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$$G_{\mathrm{NL}} \approx 1.2\times10^{-5}/m_p^2.$$

This value is tantalizingly close to the value of the universal Fermi constant

$$G \approx 1.0\times10^{-5}/m_p^2.$$

But it is difficult to see how one can attach much meaning to it since in the Cabibbo current-current theory the value of G_{NL} should be $G\cos\theta\sin\theta$, rather than G itself. In fact, the value of G_{NL} obtained above is rather disturbing.

[7]Equations (14a) and (14b) in the framework of the current-current theory automatically lead to the result that the vector and pseudoscalar meson pole diagrams *completely* represent the amplitudes of the parity-violating and of the parity-conserving weak decays, respectively. Hence the name "extreme vector-meson dominance" for it.

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Comment on the $\pi\pi\gamma$ Contribution to the Absorptive Amplitude for $K_L \to \mu\bar{\mu}$

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The contribution of the $\pi\pi\gamma$ intermediate state to the absorptive part of the $K_L \to \mu\bar{\mu}\ {}^1S_0$ amplitude is reevaluated by allowing for more general dependence of the $K_L \to \pi\pi\gamma$ invariant amplitude on the relevant Mandelstam variables than that considered in previous work. Allowance is also made for the corrections to the $K_L \to \pi\pi\gamma$ rate, which arise due to the dependence of the experimental detection efficiency on the nature of the photon spectrum. For a wide range of variation of the behavior of the $K_L \to \pi\pi\gamma$ amplitude, it is found that the $\pi\pi\gamma$ contribution to the rate of $K_L \to \mu\bar{\mu}$ decay (relative to that of the 2γ state) is less than 2–4%, if the pion electromagnetic form factor may be represented by $(1-s/m_\rho^2)^{-n}$, $n\leq 3$. Some pathological situations are also considered; in these cases the contribution can still be bounded by ten or fifteen percent as overestimated upper bounds.

I. INTRODUCTION

Several estimates[1-5] have appeared in the literature on the possible upper limit of the $\pi\pi\gamma$ intermediate-state contribution to the absorptive part of the $K_L \to \mu\bar{\mu}\ {}^1S_0$ amplitude relative to the 2γ intermediate-state contribution to the same. The purpose[6] of this note is to correct an error[7] of a factor of two in Ref. 2 and to improve the estimate by allowing for more general dependence[8] of the $K_L \to \pi\pi\gamma$ invariant amplitude on the relevant Mandelstam variables s, t, and u than considered previously.[2-5] We also consider corrections to the $K_L \to \pi\pi\gamma$ rate, which arise due to the dependence of the experimental detection efficiency on the nature of the photon spectrum. We of course allow for the variation of the pion electromagnetic form factor $f_\pi(s)$, which enters into the calculation. This is chosen to have the form $(1-s/m_\rho^2)^{-n}$. We find that the contribution of the $\pi\pi\gamma$ state to the *rate* of $K_L \to \mu\bar{\mu}$ decay (relative to that of the 2γ state) is less than 2–4%, if the pion electromagnetic form factor is no more singular than a triple pole at the ρ mass. The only *exception* to the above limit arises when the amplitude for $K_L \to \pi\pi\gamma$ has a zero in the physical region; in this case the contribution can still be bounded by ten or fifteen percent. In obtaining the latter bound, the correction due to nonuniformity of experimental efficiency plays a major role. These limits are more general than those previously obtained[1-5]; they reaffirm the previous conclusion regarding the unimportance of the $\pi\pi\gamma$ contribution to the $K_L \to \mu\bar{\mu}$ decay compared to that of the 2γ contribution. They agree with the estimates of Refs. 4 and 5, if we ignore the corrections involving experimental efficiency and nonlinear terms in the $K_L \to \pi\pi\gamma$ amplitude.

