This may be recombined to give the analytic function

$$f(\kappa)=-\frac{1}{\pi}\Gamma(-\alpha_1)\Gamma(-\alpha_2)(-\kappa)^{-\alpha_1}\times U(-\alpha_1,\,-\alpha_1+\alpha_2+1,\,-\kappa)\,. \qquad (5.6)$$

The same result is obtained by performing the Cauchy integral,[12] and is unique up to an arbitrary polynomial in $\kappa$.

Including (5.6) with the remainder of (4.11) gives the analytic extension in $M^2$ or $\kappa$. The signature structure of the Regge exchanges $\alpha_1$ and $\alpha_2$ can be found by adding diagrams like Fig. 2 but with exchanges $(a \leftrightarrow \bar{a}')$, $(b \leftrightarrow \bar{b}')$, and $(a \leftrightarrow \bar{a}', b \leftrightarrow \bar{b}')$. Care must be taken to establish that the diagram of Fig. 2 and the exchanged diagrams from $(a \leftrightarrow \bar{a}')$ and $(b \leftrightarrow \bar{b}')$ are evaluated above the $\kappa$ cut, but the diagram from the exchange $(a \leftrightarrow \bar{a}', b \leftrightarrow \bar{b}')$ is evaluated below the $\kappa$ cut. This has been carried out in Ref. 3.

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# A Statistical Theory of Particle Production*

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A statistical theory of particle production is formulated in analogy with the generalized Ginzburg-Landau theory of phase transitions in superconductors and intensity fluctuations in lasers.

## I. INTRODUCTION

At present accelerator energies the number of secondaries produced in hadron-hadron collisions is large enough so that one can consider statistical theories of particle production. In this paper we shall present such a theory.[1]

Our basic approach is as follows: We do not attempt to treat in detail the fundamental dynamics underlying particle production. Instead, we imagine integrating out the microscopic degrees of freedom and representing them in terms of a small number of phenomenological parameters. We are then left with a statistical theory of the macroscopic observables.

A well-known example of this approach is the Ginzburg-Landau theory of superconductivity.[2] In its generalized form[3] it provides a statistical theory of the superconducting order parameter which depends upon a small set of experimentally determined parameters. Once a microscopic theory of superconductivity was formulated by BCS,[4] it was possible to derive the Ginzburg-Landau theory from it and to obtain expressions for the phenomenological parameters in terms of the microscopic properties of the electron-phonon system.[5] Another example of this type of theory is the recent treatment of the correlations in a laser field near threshold.[6] In this case, after the atomic degrees of freedom are integrated out, one obtains a statistical theory for the macroscopic $E$ field. It should be emphasized that the laser is far from thermal equilibrium; nevertheless, this problem can be formulated in essentially the same manner as the generalized Ginzburg-Landau theory. This is important since we do not envision the hadronic matter reaching thermal equilibrium during a high-energy collision. The generalized Ginzburg-Landau formalism provides a particularly simple parametrization of the statistical properties of the superconducting pair field near the critical point and the laser field near threshold. In this paper we seek to develop this type of phenomenological theory for multiparticle production in high-energy hadron-hadron collisions.

In the particle production problem the relevant variable is the probability amplitude, $\Pi(y)$, for the production of a secondary with rapidity $y$.[7] $\Pi(y)$ will be treated as a random variable. We again emphasize that the hadronic matter is not pictured as reaching thermal equilibrium during the collision process. Nevertheless, it is possible to introduce a functional of $\Pi$, $F[\Pi]$, which plays a role analogous to the free energy in equilibrium statistical mechanics. Namely, all the quantities of physical interest such as the inclusive and exclusive cross sections can be calculated by taking ensemble average suitably weighted by $F[\Pi]$. In principle $F[\Pi]$ could be calculated from a knowledge of the underlying dynamics. Instead, following Ginzburg and Landau, we express $F[\Pi]$ in terms of a small number of phenomenological parameters.

Our approach makes contact with more familiar models of particle production at two points. First, several authors have pointed out the formal analogy between the problem of particle production and the statistical mechanics of a one-dimensional fluid.[8-10] The techniques employed here have been used to treat the statistical mechanics of such one-dimensional systems as superconducting wires[3,11] and lasers operating near threshold.[6] Second, the types of interactions considered here lead only to short-range intensity correlations among the produced particles. As a result, it is hardly surprising that the inclusive cross sections have the same form as is obtained in the Mueller-Regge formalism[12] when Regge poles, but not Regge cuts, are included. In the present model there are always an infinite number of Regge poles, but the positions of the poles and their coupling constants are given in terms of a small number of input parameters.

The outline of the rest of the paper is as follows: In Sec. II a general formulation of the theory is given, and techniques are developed for calculating the inclusive and exclusive cross sections and the factorial moments of the multiplicity distribution. In Sec. III two limiting cases are considered which can be solved analytically. In Sec. IV the exact solution of the model is discussed. Finally, in Sec. V, generalizations of the theory are considered as well as possible physical implications of our results.

## II. FORMULATION OF THE THEORY

Let us focus our attention on a single type of secondary that is produced in a high-energy hadron-hadron collision. We shall refer to this particle as a pion although we shall ignore quantum numbers. The probability amplitude for the production of a pion with rapidity $y$ is denoted by $\Pi(y)$.[7] In the laboratory system $y$ has the approximate range $0 \leqslant y \leqslant Y$, where $Y$ is the rapidity of the incident hadron. In our work $\Pi(y)$ will be treated as a random variable.

In order to proceed it is necessary to know how to weight the various configurations of the random field variable $\Pi(y)$. We therefore introduce a functional of $\Pi$, $F[\Pi]$, which plays a role analogous to the free energy for a system in thermal equilibrium. All quantities of interest can be calculated in terms of ensemble averages appropriately weighted by $F[\Pi]$. For example, the inclusive cross section for the production of a single pion with rapidity, $y$, $d\sigma/dy$, is given by

$$\frac{1}{\sigma}\frac{d\sigma}{dy}=\langle\Pi^2(y)\rangle$$
$$=\frac{1}{Z}\int\delta\Pi\, e^{-F[\Pi]}\,\Pi^2(y)\,. \tag{1}$$

$\sigma$ is the total cross section, and $\int\delta\Pi$ indicates a functional integration over all possible forms for the function $\Pi(y)$. The normalization factor $Z$ is given by

$$Z=\int\delta\Pi\, e^{-F[\Pi]}\,. \tag{2}$$

More generally

$$\frac{1}{\sigma}\frac{d\sigma}{dy_1 dy_2\cdots dy_n}\equiv\rho(y_1,y_2,\ldots,y_n)$$
$$=\langle\Pi^2(y_1)\Pi^2(y_2)\cdots\Pi^2(y_n)\rangle$$
$$=\frac{1}{Z}\int\delta\Pi\, e^{-F[\Pi]}$$
$$\times\Pi^2(y_1)\Pi^2(y_2)\cdots\Pi^2(y_n)\,. \tag{3}$$

