $n(R_T)=(|Q|-\mathfrak{M})/(|Q|+\mathfrak{M})>0$. The shell passes through the single event horizon, region III_-, and emerges in a different space. This is depicted in Fig. 4.

(c) $0<R_T<\frac{1}{2}(\mathfrak{M}+|Q|)$. If $R_T>\mathfrak{M}$, $R_->R_T$, and $n(R_T)>0$, the shell passes the event horizons and through region III_+ and reemerges in I_+. If $0<R_T<\mathfrak{M}$, everything is the same except that $n(R_T)<0$ and the shell passes through III_-. Both cases are shown in Fig. 5; if $R_T=\mathfrak{M}$, the curve passes directly through the point where regions III_- and III_+ meet.

(3) $\mathfrak{M}>M$. There must again be a turning point and $\mathfrak{M}^2>Q^2$; the turning point is now a maximum radius. As in the previous case, it is convenient to express M in terms of the turning radius R_T. There are now two cases:

(a) $R_T>\frac{1}{2}(\mathfrak{M}+|Q|)$. If $R_T>\mathfrak{M}$, $R_T>R_+$, and $n(R_T)>0$, the turning point is in a I_+ region, and n changes sign as the shell collapses into or emerges from the III_- regions. The shell has expanded from a point, producing a finite space with a singularity on the *outside* at $R=0$; it continues expanding until an exterior is formed into which it briefly emerges only to collapse along with the space.

If $M>R_T>\frac{1}{2}(\mathfrak{M}+|Q|)$, $R_T>R_+$, and $n(R_T)<0$, the turning point is in a I_- region and n does not change sign during the collapse. In this case, the shell never emerges from its hole into the normal space. The shell is observable from outside in the infinite past as it traverses the III_- and II_- regions, but not afterward.

If $R_T=\mathfrak{M}$, the shell passes through the point where regions I_+ and I_- touch. The space-time is depicted for $R_T>\mathfrak{M}$ in Fig. 6; if $R_T<\mathfrak{M}$, the world

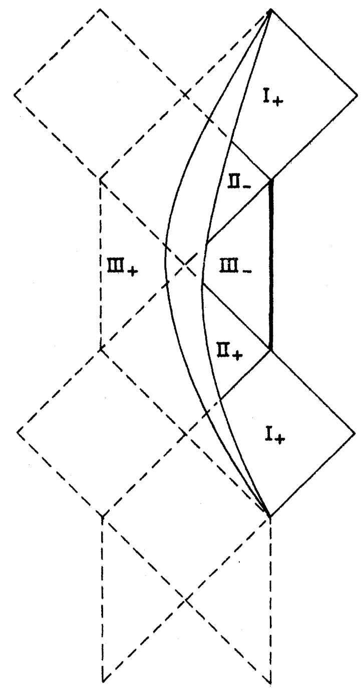

FIG. 5. Space-time for $|Q|>\mathfrak{M}$, $R_T<\frac{1}{2}(\mathfrak{M}+|Q|)$. The left curve corresponds to $R_T>\mathfrak{M}$; the right curve to $0<R_T<\mathfrak{M}$.

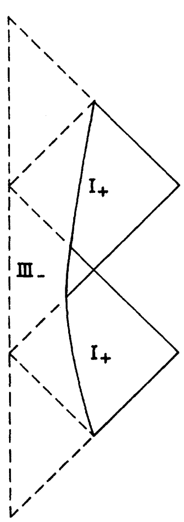

FIG. 4. Space-time for $M=|Q|>\mathfrak{M}$.

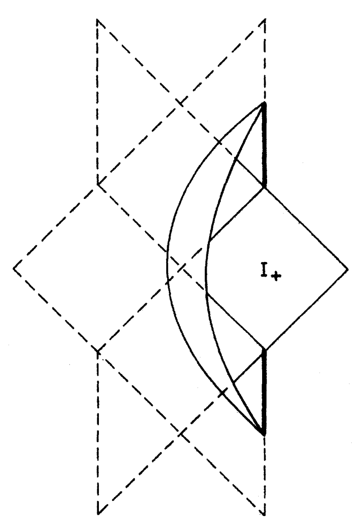

FIG. 6. Space-time for $\mathfrak{M}>|Q|$, $R_T>\frac{1}{2}(\mathfrak{M}+|Q|)$.

line should be moved into the I_- region. The total mass, M, is equal to $\mathfrak{M}-(\mathfrak{M}^2-Q^2)/2R_T \geq |Q|$ and is positive, as it must be if the space is asymptotically flat.[7]

(b) $0<R_T<\frac{1}{2}(\mathfrak{M}+|Q|)$. For $R_T>\frac{1}{2}(\mathfrak{M}-|Q|)$, the extended metric contains no event horizons. The outside of the shell is finite, r decreases going away from the shell, and there is a singularity with charge $-Q$ outside the shell.

For $R_T<\frac{1}{2}(\mathfrak{M}-|Q|)$, the extended metric contains event horizons but they are never realized since they are always to the left of the shell.

In neither case is there ever an exterior asymptotically flat space of the region-I type; such conditions have (presumably) nothing to do with our universe.

No naked singularities have appeared for any positive values of $\mathfrak{M}$. Although negative values of $\mathfrak{M}$ are unphysical, it is of interest to consider the case because collapse to a naked singularity may then appear.

(4) $0>\mathfrak{M}$, $Q^2>\mathfrak{M}^2$, $\frac{1}{2}(|Q|-|\mathfrak{M}|)<R_T<(Q^2-\mathfrak{M}^2)/2|\mathfrak{M}|$. The shell now collapses from its turning point—the total force is outward but the shell has negative inertia. From the equation for M in terms of R_T, Eq. (22), M lies between 0 and $|Q|$, hence $|Q|>M>0$ and no event horizon appears. That is, the shell has collapsed to the origin, without the appearance of an event horizon—a naked singularity has appeared.

To conclude: Collapse to a naked singularity is possible (though not required) if the proper mass of the shell is negative; hence any proof of the impossibility of the appearance of such a singularity must include some assumption equivalent to a positive definite matter energy density.

ACKNOWLEDGMENTS

I would like to thank J. Hartle, J. Bardeen, H. Siegel, and K. Thorne for discussions on this topic, D. Wilkins for comments on the manuscript, and S. Deser for extensive discussions and comments.

*Work supported in part by the U. S. Atomic Energy Commission.

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