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Note on Motion in the Schwarzschild Field*

Allen I. Janis
Department of Physics, University of Pittsburgh, Pittsburgh, Pennsylvania 15260
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It is often said that the speed of a freely falling test particle in the Schwarzschild field approaches the speed of light at the Schwarzschild radius. It is shown that this is not the case.

Many discussions of motion in the Schwarzschild field say that the speed of a freely falling test particle approaches the speed of light as the particle approaches the Schwarzschild radius; Refs. 1–5 indicate a sampling of such discussions. One might infer from these sources that a test particle does, in fact, cross the Schwarzschild radius with the speed of light. It is the purpose of this note to emphasize that this is not the case. (For simplicity, only radial geodesics will be discussed.)

Before giving what I consider to be the correct description, it may be instructive to examine briefly two of the "standard" treatments; other treatments may be analyzed in similar fashion. Zel'dovich and Novikov say[6] that the velocity they use "has direct physical significance. It is the velocity measured by an observer who is at rest (r, θ, ϕ constant) at the point which the particle is passing." The coordinates referred to here are the usual Schwarzschild coordinates; with a suitable choice of units, the line element may be written as

$$ds^2 = (1-2m/r)dt^2 - (1-2m/r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1)$$

It is clear from Eq. (1) that an observer "at rest" at the Schwarzschild radius, $r=2m$, must move with the speed of light. One might say, then, that the reason a test particle's speed approaches the speed of light is that it is measured by a family of observers whose speeds approach the speed of light. There is no reason to conclude from such measurements that the particle actually reaches the speed of light at $r=2m$.

Landau and Lifshitz say[7] that the velocity they use "is measured in terms of the proper time, as determined by clocks synchronized along the trajectory of the particle." It may seem at first sight that conclusions based on a velocity thus defined are independent of the choice of coordinates. However, the process of synchronizing clocks along the particle's trajectory involves the determination of simultaneous events at different spatial locations on the trajectory, and this determination is certainly frame-dependent. In fact, in another of the coordinate systems exhibited by Landau and Lifshitz[8] (with no mention, though, of its relevance to the present question), test particles at rest relative to the reference system are freely falling particles, and their velocity (calculated from the Landau-Lifshitz prescription) is zero everywhere, including at the Schwarzschild radius. Thus again there is no reason to conclude that a test particle reaches the speed of light at $r=2m$.

How, then, should one find a physically reasonable velocity for a test particle? It seems to me that the best, most unambiguous way of doing so is to introduce a locally Minkowskian frame at the particle's instantaneous location. (Other velocities, including those I have criticized, may nevertheless be appropriate for particular purposes.) One might expect that one would then find a speed which does, of course, depend on the particular Minkowski frame chosen, but which, in every such frame, is less than the speed of light. It will be shown in the following that this is the case.

In the coordinate system of Eq. (1), with $x^0=t$, $x^1=r$, $x^2=\theta$, and $x^3=\phi$, the tangent vector to radially inward, timelike geodesics, expressed with respect to an affine parameter, may be written as

$$t^\mu = a(1-2m/r)^{-1}\delta_0^\mu - [a^2-(1-2m/r)]^{1/2}\delta_1^\mu\,, \tag{2}$$

where a is a positive constant. [If the particle comes from infinite r with $dr/dt=-v_0$ at infinity, then $a=(1-v_0^{\,2})^{-1/2}$; if the particle starts at $r=r_0$ with $dr/dt=0$ there, then $a=(1-2m/r_0)^{1/2}$. These relations facilitate comparison of Eq. (2) with other treatments.] The Landau-Lifshitz prescription for constructing the velocity[7] leads to the expression

$$V^2 = (g_{01}{}^2 - g_{00}g_{11})\left(g_{00}+g_{01}\frac{dx^1}{dx^0}\right)^{-2}\left(\frac{dx^1}{dx^0}\right)^2 \tag{3}$$

for the square of the velocity of a radially moving particle. With the metric of Eq. (1) and the motion specified by Eq. (2), Eq. (3) yields

$$V^2 = [a^2-(1-2m/r)]/a^2\,. \tag{4}$$

It is clear from Eq. (4) that $V^2\to 1$ as $r\to 2m$.