The factorial moments of the multiplicity distribution are given by

$$S_q \equiv \langle N(N-1)\cdots(N-q+1)\rangle_y = \frac{1}{\sigma}\int dy_1\cdots dy_q \frac{d\sigma}{dy_1\cdots dy_q}. \tag{4}$$

Following Mueller[9] we introduce a generating function $\Omega(z)$.

$$\Omega(z)=\sum_{q=0}^{\infty}\frac{(z-1)^q}{q!}S_q = \sum_{n=0}^{\infty} z^n P_n, \tag{5}$$

where $P_n=\sigma_n/\sigma$, and $\sigma_n$ is the semi-inclusive cross section for the production of $n$ pions plus any number of other secondaries. In a world containing only pions, $\sigma_n$ would of course be the exclusive cross section. Making use of Eqs. (4) and (5) one sees that

$$\Omega(z)=\left\langle \exp\left[(z-1)\int_0^Y dy\, \Pi^2(y)\right]\right\rangle. \tag{6}$$

So far we have done nothing more than state the consequences of treating $\Pi(y)$ as a random variable. The real physics enters the problem when we specify the form of $F[\Pi]$. Ideally one would like to derive $F[\Pi]$ from a knowledge of the underlying dynamics. Lacking such knowledge we follow Ginzburg and Landau[2] in retaining the first few terms in a series expansion of $F[\Pi]$ in powers of $\Pi$ and its derivatives.

$$F[\Pi]=\int_0^Y dy\left[a\Pi^2(y)+b\Pi^4(y)+c\left(\frac{\partial \Pi}{\partial y}\right)^2\right]. \tag{7}$$

A constant term in $F[\Pi]$ would not have any physical effect, and terms linear in $\Pi$ are forbidden by symmetry considerations. One could easily extend our technique to more general forms for $F[\Pi]$, but in the present work we shall restrict ourselves to the form given in Eq. (7). The chief justification for the use of the form for $F$ given in Eq. (7) is its flexibility. By choosing different values of the parameters $a$, $b$, and $c$, one is able to sweep over a wide class of random fields, ranging from Gaussian random fields ($b=0$) to highly coherent fields ($a<0,\ b>0,\ c\gg 1$).

The next step is to evaluate the functional integrals. In the original Ginzburg-Landau theory[2] no functional integrals were performed. Instead $\ln Z$ was taken as proportional to $F[\Pi_0]$, where $\Pi_0$ was determined from the Ginzburg-Landau equation $\delta F/\delta\Pi_0=0$. This corresponds to taking a saddle-point integration in $\Pi$. By the generalized Ginzburg-Landau theory,[3] we mean that a functional integration over all forms for the field $\Pi(y)$ is to be taken. For a field which depends on only one variable, the method for carrying this out is well known.[13] We review it here.

It is convenient to break up the region $0\leq y\leq Y$ into $N$ equal intervals of length $\Delta y$ so that $\Delta y N=Y$. Then

$$F[\Pi]=\lim_{\Delta y\to 0}\sum_{l=1}^{N}\Delta y\left[a\Pi_l^2+b\Pi_l^4+c\left(\frac{\Pi_l-\Pi_{l-1}}{\Delta y}\right)^2\right], \tag{8}$$

with

$$\Pi_l=\Pi(l\Delta y),\quad l=0,1,2,\ldots,N. \tag{9}$$

Since functional integration means a sum over all possible forms for the function $\Pi(y)$, we can write

$$Z(a)=\int \delta\Pi\, e^{-F[\Pi]} = \lim_{\Delta y\to 0}(c/\Delta y\pi)^{N/2}\int_{-\infty}^{\infty}\prod_{l=0}^{N} d\Pi_l \exp\left\{-\Delta y\sum_{l=1}^{N}\left[a\Pi_l^2+b\Pi_l^4+c\left(\frac{\Pi_l-\Pi_{l-1}}{\Delta y}\right)^2\right]\right\}. \tag{10}$$

The factor $(c/\Delta y\pi)^{N/2}$ defines the measure of the functional integration. Notice that in allowing $\Pi_0$ and $\Pi_N$ to take on all values with equal weight we are making a definite choice of boundary conditions. It seems most natural to allow $\Pi(y)$ to be free on the boundaries; that is certainly what one would do in the case of the analogous Feynman fluid. In any case, other choices of boundary conditions will not change the qualitative features of our models.

Equation (10) can be rewritten in the form

$$Z(a)=\lim_{\Delta y\to 0}(c/\Delta y\pi)^{N/2}\sum_{m,n}\int_{-\infty}^{\infty} d\bar{\Pi}_0 d\bar{\Pi}_N\prod_{l=0}^{N} d\Pi_l \psi_m(\Pi_0)\psi_m(\bar{\Pi}_0)\psi_n(\Pi_N)\psi_n(\bar{\Pi}_N) \times\exp\left\{-\Delta y\sum_{l=1}^{N}\left[a\Pi_l^2+b\Pi_l^4+c\left(\frac{\Pi_l-\Pi_{l-1}}{\Delta y}\right)^2\right]\right\}, \tag{11}$$

where the $\psi_m$ are a complete set of real functions normalized so that

$$\sum_m \psi_m(\Pi_0)\psi_m(\overline{\Pi}_0) = \delta(\Pi_0 - \overline{\Pi}_0). \tag{12}$$

Let us first consider the $\Pi_0$ integration. Working to leading order in $\Delta y$ we find

$$(c/\Delta y\pi)^{1/2}\int_{-\infty}^{\infty} d\Pi_0 \exp[-c(\Pi_1 - \Pi_0)^2/\Delta y]\,\psi_m(\Pi_0) \simeq \left(1 + \frac{\Delta y}{4c}\frac{\partial^2}{\partial \Pi_1{}^2}\right)\psi_m(\Pi_1). \tag{13}$$

We now choose the $\psi_m(\Pi)$ to be eigenstates of the Hamiltonian

$$H(\Pi) \equiv -\frac{1}{4c}\frac{\partial^2}{\partial \Pi^2} + a\Pi^2 + b\Pi^4, \tag{14}$$

such that

$$H\psi_m = \mathcal{E}_m\psi_m. \tag{15}$$

Since we need only work to leading order in $\Delta y$, we can write

$$\exp[-\Delta y(a\Pi_1{}^2 + b\Pi_1{}^4)] \simeq 1 - \Delta y(a\Pi_1{}^2 + b\Pi_1{}^4).$$

