η-X Mixing and Chiral-Symmetry Breaking

B. G. Kenny*
Research School of Physical Sciences, The Australian National University, Canberra, Australia and Department of Physics, University of Virginia, Charlottesville, Virginia 22901

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Certain properties of the pseudoscalar mesons and the strange scalar mesons are discussed in the context of the chiral-symmetry-breaking model of Glashow, Weinberg, Gell-Mann, Oakes, and Renner. In particular, the approach of Glashow and Weinberg is followed, with particular attention being paid to η-X mixing. In order to obtain reasonable agreement with experiment, it is found that unequal wave-function renormalization constants must be assumed for the fields which belong to the representation breaking SU(3) $\times$ SU(3) symmetry. The η-X mixing angle is predicted to be zero and this result is compared with predictions of other authors. Mechanisms other than η-X mixing are conjectured for the enhancement of the decay rate $\eta \to 2\gamma$.

I. INTRODUCTION

Following the pioneering work of Glashow and Weinberg[1] and Gell-Mann, Oakes, and Renner[2] on chiral-symmetry breaking, numerous authors have investigated certain properties of the pseudoscalar-meson nonet. Other authors, using specific models for the chiral-invariant part of the Hamiltonian, have also analyzed the conjectured scalar-meson nonet.

In this paper, we assume no specific model for the chiral-invariant part of the Hamiltonian. The approach of Glashow and Weinberg is followed to a large extent. In their work, the wave-function renormalization constants for the various mesons were assumed to be different. With such restrictive assumptions, it is difficult to make many specific predictions.

A simple and economical approach is adopted in this paper in attempting to obtain predictions in reasonable agreement with experiment. Initially, a simple model is assumed for the renormalization and mixing of the mesons. Then restrictions are applied to the simple model. This is done in a step-by-step process in order to finally arrive at a point where agreement with experiment is reasonable. In other words, we try to derive the least complex model of the Glashow-Weinberg type that seems to agree with experiment.

In Sec. II, we discuss the assumption of the equality of the wave-function renormalization constants $Z_\pi = Z_K = Z_\kappa = Z$, and attempt to derive all possible consequences of this assumption. Elsewhere, we have suggested[3] that this assumption is correct to a good approximation.

In Sec. III, we introduce the η and X, neglect η-X mixing, and assume $Z = Z_\eta = Z_X$. We are able to calculate a reasonable value for f_K/f_π and m_κ, but make an incorrect prediction for m_X.

In Sec. IV, η-X mixing is assumed together with the equality $Z = Z_\eta = Z_X$, and m_X is no longer predicted. We predict a reasonable value of m_κ, but an unreasonable value of f_K/f_π.

In Sec. V, η-X mixing is assumed, but the constraint of equal wave-function renormalization for the singlet state is dropped. Since this introduces an unknown parameter, we make an assumption about the unmixed η mass in order to derive predictions. The value used for the unmixed mass is a value which is naturally suggested when not only the SU(3) breaking of the interactions is considered, but also the SU(3) breaking of the vacuum. Making this assumption enables us to make predictions for f_K/f_π and m_κ in good agreement with experiment, and a prediction for f_κ/f_π in good agreement with an independent estimate of this quantity. Surprisingly, however, zero η-X mixing is predicted. We conclude by comparing this result with related results of other authors, and conjecture mechanisms other than η-X mixing for the enhancement of the decay rate $\eta \to 2\gamma$.

In Sec. VI, we summarize the paper and make a few additional comments.

II. SOME RESULTS OF CHIRAL-SYMMETRY BREAKING

We follow numerous authors in writing the strong-interaction Hamiltonian density as[1,2]

$$\mathcal{H} = \mathcal{H}_0 - \epsilon_0 \sigma_0 - \epsilon_8 \sigma_8 . \tag{1}$$

$\mathcal{H}_0$ is SU(3)$\times$SU(3) invariant, and we assume that σ_0 and σ_8 are local scalar fields belonging to the 18-dimensional $(3, \bar{3}) + (\bar{3}, 3)$ representation of

SU(3)×SU(3). We further assume that the various scalar σ_i and pseudoscalar π_i ($i=0,\ldots,8$) are good interpolating fields for the scalar and pseudoscalar meson nonets.