II. CONTRIBUTIONS OF $\pi\pi\gamma$ AND 2γ STATES

For the sake of completeness, we will define the familiar amplitudes and give the expressions for the relevant phase-space integrals, some of which are given in Refs. 1–5. The CP-conserving[9] $K_L \to \mu\bar{\mu}$ and $K_L \to \pi^+\pi^-\gamma$ amplitudes are given by

$$M(K_L \to \mu^+(p)\mu^-(p')) \equiv ieF_2\bar{u}(p')\gamma_5 v(p), \qquad (1)$$

$$M(K_L \to \pi^+(p^+)\pi^-(p^-)\gamma(k', e')) \equiv -ieG(s,t,u)\epsilon_{\alpha\beta\gamma\delta}e'_\alpha k'_\beta p^+_\gamma p^-_\delta, \qquad (2)$$

where

$$\begin{aligned} s &= (p^+ + p^-)^2, \\ t &= (p_K - p^+)^2, \\ u &= (p_K - p^-)^2, \\ s+t+u &= m_K^2 + 2m_\pi^2. \end{aligned} \qquad (3)$$

By CP invariance,

$$G(s,t,u) = G(s,u,t). \qquad (4)$$

Note that $s_{\max} = m_K^2$, $t_{\max} = u_{\max} = (m_K - m_\pi)^2$. There are two remarks which are worth noting. (i) Under the assumption that a dispersion representation for $G(s,t,u)$ is dominated by the vector mesons (ρ and K^*) and higher-mass intermediate states in the s, t, and u channels, which seems reasonable, the invariant amplitude $G(s,t,u)$, to a good approximation, may be treated as a linear function of s, t, and u. The error introduced (in the amplitude) in neglecting quadratic and higher-order terms is, in general, less than or of the order of

$$\frac{(s_{\max}/m_\rho^2)^2}{1+(s_{\max}/m_\rho^2)} \simeq 10\,\%.$$

(See also comments later.) (ii) The quadratic dependence (of G) on t and u should, in general, be less important than that on s, since $(t_{\max}/s_{\max})^2 \simeq \frac{1}{4}$.

Therefore, it appears reasonable to *neglect the quadratic dependence of G at least on t and u.* In this case, because of the ($t \leftrightarrow u$) symmetry [Eq. (4)], G can depend on t and u only through the symmetric combination $(t+u) = (m_K^2 + 2m_\pi^2) - s$. Thus, *to a very good approximation, G may be treated as a function of s alone.* From now on, we make this assumption and write

$$G(s,t,u) \equiv G_{\pi\pi\gamma}\chi(s), \quad \chi(0) = 1, \qquad (5)$$

where $G_{\pi\pi\gamma}$ is a constant. Using Eqs. (1), (2), and (5), the absorptive part of F_2 (due to the $\pi\pi\gamma$ state) and the width of $K_L \to \pi\pi\gamma$ are, respectively, given by

$$\mathrm{Abs}\,F_2^{\pi\pi\gamma} = \left(\frac{\alpha^2}{48\pi}\right)\frac{m_\mu}{(m_K^2 - 4m_\mu^2)^{1/2}}(m_K m_\rho^2 G_{\pi\pi\gamma}) \times \frac{1}{\beta}\ln\frac{1+\beta}{1-\beta} I_1, \qquad (6)$$

$$\Gamma(K_L \to \pi\pi\gamma) = \frac{\alpha\, m_K^3 m_\rho^4 G_{\pi\pi\gamma}^2}{(1536)\pi^2} I_2, \qquad (7)$$

where

$$I_1 = \int_{\sigma_{\min}}^{\sigma_{\max}} d\sigma \left(1 - \frac{\sigma}{\sigma_{\max}}\right)^2 \left(1 - \frac{\sigma_{\min}}{\sigma}\right)^{3/2} F_\pi(\sigma)\chi(\sigma), \qquad (8)$$

$$I_2 = \int_{\sigma_{\min}}^{\sigma_{\max}} d\sigma \left(1 - \frac{\sigma}{\sigma_{\max}}\right)^3 \left(1 - \frac{\sigma_{\min}}{\sigma}\right)^{3/2} \sigma\chi^2(\sigma), \qquad (9)$$

$$\beta = \left(\frac{m_K^2 - 4m_\mu^2}{m_K^2}\right)^{1/2}, \quad \sigma_{\min} = \frac{4m_\pi^2}{m_\rho^2}, \quad \sigma_{\max} = \frac{m_K^2}{m_\rho^2}. \qquad (10)$$