The $\Pi_1$ dependence of the integrand then becomes

$$\exp[-c(\Pi_1 - \Pi_2)^2/\Delta y][1 - \Delta y H(\Pi_1)]\,\psi_m(\Pi_1) \simeq \exp(-\Delta y\mathcal{E}_m)\exp[-c(\Pi_1 - \Pi_2)^2/\Delta y]\psi_m(\Pi_1). \tag{16}$$

Now the $\Pi_1$ integration can be performed in the same way as the $\Pi_0$. One finally obtains

$$\begin{aligned} Z(a) &= \sum_{m,n}\int_{-\infty}^{\infty} d\overline{\Pi}_0 d\overline{\Pi}_N d\Pi_N e^{-\mathcal{E}_m Y}\psi_m(\overline{\Pi}_0)\psi_m(\Pi_N)\psi_n(\Pi_N)\psi_n(\overline{\Pi}_N) \\ &= \sum_m e^{-\mathcal{E}_m Y} G_m{}^2, \end{aligned} \tag{17}$$

with

$$G_m \equiv \int_{-\infty}^{\infty} d\Pi\,\psi_m(\Pi). \tag{18}$$

The other integrals of interest can be evaluated by the same technique. For example, writing $y = N'\Delta y$,

$$\begin{aligned} \int \delta\Pi\, e^{-F[\Pi]}\Pi^2(y) &= \lim_{\Delta y\to 0}\sum_{n,m} G_n G_m (c/\Delta y\pi)^{N/2}\int_{-\infty}^{\infty}\prod_{l=0}^{N} d\Pi_l e^{-F[\Pi]}\psi_n(\Pi_N)\psi_m(\Pi_0)\Pi_{N'}{}^2 \\ &= \lim_{\Delta y\to 0}\sum_{l,m,n} G_n G_m (c/\Delta y\pi)^{(N-N')/2}\int_{-\infty}^{\infty}\prod_{l=N'}^{N} d\Pi_l\psi_n(\Pi_N) \\ &\qquad \times\exp\left\{-\Delta y\sum_{l=N'+1}^{N}\left[a\Pi_l{}^2 + b\Pi_l{}^4 + c\left(\frac{\Pi_l - \Pi_{l-1}}{\Delta y}\right)^2\right]\right\} \\ &\qquad \times\psi_l(\Pi_{N'})e^{-\mathcal{E}_m y}\int_{-\infty}^{\infty} d\Pi\,\psi_l(\Pi)\Pi^2\psi_m(\Pi) \\ &= \sum_{n,m} G_n e^{-\mathcal{E}_n(Y-y)} g_{n,m} e^{-\mathcal{E}_m y} G_m, \end{aligned} \tag{19}$$

where

$$g_{n,m} \equiv \int_{-\infty}^{\infty} d\Pi\,\psi_n(\Pi)\Pi^2\psi_m(\Pi). \tag{20}$$

Referring to Eq. (2) we see that the inclusive cross section for the production of $n$ pions with rapidities $0 \leqslant y_1 \leqslant y_2 \leqslant \cdots \leqslant y_n \leqslant Y$ is given by

$$\begin{aligned} \rho(y_1, y_2, \ldots, y_n) = Z(a)^{-1} &\sum_{m_1,\ldots,m_{n+1}} G_{m_{n+1}}\exp[-(Y - y_n)\mathcal{E}_{m_{n+1}}] \\ &\times g_{m_{n+1},m_n}\cdots\exp[-(y_2 - y_1)\mathcal{E}_{m_2}]\,g_{m_2,m_1}\exp(-y_1\mathcal{E}_{m_1})G_{m_1}. \end{aligned} \tag{21}$$

This is precisely the form one would obtain in the Mueller-Regge formalism when there are Regge poles, but no cuts.[12] In the present case there are an infinite number of Regge poles, one associated with each energy level of the Hamiltonian defined in Eq. (14). $g_{n,m}$ gives the coupling of the $n$ and $m$ trajectories to two pions, and $G_m$ the coupling of the $m$ trajectory to the external particles.

In our framework it is only possible to calculate the ratio of a given cross section to the total cross section. Nevertheless, it is clear that $\sigma$ is proportional to $Z(a)$. We are free to adjust the leading Regge trajectory to have any intercept. If, for example, we wish to have $\sigma$ approach a finite limit, $\sigma_\infty$, at high energies, then we can write

$$\sigma = \sigma_\infty e^{\mathcal{E}_0 Y} G_0^{-2} Z(a) = \sigma_\infty \sum_{m=0}^{\infty} \exp[-(\mathcal{E}_m - \mathcal{E}_0)Y](G_m/G_0)^2 . \tag{22}$$

In this case the intercept of the $m$th trajectory is given by

$$\alpha_m = 1 - (\mathcal{E}_m - \mathcal{E}_0) . \tag{23}$$

From Eq. (6) we see that the generating function is given by

$$\Omega(z) = Z(1+a-z)/Z(a) . \tag{24}$$

The $P_N$ and $S_q$ can now be obtained by taking appropriate derivatives of $\Omega(z)$. In most cases one is interested in calculating $P_N$ and $S_q$ to leading power in $s \sim e^Y$. In that case it is sufficient to only retain the contribution to $\Omega(z)$ associated with the lowest-energy level

$$\Omega(z) \simeq \exp\{-Y[\mathcal{E}_0(a+1-z) - \mathcal{E}_0(a)]\} \times G_0^2(a+1-z)/G_0^2(a) . \tag{25}$$

The generating function given in Eq. (25) has the form of a compound Poisson distribution. It can be interpreted in terms of a microscopic theory in which pions are produced in clusters. All the pions in a cluster have small rapidity differences and tend to be correlated, while the clusters themselves are well separated in rapidity and are uncorrelated. The probability for a cluster to decay into a given number of pions is determined by the functional form of $\mathcal{E}_0(a)$.[14] Each of the nonleading terms in the expansion of $\Omega(z)$ in powers of $s$ can be given a similar interpretation.

## III. EXAMPLES

In order to illustrate our formalism, we begin by considering two limiting cases. The first of these, appropriate when $a \gg b\langle\Pi^2\rangle$, is analogous to the case of a superconductor well above $T_c$ or a laser which is being pumped well below its operation threshold. In this case, the quartic term is unimportant, and the functional is quadratic in $\Pi$ leading to simple Gaussian averages. In our second example, we consider the mean field limit where $a<0$ and the gradient term is small. This corresponds to a superconductor which is at a temperature well below $T_c$ or a laser which is pumped well above its operation threshold. For one-dimensional systems there is, of course, no sharp phase transition. The order becomes very long range but not infinite. We will return to a discussion of the nature of this long-range order of the pion field in the conclusion, Sec. V. In this section we also obtain estimates of the phenomenological parameters $a$, $b$, and $c$. We will find that the results favor the mean field limit but are sufficiently close to the critical or threshold region that an exact solution is necessary. This is carried out in the next section, IV.