Both pion and kaon PCAC (partially conserved axial-vector current) for the axial-vector currents ($i=1,\ldots,7$) and PCVC (partially conserved vector current) for the strangeness-changing vector currents ($i=4,5,6,7$) follow naturally from the assignment of $(\epsilon_0\sigma_0+\epsilon_8\sigma_8)$ to the above representation, and the assumption that the fields belonging to the representation may be identified with the scalar and pseudoscalar meson nonets.

If we define f_π, f_K and f_κ through

$$\begin{aligned}\partial_\mu A_\mu^{1+i2}&=f_\pi m_\pi{}^2\phi_{\pi^+},\\ \partial_\mu A_\mu^{4+i5}&=f_K m_K{}^2\phi_{K^+},\\ \partial_\mu V_\mu^{4+i5}&=i f_\kappa m_\kappa{}^2\phi_{\kappa^+},\end{aligned}\tag{2}$$

where ϕ_{π^+}, ϕ_{K^+}, and ϕ_{κ^+} are *renormalized* interpolating fields, then we may relate f_π, f_K, and f_κ to ϵ_0 and ϵ_8. We shall assume equality of the wave-function renormalization constants:

$$Z_\pi=Z_K=Z_\kappa=Z\ .\tag{3}$$

The consequences of this approximation, which we summarize below, have been shown elsewhere[3] to be reasonably good. Using the Hamiltonian density, Eq. (1), we may compute the divergences of the currents A_μ^{1+i2}, A_μ^{4+i5}, and V_μ^{4+i5}. A comparison of these results with Eq. (2) leads to the consistency condition

$$f_K m_K{}^2=f_\pi m_\pi{}^2+f_\kappa m_\kappa{}^2\ ,\tag{4}$$

together with the independent relations

$$\begin{aligned}Z^{1/2}\epsilon_8&=-(\tfrac{2}{3})^{1/2}f_\kappa m_\kappa{}^2\ ,\\ Z^{1/2}\epsilon_0&=(\tfrac{1}{3})^{1/2}(f_K m_K{}^2+\tfrac{1}{2}f_\pi m_\pi{}^2)\ ,\end{aligned}\tag{5}$$

To proceed further, we make the pole approximation for the meson propagators, i.e., the inverse propagators are only linear in the square of the momentum. From the definition of the propagator in momentum space, e.g.,

$$\Delta_\pi(p^2)=i\int d^4x\,e^{-ip\cdot x}\langle 0\,|\,T\{\phi_{\pi^+}(x)\phi_{\pi^-}(0)\}\,|\,0\rangle\ ,\tag{6}$$

we find at zero momentum transfer, making use of the pole approximation,

$$m_\pi{}^{-2}=i\int d^4x\,\langle 0\,|\,T\{\phi_{\pi^+}(x)\phi_{\pi^-}(0)\}\,|\,0\rangle\ .\tag{7}$$

Using the definition of PCAC in Eq. (2), we rewrite this as

$$f_\pi=i\int d^4x\,\langle 0\,|\,T\{\partial_\mu A_\mu^{1+i2}(x)\phi_{\pi^-}(0)\}\,|\,0\rangle\ .\tag{8}$$

Integration by parts leads to

$$\sqrt{Z}\,f_\pi=\frac{2}{\sqrt{3}}\langle\sigma_0\rangle+(\tfrac{2}{3})^{1/2}\langle\sigma_8\rangle\ .\tag{9a}$$

A similar treatment of the K-meson and κ-meson propagators leads to the relations

$$\sqrt{Z}\,f_K=\frac{2}{\sqrt{3}}\langle\sigma_0\rangle-\frac{1}{\sqrt{6}}\langle\sigma_8\rangle\tag{9b}$$

and

$$\sqrt{Z}\,f_\kappa=-(\tfrac{3}{2})^{1/2}\langle\sigma_8\rangle\ .\tag{9c}$$

$\langle\sigma_0\rangle$ and $\langle\sigma_8\rangle$ are the vacuum expectation values of the fields σ_0 and σ_8.