$F_\pi(\sigma)$ is the pion electromagnetic form factor, $F_\pi(0) = 1$, and $\sigma = s/m_\rho^2$. The 2γ intermediate-state contribution to $\mathrm{Abs}F_2$ is given by[1]

$$\mathrm{Abs}F_2^{2\gamma} = \sqrt{4\pi}\left(\frac{m_\mu}{m_K}\right)\left[\frac{\Gamma(K_L \to 2\gamma)}{m_K}\right]^{1/2}\frac{1}{\beta}\ln\frac{1+\beta}{1-\beta}. \qquad (11)$$

Thus the ratio of ($\pi\pi\gamma$) and (2γ) contributions (through absorptive parts only) to the *rate* of $K_L \to \mu\bar{\mu}$ decay is given by

$$\begin{aligned} R &\equiv |2\,\mathrm{Abs}F_2^{\pi\pi\gamma}/\mathrm{Abs}F_2^{2\gamma}| \\ &= \left[\frac{\alpha}{6\pi}\,\frac{\Gamma(K_L \to \pi\pi\gamma)}{\Gamma(K_L \to 2\gamma)}\right]^{1/2}\left[\frac{2m_K}{(m_K^2 - 4m_\mu^2)^{1/2}}\right]\left|\frac{I_1}{\sqrt{I_2}}\right|. \end{aligned} \qquad (12)$$

First let us assume (without worrying about the experimental efficiency factor) that the true upper limit on $\Gamma(K_L \to \pi\pi\gamma)/\Gamma(K_L \to \text{all})$ is 4×10^{-4}, as quoted in Refs. 10 and 11, and $\Gamma(K_L \to 2\gamma)/\Gamma(K_L \to \text{all}) = (5\pm1)\times10^{-4}$. In this case, Eq. (12) gives

$$R \leq (3.9 \pm 0.4)\times10^{-2}|I_1/\sqrt{I_2}|. \qquad (13)$$

III. EVALUATIONS OF I_1, I_2, AND R

To evaluate I_1 and I_2, we first choose a linear representation for $\chi(\sigma)$ and a multipole form for

$F_\pi(\sigma)$. We obtain later more general bounds allowing for nonlinearity of $\chi(\sigma)$. Thus, we adopt

$$\chi(\sigma)=(1+l\sigma), \tag{14}$$

$$F_\pi(\sigma)=(1-\sigma)^{-n} \quad (n=1,2,3). \tag{15}$$

We vary l over all possible values from $+\infty$ to $-\infty$. [Note that for $l\to+\infty$, $G_{\pi\pi\gamma}\to 0$, so that $G_{\pi\pi\gamma}(1+l\sigma)$ = (finite constant) $\times\sigma$ in this case.] Over the entire range of variation of l, I_1 and I_2 individually vary considerably; but the ratio $(I_1/\sqrt{I_2})$ is found to remain remarkably constant (within 10%) for a given value of n. This leads to

$$R\le\begin{cases}1.5\%, & n=1\\ 2\%, & n=2\\ 3\%, & n=3.\end{cases} \tag{16}$$

Note that the inequalities above would be replaced by near equalities, if we knew the actual value of $\Gamma(K_L\to\pi\pi\gamma)$ rather than its upper limit. For an indication of the possible range of values of the slope parameter l and also the effect of quadratic terms in $\chi(\sigma)$, it is useful to represent $G(s,t,u)$ by the (ρ, K^*) pole terms and set $m_{K^*}=m_\rho$ (for simplicity), i.e.,

$$G(s,t,u)\simeq\alpha_0 m_\rho^2(m_\rho^2-s)^{-1} + \beta m_\rho^2[(m_\rho^2-t)^{-1}+(m_\rho^2-u)^{-1}] \tag{17.a}$$

$$\begin{aligned}&=\alpha_0(1+\sigma)+\beta(\xi-\sigma)\\ &\quad+[\text{quadratic terms}+\cdots]\\ &\simeq\alpha_0(1+\gamma\xi)[1+(1-\gamma)(1+\gamma\xi)^{-1}\sigma].\end{aligned} \tag{17.b}$$