*Case 1:* $b=0,\ a, c>0$

In this case the Hamiltonian defined in Eq. (14) is just that of the simple harmonic oscillator, so all quantities of interest can be obtained in closed form. For example,

$$\begin{aligned} &\mathcal{E}_m = (a/c)^{1/2}(m+\tfrac{1}{2}) ,\\ &\psi_m(\Pi) = \left[\frac{(4ac)^{1/4}}{2^m m!\sqrt{\pi}}\right]^{1/2} H_m((4ac)^{1/4}\Pi) \times \exp[-(ac)^{1/2}\Pi^2] ,\\ &m = 0,1,2,\ldots , \end{aligned} \tag{26}$$

where $H_m$ is the Hermite polynomial of order $m$. Referring to Eqs. (18) and (20) one sees that

$$\begin{aligned} &G_{2m} = [(4/ac)^{1/4}\Gamma(m+\tfrac{1}{2})/m!]^{1/2} ,\\ &G_{2m+1} = 0 ,\\ &g_{n,m} = \frac{1}{4(ac)^{1/2}}\{\delta_{n,m-2}[m(m-1)]^{1/2} + \delta_{n,m+2}[(m+1)(m+2)]^{1/2} + \delta_{n,m}(2m+1)\} . \end{aligned} \tag{27}$$

Notice that the only energy levels that contribute to the inclusive cross sections are those for which $\psi_m(\Pi)$ is an even function of $\Pi$. This is a general result which depends only on our choice of boundary conditions for $\Pi(y)$.

From Eqs. (17) and (27) one sees that

$$Z(a) = \left(\frac{4}{ac}\right)^{1/4} e^{-\kappa Y/2} \times \sum_{m=0}^{\infty} e^{-2m\kappa Y}\Gamma(m+\tfrac{1}{2})/m! = \left(\frac{\pi}{\sinh \kappa Y}\right)^{1/2}(ac)^{-1/4} , \tag{28}$$

with $\kappa \equiv (a/c)^{1/2}$. As a result, the generating function is given by

$$\Omega(z) = \left[\frac{\sinh(\kappa Y)}{\sinh[\kappa Y(1+(1-z)/a)^{1/2}]}\right]^{1/2} \times [1+(1-z)/a]^{-1/4} \simeq \exp\{-\tfrac{1}{2}\kappa Y[(1+(1-z)/a)^{1/2}-1]\} \times [1+(1-z)/a]^{-1/4}. \quad (29)$$

The multiplicity moments $f_n$ are defined by

$$\Omega(z) = \exp\left[\sum_{n=1}^{\infty} \frac{f_n(z-1)^n}{n!}\right], \quad (30)$$

so working to leading power in $s \sim e^Y$ we have

$$f_n = \tfrac{1}{4}\kappa Y a^{-n}\frac{\Gamma(n-\frac{1}{2})}{\Gamma(\frac{1}{2})} + \tfrac{1}{4}a^{-n}(n-1)!\ . \quad (31)$$

In particular

$$f_1 = \langle N\rangle_y = (\kappa Y+1)/4a, \quad f_2 = \langle N(N-1)\rangle_y - (\langle N\rangle_y)^2 = \kappa Y/8a^2 + 1/4a^2. \quad (32)$$

In order to calculate the partial cross sections $P_N$, it is convenient to introduce an auxiliary generating function

$$\overline{\Omega}(z) \equiv \exp\left\{-\tfrac{1}{2}\kappa Y\left[\left(1+\frac{1-z}{a}\right)^{1/2}-1\right]\right\}. \quad (33)$$

Then for $N \geq 1$

$$\overline{P}_N \equiv \frac{1}{N!}\frac{d^N}{dz^N}\overline{\Omega}(z)\Big|_{z=0} = \frac{\exp\{-\frac{1}{2}\kappa Y[(1+1/a)^{1/2}-1]\}}{2^{2N}N!\,(1+a)^{N-1/2}}\frac{\kappa Y}{a^{1/2}}\sum_{k=0}^{N-1}\frac{(2N-k-2)!\,[\kappa Y(1+1/a)^{1/2}]^k}{k!\,(N-k-1)!} = \frac{e^{\kappa Y/2}(1+a)^{1/2}}{2^{2N}\pi^{1/2}N!}\left[\frac{\kappa Y}{[a(1+a)]^{1/2}}\right]^{N+1/2} K_{N-1/2}(\tfrac{1}{2}\kappa Y(1+1/a)^{1/2}), \quad (34)$$

and

$$\overline{P}_0 = \exp\{-\tfrac{1}{2}\kappa Y[(1+1/a)^{1/2}-1]\}. \quad (35)$$

Here $K_{N-1/2}$ is the modified Bessel function of order $N-\frac{1}{2}$. The partial cross sections are given to leading power in $s$ by

$$P_N = \left(1+\frac{1}{a}\right)^{-1/4}\sum_{l=0}^{N}(1+a)^{-l}\overline{P}_{N-l}\frac{\Gamma(l+\frac{1}{4})}{l!\,\Gamma(\frac{1}{4})}. \quad (36)$$

Writing out the first few terms

$$P_0 = \exp\{-\tfrac{1}{2}\kappa Y[(1+1/a)^{1/2}-1]\}(1+1/a)^{-1/4}, \quad P_1 = P_0\{\tfrac{1}{4}\kappa Y[a(1+a)]^{-1/2} + \tfrac{1}{4}(1+a)^{-1}\}, \quad P_2 = P_0\{\tfrac{1}{16}(\kappa Y)^2[a(1+a)]^{-1} + \tfrac{3}{16}\kappa Y a^{-1/2}(1+a)^{-3/2} + \tfrac{5}{32}(1+a)^{-1}\}. \quad (37)$$