The first suggestion that this result may be due to the choice of coordinates comes from looking at $g_{\mu\nu}t^\mu t^\nu$, which is easily seen from Eqs. (1) and (2) to have the value unity everywhere. Thus the trajectory does not appear to become lightlike in the limit $r\to 2m$. (As the trajectory can, in fact, be continued through the Schwarzschild radius, it seems clear that it must remain timelike there.) Since the coordinate system of Eq. (1) is not suitable for describing the manifold at $r=2m$, it is natural to try a coordinate system that does not have this defect. If we transform from the coordinate system (t, r, θ, ϕ) to the system (v, r, θ, ϕ), where

$$v = t + r + 2m\ln(r-2m)\,, \tag{5}$$

the line element of Eq. (1) takes the form

$$ds^2 = (1-2m/r)dv^2 - 2dv\,dr - r^2(d\theta^2+\sin^2\theta\, d\phi^2)\,. \tag{6}$$

In this coordinate system (advanced Eddington-Finkelstein coordinates[9]), there are no difficulties at $r=2m$; although the transformation (5) is valid only for $r>2m$, the line element (6) may be considered to describe the manifold for all $r>0$. The tangent vector of Eq. (2) transforms into

$$t^\mu = \{a-[a^2-(1-2m/r)]^{1/2}\}(1-2m/r)^{-1}\delta_0^\mu - [a^2-(1-2m/r)]^{1/2}\delta_1^\mu\,, \tag{7}$$

with $x^0=v$, $x^1=r$, $x^2=\theta$, and $x^3=\phi$. It may be verified from Eq. (3) that, in this coordinate system, it is still the case that $V^2\to 1$ as $r\to 2m$.[10] But consider the coordinate transformation from (v, r, θ, ϕ) to (x^0, x^1, x^2, x^3), defined by

$$\begin{aligned} x^0 &= \frac{1}{\sqrt{2}}\left[v\left(1+\frac{v}{8m}\right)-(r-2m)\left(1-\frac{v}{4m}\right) - m(\theta-\tfrac{1}{2}\pi)^2 - m\phi^2\right],\\ x^1 &= \frac{1}{\sqrt{2}}\left[v\left(1+\frac{v}{8m}\right)+(r-2m)\left(1-\frac{v}{4m}\right) - m(\theta-\tfrac{1}{2}\pi)^2 - m\phi^2\right],\\ x^2 &= r(\theta-\tfrac{1}{2}\pi)\,,\\ x^3 &= r\phi\,. \end{aligned} \tag{8}$$

Equations (8) transform the point with coordinates $v=0$, $r=2m$, $\theta=\frac{1}{2}\pi$, and $\phi=0$ into the origin of the new coordinate system, and transform the metric tensor at the origin of the new coordinate system into the form $g_{00}=-g_{11}=-g_{22}=-g_{33}=1$, with $g_{\mu\nu}=0$

for $\mu \neq \nu$, and with all first derivatives of the metric tensor vanishing at that point; i.e., in the new coordinate system the metric is locally Minkowskian at a point on the Schwarzschild surface.[11] The tangent vector of Eq. (7), when transformed by Eqs. (8) and evaluated at the origin of the new coordinate system, becomes

$$t^\mu = \frac{2a^2+1}{2\sqrt{2}\,a}\delta_0^\mu - \frac{2a^2-1}{2\sqrt{2}a}\delta_1^\mu \,. \tag{9}$$

Equation (3) for the square of the velocity now gives what one would expect from Eq. (9) with a Minkowskian metric, namely

$$V^2 = \left(\frac{2a^2-1}{2a^2+1}\right)^2, \tag{10}$$

which is less than unity for every finite positive value of a. Every other locally Minkowskian frame would, of course, also lead to a speed less than the speed of light for the test particle at this point on the Schwarzschild surface.

It thus seems to me to be clear that freely falling test particles in the Schwarzschild field cross $r = 2m$ with a speed less than the speed of light.

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[6]Reference 2, p. 94.

[7]Reference 1, p. 250.

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[9]See, for example, D. Pajerski and E. Newman, J. Math. Phys. 12, 1929 (1971).

[10]It is clear from Eq. (3) that $V^2 = 1$ whenever $g_{00} = 0$. Thus the Landau-Lifshitz prescription cannot give a proper description of a timelike trajectory in any coordinate system in which g_{00} can vanish.

[11]For the present purposes, the requirement that the first derivatives of the metric tensor vanish at the origin is actually unnecessary, and Eqs. (8) could be replaced, for example, by the simpler transformation equations

$$x^0 = [v - (r - 2m)]/\sqrt{2},$$
$$x^1 = [v + (r - 2m)]/\sqrt{2},$$
$$x^2 = 2m(\theta - \pi/2),$$
$$x^3 = 2m\phi.$$