α_0 and β are the residues of s- and t-channel pole terms at $s=m_\rho^2$ and $t=m_\rho^2$, respectively; $\gamma\equiv\beta/\alpha_0$ and $\xi\equiv(m_K^2+2m_\pi^2)/m_\rho^2+2\simeq 2.5$. Comparing Eqs. (5), (14), and (17), the slope parameter $l=(1-\gamma)\times(1+\gamma\xi)^{-1}$. One may expect γ to be a number of order unity, since it is the ratio of residues of similar pole terms in s and t channels. Note that for almost all values of γ (except $-1\lesssim\gamma\lesssim 0$), l varies slowly and lies between -1 and $+1$. In the exceptional range, as γ approaches -0.4 from higher and lower values, l goes to $+\infty$ and $-\infty$, respectively. In this case (as noted before), the constant term in G vanishes due to a cancellation between s- and t-channel poles and the quadratic terms tend to be relatively more important. However, it is easy to estimate from Eq. (17a) that even in this case, the correction to the amplitude due to neglect of quadratic and higher terms is less than 20%. Furthermore, the error due to neglect of such terms is considerably reduced in any case in evaluating the ratio $I_1/\sqrt{I_2}$.

IV. MORE GENERAL BOUNDS

In order to obtain more general bounds, allowing for the presence of higher-order terms, we also made estimates as follows. The linear terms in the decay distributions were calculated exactly. The contributions of higher-order terms are then bounded by using inequalities of the type

$$\langle\sigma\rangle^2\le\langle\sigma^2\rangle\le\sigma_{\max}^2, \text{ etc.}, \tag{18}$$

where the mean value is taken with respect to the relevant phase-space function (see I_1 and I_2). If we parametrize the form factors by $\chi(\sigma)=1+l\sigma$ and $F_\pi(\sigma)=1+n\sigma$, we obtain

$$R\le\begin{cases}1.01+(0.38)n, & l\ge-3\\ 1.01+(0.47)n, & l\le-10\\ 1.0+(0.92)n, & -3\le l\le-10.\end{cases} \tag{19}$$

The region which leads to the largest bound, $-3\le l\le-10$, corresponds to a zero in the $K_L\to\pi\pi\gamma$ amplitude in the physical region. Using a linear form for $F_\pi(\sigma)$ underestimates the contribution with respect to a multipole form. The correction is between a factor 1.3 and 2 for $n=3$, 20% and 50% for $n=2$, and less than 20% for $n=1$.

To take into account the fact that quadratic term in $\chi(\sigma)$ may not be negligible if there is a zero in the physical region, we considered the form

$$\chi(\sigma)=1+\alpha'\sigma+\beta'[(1-\sigma)^{-1}-1-\sigma]. \tag{20}$$

This is equivalent to (17a) plus a possible constant (i.e., slowly varying) background, where the quadratic dependence on (t,u) has been neglected. Again using mean-value inequalities, the bounds of Eq. (19) are obtained, provided that l is interpreted as an "effective" slope

$$\begin{aligned}&l_{\text{eff}}=\alpha'+\eta\beta',\\ &0.316\le\eta\le0.75.\end{aligned} \tag{21}$$

Since the correction is dominated by the quadratic term and it is unlikely that other forms for higher-order terms would significantly increase the bounds, these results appear to be fairly general. Thus with $F_\pi(\sigma)=(1-\sigma)^{-n}$, we obtain

$$R<\begin{cases}1.8\%, & n=1\\ 3\%, & n=2\\ 5\%, & n=3\end{cases} \tag{22}$$

unless there is a zero in $\chi(\sigma)$, in which case

$$R<\begin{cases}2.4\%, & n=1\\ 4.2\%, & n=2\\ 7.4\%, & n=3.\end{cases} \tag{23}$$

We should *emphasize that the bounds (21), (22), and (23) are indeed overestimates* due to the use of the inequalities Eq. (18), whereas the estimates in Eq. (16) are exact for the corresponding parametrization. Thus Eqs. (21), (22), and (23) should be regarded as safe upper bounds.