For $N$ and $Y$ both large it is convenient to introduce the variable $x = \frac{1}{2}\kappa Y(1+1/a)^{1/2}$. Then making use of the asymptotic form for the $K$ function for large index and argument one finds that for $N \lesssim x$

$$P_N \simeq \overline{P}_N \simeq \left[\frac{N+(N^2+x^2)^{1/2}}{2(1+a)}\right]^N \frac{1}{N!}x\{(N^2+x^2)^{1/2}[N+(N^2+x^2)^{1/2}]\}^{-1/2}\exp[x(1+1/a)^{-1/2}]\exp[-(N^2+x^2)^{1/2}] \simeq \{2\pi x\gamma(1+\gamma^2)^{1/2}[\gamma+(1+\gamma^2)^{1/2}]\}^{-1/2}\exp\left[-\frac{(N-\gamma x)^2}{2x}\frac{(1+\gamma^2)^{1/2}-\gamma}{\gamma(1+\gamma^2)^{1/2}}\right], \quad (38)$$

where $\gamma \equiv \frac{1}{2}[a(1+a)]^{-1/2}$. On the other hand, for $N \gg \gamma x$ one finds

$$P_N \simeq \frac{\Gamma(l_0+\frac{1}{4})(1+1/a)^{-1/4}}{\Gamma(l_0+1)\Gamma(\frac{1}{4})}\left[\frac{4\pi\gamma(1+\gamma^2)^{1/2}}{(1+\gamma^2)^{1/2}-\gamma}\right]^{1/2}\exp[-(N-\gamma x)\ln(1+a)], \quad (39)$$

with

$$l_0 = N - \gamma x - \frac{2x\gamma(1+\gamma^2)^{1/2}}{(1+\gamma^2)^{1/2}-\gamma}\ln(1+a). \quad (40)$$

In order to estimate the parameters $a$ and $c$ we take the correlation length $\xi_2 = 2$, so that

$$\xi_2^{-1} = \mathcal{E}_2 - \mathcal{E}_0 = 2(a/c)^{1/2} = \tfrac{1}{2}. \quad (41)$$

For negatively charged particles the coefficient of $Y$ in the average multiplicity is approximately 1.15.[14,15] Using this value gives

$$a \simeq 0.055, \quad c \simeq 0.87. \quad (42)$$

This low a value for $a$ indicates that the $b\Pi^4$ term in $F[\Pi]$ ought not be neglected.

*Case 2:* $a<0,\ b,c>0$

For $a<0$ and $b>0$ the potential

$$V(\Pi)=a\Pi^2+b\Pi^4 \tag{43}$$

has a minimum at

$$\Pi_\pm=\pm(-a/2b)^{1/2}. \tag{44}$$

In order to orient our thinking, let us first expand the potential about the minima. Retaining only quadratic terms we have

$$V(\Pi)\simeq -a^2/4b+2a(\Delta\Pi_+)^2\theta(\Pi)+2a(\Delta\Pi_-)^2\theta(-\Pi), \tag{45}$$

where $\Delta\Pi_\pm=\Pi-\Pi_\pm$ and $\theta(x)=1$ for $x>0$ and vanishes for $x<0$. We shall see below the data indicates that $c>>b,|a|$, so there is little tunneling between the two minima. As a result, the energy levels whose wave functions are even in $\Pi$ are to lowest order

$$\mathcal{E}_{2n}\simeq -a^2/4b+(n+\tfrac{1}{2})(-2a/c)^{1/2},\quad n=0,1,2,\ldots \tag{46}$$

and the ground-state wave function is given by

$$\Psi_0\simeq 2^{-1/2}[\psi_0(\Delta\Pi_+)+\psi_0(\Delta\Pi_-)]. \tag{47}$$

Here $\psi_0$ is given by Eq. (26) with $a$ replaced by $(-2a)$. Again working to leading order in $s$ we find

$$\Omega(z)\simeq\left(1+\frac{1-z}{a}\right)^{-1/4}\times\exp\left(Y\left\{-\frac{a}{2b}(z-1)+\frac{(z-1)^2}{4b}-\left(-\frac{a}{2c}\right)^{1/2}\left[\left(1+\frac{1-z}{a}\right)^{1/2}-1\right]\right\}\right). \tag{48}$$

The moments of the multiplicity distribution are

$$\begin{aligned}
f_1&=Y\left[-\frac{a}{2b}+\frac{(-a/2c)^{1/2}}{2a}\right]+\frac{1}{4a},\\
f_2&=Y\left[\frac{1}{2b}+\frac{(-a/2c)^{1/2}}{4a^2}\right]+\frac{1}{4a^2},\\
f_n&=\tfrac{1}{2}Y\left(-\frac{a}{2c}\right)^{1/2}a^{-n}\frac{\Gamma(n-\frac{1}{2})}{\Gamma(\frac{1}{2})}+\tfrac{1}{4}a^{-n}(n-1)!\,,\\
n&=2,3,4,\ldots\,.
\end{aligned} \tag{49}$$

In order to estimate the parameters $a$, $b$, and $c$ we once more take the correlation length $\xi_2=(\mathcal{E}_2-\mathcal{E}_0)^{-1}=2$. For negatively charged particles, the coefficients of $Y$ in $f_1$ and $f_2$ are approximately 1.15 and 0.69, respectively.[15] Using these numbers we find

$$\begin{aligned}
a&\simeq-1.87,\\
b&\simeq0.77,\\
c&\simeq15.0.
\end{aligned} \tag{50}$$

For these parameters, it follows that a reasonable approximation for the generating function $\Omega(z)$ can be obtained by neglecting the zero-point contribution $-\frac{1}{2}\sqrt{-2a}/c$. The generating function can then be simply expressed as

$$\Omega(z)=\frac{\exp[\langle N\rangle_y(z-1)+\frac{1}{2}f_2(z-1)^2]}{[1+(1-z)/a]^{1/4}}. \tag{51}$$

This numerator is just the generating function derived by Mueller for a two trajectory model and leads to a set of $N$ pion cross sections $\overline{P}_N=\overline{\sigma}_N/\overline{\sigma}$ which are Hermite polynomials of imaginary argument. As previously noted, Eq. (36), the denominator of Eq. (51) mixes these to give the final $P_N$.

Although the parameters given by Eq. (50) suggest that this mean field limit is a useful approximation, it is important to remember that the partial cross sections are obtained by expanding $\Omega(z)$ about $z=0$. This means that the effective value for $a$ which enters in determining $\Omega$ is $a+1$. Thus we are again pushed towards the critical region which lies between the two limiting cases we have discussed. We therefore consider next the form of the exact solution.

## IV. NUMERICAL SOLUTION

An exact solution of our statistical theory of pion production can be obtained from the eigenenergies and eigenstates of the anharmonic-oscillator Hamiltonian, Eq. (14). There exist a variety of methods for obtaining such solutions. Here we have used the results obtained from a truncated matrix representation. $H$ is represented in terms of a basis of $n$ harmonic-oscillator states with the scale length of the basis states set by minimizing the harmonic-oscillator–ground-state expectation value of $H$ for $a=0$. The matrix is then diagonalized numerically, and the resultant eigenvalues and eigenvectors determined. The number of basis states $n$ was varied to obtain numerical convergence.

In order to deal with three parameters, we rescale the problem by setting

$$x=(8bc)^{1/6}\Pi. \tag{52}$$

Then the Hamiltonian becomes

$$H=\frac{b^{1/3}}{c^{2/3}}\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2}+\tfrac{1}{4}x^4+\alpha x^2\right) \tag{53}$$

with

$$\alpha = \frac{1}{2}\frac{ac^{1/3}}{b^{2/3}}. \tag{54}$$

The energy eigenvalues and eigenstates of the reduced Hamiltonian

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \tfrac{1}{4}x^4 + \alpha x^2\right)\phi_n(x) = E_n\phi_n, \tag{55}$$

depend only on one parameter, $\alpha$. From $E_n(\alpha)$ we can obtain the desired eigenenergies $\mathcal{E}_n(a, b, c)$.

$$\mathcal{E}_n(a, b, c) = \frac{b^{1/3}}{c^{2/3}} E_n\left(\frac{1}{2}\frac{ac^{1/3}}{b^{2/3}}\right). \tag{56}$$

The parameters $a$, $b$, and $c$ are now obtained by fitting the data. The correlation length, $\xi_2$, which appears in the inclusive cross sections is given by

$$\frac{1}{\xi_2} = \frac{b^{1/3}}{c^{2/3}}[E_2(\alpha) - E_0(\alpha)]. \tag{57}$$

Next, notice that to leading order in $s$, the $f_n$ coefficients have the form

$$f_n = a_n Y + b_n. \tag{58}$$

The $a_n$ coefficients can be related to derivatives of $\mathcal{E}_0$. From Eqs. (25) and (30) one sees that

$$\mathcal{E}_0(a + 1 - z) - \mathcal{E}_0(a) = \sum_{n=1}^{\infty} \frac{(z-1)^n}{n!} a_n, \tag{59}$$

so

$$a_n = (-)^{n+1}\frac{d^n}{da^n}\mathcal{E}_0(a). \tag{60}$$

Finally, using the scaled form of $E_0$, Eq. (56), the $a_n$ can be related to the $\alpha$ derivatives of the reduced ground-state eigenvalue $E_0(\alpha)$.

$$a_n = (-1)^{n+1}\frac{b^{1/3}}{c^{2/3}}\left(\frac{1}{2}\frac{c^{1/3}}{b^{2/3}}\right)^n \frac{d^n}{d\alpha^n}E_0(\alpha). \tag{61}$$

The first two coefficients are

$$a_1 = \frac{1}{2(bc)^{1/3}} E_0'(\alpha), \tag{62}$$

$$a_2 = -\frac{1}{4b} E_0''(\alpha). \tag{63}$$

Equations (62) and (63) combined with the correlation length condition, Eq. (57), can be used to determine the parameters $a$, $b$, and $c$ in terms of the experimental values of $\xi_2$, $a_1$, and $a_2$. Multiplying Eq. (57) times Eq. (62) gives

$$c = \frac{\xi_2}{2a_1} E_0'(\alpha)[E_2(\alpha) - E_0(\alpha)] \tag{64}$$

and solving Eq. (63) for $b$ gives

$$b = -\frac{E_0''(\alpha)}{4a_2}. \tag{65}$$

Combining these to obtain an expression for $(bc)^{1/3}$ and substituting this into Eq. (62), we obtain an implicit equation relating $\alpha$ to $\xi_2$, $a_1$ and $a_2$.

$$\frac{a_1}{(a_1a_2/\xi_2)^{1/3}} = \frac{E_0'(\alpha)}{\{-E_0''(\alpha)E_0'(\alpha)[E_2(\alpha) - E_0(\alpha)]\}^{1/3}}. \tag{66}$$

Once $\alpha$ is determined, $b$ and $c$ are obtained from Eqs. (65) and (66). Then $a$ can be found from $\alpha$, $b$, and $c$ using Eq. (54).

The reduced ground-state eigenvalue $E_0(\alpha)$ and the energy difference $E_2(\alpha) - E_0(\alpha)$ which sets the inverse correlation length are plotted in Figs. 1 and 2, respectively. The dashed curve for $\alpha > 0$ shows the $b = 0$ behavior calculated as Case 1 in Sec. III, and the dashed line for $\alpha < 0$ is the mean field result, Case 2, including the zero-point contribution. The correlation length diverges for $\alpha = 0$ in the two limiting cases. However, for $b > 0$, the exact calculation shows that the correlation length reaches a maximum for $\alpha = -1.25$.

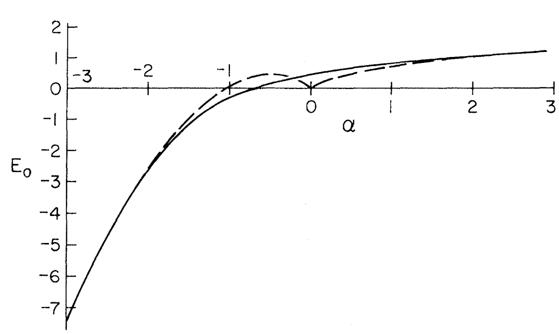


FIG. 1. The solid line is the ground-state energy $E_0$ plotted versus $\alpha$ for the anharmonic oscillator. The dashed line for $\alpha > 0$ is the harmonic approximation $b = 0$, and the dashed line for $\alpha < 0$ is the mean-field approximation including the zero-point energy.

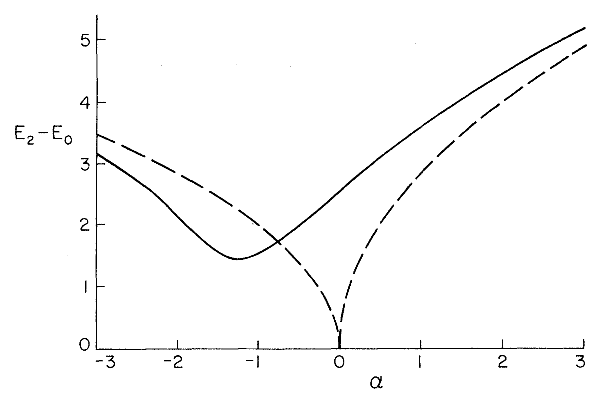


FIG. 2. The energy splitting between the first two even states of the anharmonic oscillator is plotted versus $\alpha$ (solid line). The dashed lines give the harmonic ($b = 0$) approximation for this splitting for $\alpha > 0$ and the mean-field zero-point approximation for $\alpha < 0$.