Finally we consider the question of experimental efficiency $\epsilon(\sigma)$. Since the limit on $\Gamma(K_L \to \pi\pi\gamma)$ quoted in Ref. 10 was extracted by assuming a constant invariant amplitude $G(s,t,u)$, the true value is given by

$$\Gamma_{\text{true}} = \chi^2 \Gamma_{\text{quoted}}, \qquad \chi^2 \equiv \langle \epsilon(\sigma)\rangle \langle \chi^2(\sigma)\rangle / \langle \epsilon(\sigma)\chi^2(\sigma)\rangle, \tag{24}$$

where $\langle f(\sigma)\rangle$ denotes the weighted average of $f(\sigma)$ over the phase space for $K_L \to \pi\pi\gamma$. Then the relative contribution of the $(\pi\pi\gamma)$ state must be corrected by the factor X; $R_c \equiv XR$. We parametrized the published efficiency curve by two straight lines such that (normalization is arbitrary)

$$\epsilon(k'=20)=0, \quad \epsilon(k'=80)=4,$$
$$\epsilon(k'=170.5)=1,$$

where k' is the photon energy in MeV and $\sigma = (M_k{}^2 - 2m_K k')/m_\rho{}^2$. Using the same procedure as before, we find that X is close to unity except for $-10 \leqslant l_{\text{eff}} \leqslant -3$, where its maximum value is bounded by 2.9. Maximizing the product XR, we find

$$R_c \equiv XR < \begin{Bmatrix} 8.6\%, & F_\pi = 1 + n\sigma \\ 17.2\%, & F_\pi = (1-\sigma)^{-n} \end{Bmatrix}, \quad n < 3. \tag{25}$$

Again this is an overestimated upper bound.

To conclude, we agree with the numerical results of Refs. 4 and 5. We have improved these results by taking into account the problem of nonuniformity of experimental efficiency and the effect of quadratic terms. For reasonable form factors [F_π no more singular than a dipole, no zero in $\chi(\sigma)$ in the physical region], the contribution of the $\pi\pi\gamma$ state to the absorptive amplitude for $K_L \to \mu\bar{\mu}$ can be at most two or three percent. Allowing for a more pathological behavior an upper limit of 10–15% for the contribution of the interference term to the rate appears quite safe. This result is more general than previous work, but does not change the conclusion[1-5] that the $\pi\pi\gamma$ state cannot account for the discrepancy between the Berkeley result[12] and the theoretical bound.[13]

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[7]The calculations of Ref. 2 should be divided by a factor of 2. The remaining discrepancy between the present estimates and that of Ref. 2 is primarily because the amplitude as well as the decay distributions were approximated in Ref. 2 by linear terms. In other words, in Ref. 2 only linear terms were retained for $\chi(\sigma)$ (to evaluate I_1) and $\chi^2(\sigma)$ (to evaluate I_2) (see definitions in text); this leads to an overestimate for negative values of l [see Eq. (14)].

[8]Independently of the nature of the $K_L \to \pi\pi\gamma$ invariant amplitude, an upper limit on the $\pi\pi\gamma$ contribution can be obtained from the decay rate of $K_L \to \pi\pi\gamma$ by using the Schwarz inequality (see Ref. 1), provided the pion electromagnetic form factor is known. This, however, invariably overestimates the upper bound. Furthermore, as the experimental detection efficiency is not uniform, the upper limit on the $K_L \to \pi\pi\gamma$ rate extracted from experiment (Ref. 10) depends on the assumed form of the decay distribution. Much better bounds are obtained, therefore, by using the explicit nature of the decay distribution throughout the calculation.

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