Solving Eq. (66) when $\xi_2 = 2$, $a_1 = 1.15$, and $a_2 = 0.69$ we find $\alpha = -2.6$ and

$$a = -1.91, \quad b = 0.76, \quad c = 11.7 \;. \qquad (67)$$

These are in reasonable agreement with the simple mean field results, Eq. (50). Using these parameters the coefficients $a_3$, $a_4$, and $a_5$ have been determined from Eq. (61). These are listed in Table I in the second column labeled $\alpha = -2.6$. The third column labeled $\alpha = -2.0$ corresponds to taking $\xi_2 = 2$, $a_1 = 0.93$, and $a_2 = 0.69$. The last column shows that $a_n$ obtained when $\xi_2 = 2$, $a_1 = 0.8$, and $a_2 = 0.69$. In the first column is shown estimates of the experimental coefficients for the production of negatively charged particles taken from a data analysis by Frazer.[15] In view of the fact that $a_3$, $a_4$, and $a_5$ are not accurately determined by the present experimental data[15] and the fact that we have not included isospin effects which are expected to make quantitative changes in our results, we have not attempted to obtain a best fit to the data. Notice, however, the over-all agreement with respect to sign and magnitude of the $a_n$ in the last column. The point we wish to emphasize is that we are operating in a region where our results are very sensitive to changes in $\alpha$.

Perhaps the best way to illustrate the structure of this type of theory is to plot the derivatives of $E_0(\alpha)$. These are shown in Figs. 3 and 4. Notice that as $\alpha$ goes from $-2.6$ towards $-1.7$ one is approaching the maximum value of $-E_0''$ which occurs at $\alpha = -1.56$. This value of $\alpha$ corresponds to maximum intensity fluctuations of $\Pi(y)$. From Fig. 2 one sees that the point at which the reduced correlation length $(E_2 - E_0)^{-1}$ becomes a maximum is still further towards zero at $\alpha = -1.24$.

As we have noted, the formalism presented here has also been used to study lasers near threshold and "one-dimensional" superconducting wires near the critical point. For the laser the present parameters correspond to a region just above the

TABLE I. Calculation of the coefficients $a_n$ for various values of the input parameters. The column labeled exp is taken from an analysis of the negatively charged particle data by Frazer.[15] He calculates the $a_n$ under the assumption that the $f_n$ have the form $f_n = a_n Y + b_n$. The value $a_1 = 1.15$ arises when $s^{-1/2}$ corrections are included in $f_1$.

| | exp | $\alpha = -2.6$ | $\alpha = -2.0$ | $\alpha = -1.7$ |
|---|---|---|---|---|
| $a_1$ | 0.93 (1.15) | 1.15 | 0.93 | 0.80 |
| $a_2$ | 0.69 | 0.69 | 0.69 | 0.69 |
| $a_3$ | −0.405 | −0.058 | −0.26 | −0.15 |
| $a_4$ | −1.7 | 0.247 | 0.45 | −1.34 |
| $a_5$ | 4.2 | −1.2 | 4.5 | 6.0 |

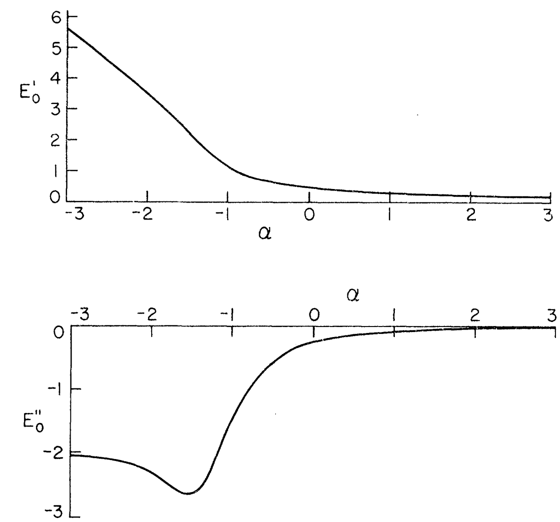


FIG. 3. The first and second derivatives of $E_0(\alpha)$ are plotted versus $\alpha$. These derivatives are related to the $a_n$ coefficients as follows: $a_1 \sim E_0'$, $a_2 \sim -E_0''$.

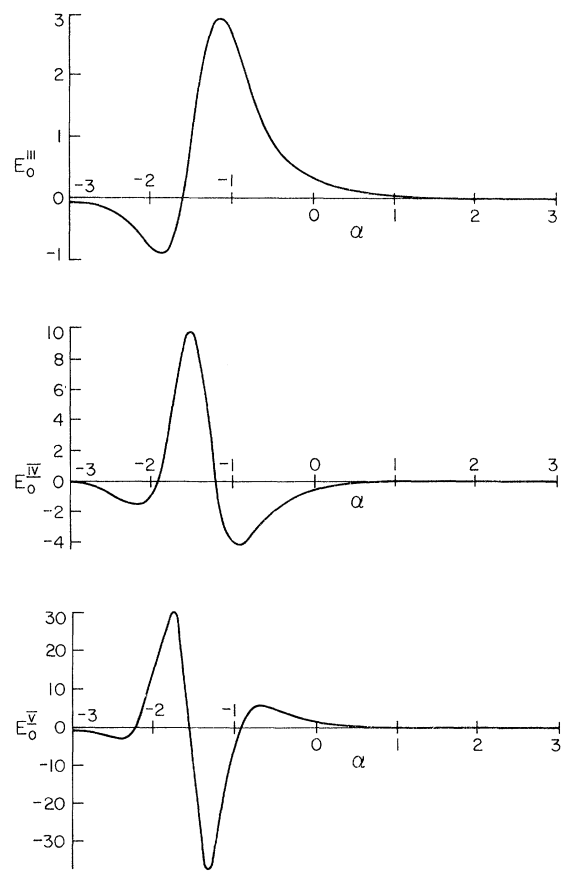


FIG. 4. Further derivatives of $E_0(\alpha)$ versus $\alpha$. These derivatives are related to the $a_n$ coefficients as follows: $a_3 \sim E_0'''$, $a_4 \sim -E_0^{IV}$, $a_5 \sim E_0^{V}$.

laser threshold, and for the superconductor they correspond to the region of $T$ just below $T_c$. The possible physical significance of these features will be discussed in the next section.

## V. DISCUSSION

The qualitative features of our theory have a familiar interpretation. The inclusive cross sections have the form that one would obtain in a Mueller-Regge model in which there were an infinite number of factorizable Regge poles but no Regge cuts. In our case the positions of all of the poles and the values of all of their coupling constants are determined in terms of three phenomenological parameters. Our generating function corresponds to a production mechanism in which the particles come off in clusters. All particles in the same cluster have small relative rapidities, while the clusters themselves are well separated in rapidity space and are uncorrelated.

We have presented our theory in its simplest possible form. There are several generalizations which should be mentioned. First, it is possible to take into account the quantum numbers of the produced particles. For example, the isospin of the pions could be included by treating $\Pi(y)$ as a three component field. Although this would not change the qualitative features mentioned above, it is expected to make a quantitative difference. This is because we are operating in a region where the $\Pi^4$ term in $F[\Pi]$ is important, and as a result, there will be significant interactions among the various components of $\Pi(y)$. For this reason we have postponed making a detailed comparison of our theory with experiment until isospin effects have been included.

A second possible generalization has to do with the boundary conditions on $\Pi(y)$. Although it seems natural to take $\Pi(y)$ to be free on the boundaries, it should be pointed out that any choice of boundary conditions implies a definite form for the coupling of $\Pi(y)$ to the incident hadrons. More generally one could introduce a form factor, $N(\Pi)$, which describes how the incident hadron couples to the pion field. The only change in our formalism would be to replace the coupling constant $G_n$ defined in Eq. (18) by

$$G_n' = \int_{-\infty}^{\infty} d\Pi\, \psi_n(\Pi) N(\Pi)\,. \qquad (68)$$

As a result, there would be no qualitative changes so long as $N(\Pi)$ is an even function of $\Pi$. We expected our statistical theory to work best in the central region of the rapidity plot where most of the particles are produced. Certainly some edge effects, such as the diffractive excitation of resonances which then decay into a small number of pions, are not expected to be well described. One might hope to include such effects in the form factor $N(\Pi)$.

It is possible to consider more general forms for the functional $F[\Pi]$ within the framework presented here. One could, for example, replace the potential

$$V(\Pi) = a\Pi^2 + b\Pi^4$$

by a general even function of $\Pi$. So long as the Hamiltonian,

$$H(\Pi) = -\frac{1}{4c}\frac{\partial^2}{\partial \Pi^2} + V(\Pi)\,,$$

has a discrete spectrum, the qualitative features of the theory will not be changed. A continuous spectrum for $H(\Pi)$ would give rise to Regge-cut terms in the inclusive cross sections. However, these enter in a more natural manner when the dependence of the $\Pi$ field on the impact parameter $\vec{b}$ as well as $y$ is included. This increase in dimension of the $\Pi$ field variables to three dimensions leads to a $\Pi^4$ field theory rather than the simple anharmonic Hamiltonian of Eq. (14). However, various methods have recently been developed for treating such systems near critical points.[16]

Turning next to the physical implications of our formalism, the most important is not readily visible in what we have calculated in the preceding sections. The $\Pi$ field has long-range phase order over the rapidity space. We have concentrated on the intensity correlation functions since they determine the inclusive cross sections and the generating function. The intensity correlations are short range having $\xi_2 \sim 2 \ll Y$ at high energies. However, the $\Pi$ field correlation function

$$\langle \Pi(y_1)\Pi(y_2)\rangle \sim |\langle 1|\Pi|0\rangle|^2 \exp\left(-\frac{|y_1-y_2|}{\xi_1}\right) \qquad (69)$$

has $\xi_1 \gtrsim Y$ with

$$\xi_1^{-1} = \frac{b^{1/3}}{c^{2/3}}(E_1 - E_0)\,. \qquad (70)$$

Here $E_1$ is the eigenvalue of the first excited state $|1\rangle$ of the anharmonic oscillator. We have not needed this state previously because the intensity fluctuations only coupled the even states. As $a$ decreases below zero, the splitting between $E_1$ and $E_0$ decreases rapidly. It is related to the inverse time for a particle of mass $2c$ to tunnel between the potential minima of $\alpha\Pi^2+\frac{1}{4}\Pi^4$. Figure 5 shows a combined plot of $\xi_1$ and $\xi_2$ versus $\alpha$. For all the cases listed in Table I, $\xi_1>10$.

As we have noted, our theory is similar to both the theory of a superconducting wire near $T_c$ and a laser near threshold. This naturally leads to the question of whether this is simply a formal similarity and our $F[\Pi]$ just one more way to parametrize the data, or whether there is a physical basis for the similarity. We would like to suggest that in fact there may be a physical basis for the similarity.

Consider two protons approaching each other in the center-of-mass system. At high energies their matter distributions will be in the shape of Lorentz-contracted disks. Just after the collision we have a highly excited system localized in a region whose dimension along the beam is of order $Rs^{-1/2}$ and perpendicular to the beam is $R$. The most important mode of energy radiation is pion emission. Now, just as it is possible to construct a single pass laser, it should be possible for coherent emission of pions to occur from the excited region of the disk. This type of coherent production should saturate very near threshold conditions since once it starts the resulting pion emission, in the absence of a "resonance cavity," prevents the further buildup of the pion fields. Furthermore, the coherence of the pion field over the emission region of the disk leads naturally to long-range phase order of $\Pi$ over a region of order $Y$.

*Note added in proof.* Here we have treated $\Pi(y)$ as a statistical variable. An alternative approach[1] would be to start from the pion-field density matrix. Then, using a coherent state representation, one can obtain all of the results of Sec. II. In this framework, $\Pi(y)$ is the eigenvalue of the pion-field annihilation operator, which in general can be complex.

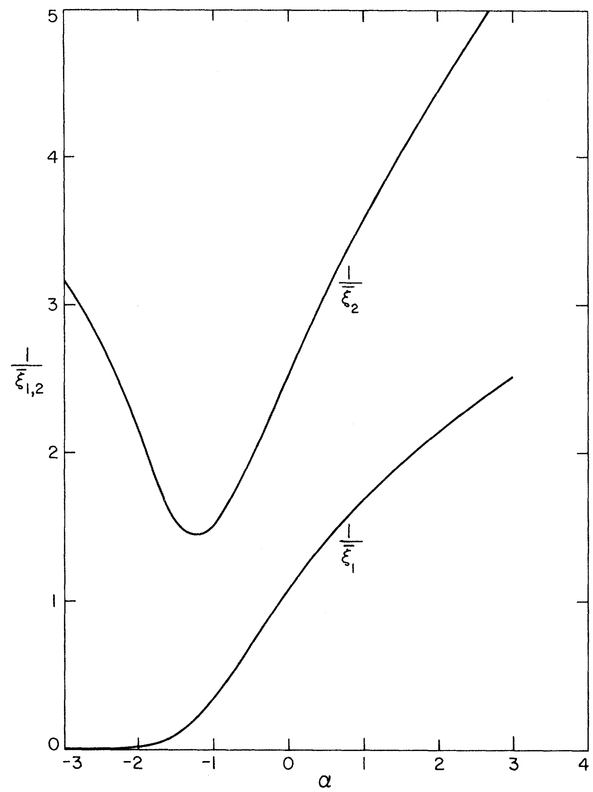


FIG. 5. The reduced correlation lengths $\xi_1^{-1}=E_1(\alpha)-E_0(\alpha)$ and $\xi_2^{-1}=E_2(\alpha)-E_0(\alpha)$ are plotted versus $\alpha$. Note that while the intensity-intensity correlation length $\xi_2$ reaches a maximum at $\alpha\sim-1.5$, the field-field correlation length $\xi_1$ increases indefinitely as $\alpha$ decreases.


## ACKNOWLEDGMENTS

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