For Eqs. (9) to be consistent, we must have the relation

$$f_K=f_\pi+f_\kappa\ .\tag{10}$$

If we combine the consistency conditions, Eqs. (4) and (10), we find

$$\frac{f_K}{f_\pi}=\frac{m_\kappa{}^2-m_\pi{}^2}{m_\kappa{}^2-m_K{}^2}\ .\tag{11}$$

The above results are essentially contained in the work of Glashow and Weinberg[1], who consider the more general case of unequal renormalization constants. In addition, using techniques similar to the ones we have discussed above for the two-point function, Glashow and Weinberg[1] investigate the three-point function for the $K\pi\kappa$ vertex. By assuming smooth behavior for this function in the momentum of the various particles, they are able to show that[4]

$$\sqrt{2}f_+(0)=1\tag{12}$$

in the case where the renormalization constants Z_π, Z_K, and Z_κ are all equal, i.e., $f_+(0)$ assumes its exact SU(3) value. This possibility is rejected by Glashow and Weinberg because of expected renormalization effects due to the existence of vector and axial-vector mesons. We have shown recently,[3] however, that Eqs. (4), (10), and (12) seem to hold to a good approximation. This suggests that the assumption $Z_\pi=Z_K=Z_\kappa$ is a good approximation in nature, despite the above-mentioned renormalization effects which could destroy the equality.

Throughout this paper, we shall assume this equality together with all its consequences, which we summarize here as

$$f_K m_K{}^2=f_\pi m_\pi{}^2+f_\kappa m_\kappa{}^2\ ,\tag{4}$$

$$f_K=f_\pi+f_\kappa\ ,\tag{10}$$

$$\frac{f_K}{f_\pi}=\frac{m_\kappa^2-m_\pi^2}{m_\kappa^2-m_K^2}, \tag{11}$$

$$\sqrt{2}f_+(0)=1\,. \tag{12}$$

In the literature, there are many authors who implicitly or explicitly assume equality of renormalization constants but do not treat the above results as necessary theoretical consequences of this theoretical assumption. *All* of these results must be used in conjunction with one another in order to be theoretically consistent. For example, under our assumptions, there is only one unknown in the above relations, e.g., f_K/f_π, f_κ/f_π, or m_κ. Once we have fixed one of these quantities, the others are also fixed and may not be varied.

Before concluding this section, we note that Eqs. (9) lead to the relations

$$\begin{aligned} Z^{1/2}\langle\sigma_0\rangle &= \frac{1}{2\sqrt{3}}(2f_K+f_\pi)\,,\\ Z^{1/2}\langle\sigma_8\rangle &= -(\tfrac{2}{3})^{1/2}(f_K-f_\pi)\,, \end{aligned} \tag{13}$$

from which we deduce

$$\frac{\langle\sigma_8\rangle}{\langle\sigma_0\rangle}=\frac{-2\sqrt{2}\,(f_K-f_\pi)}{2f_K+f_\pi}\,. \tag{14}$$

In the notation of Mathur and Okubo,[5]

$$\begin{aligned} \frac{\langle\sigma_8\rangle}{\langle\sigma_0\rangle} &= \sqrt{2}\,b\,,\\ \frac{\epsilon_8}{\epsilon_0} &= \sqrt{2}\,a\,. \end{aligned} \tag{15}$$

It is necessary to consider the pseudoscalar mesons η and X to obtain further results. We first consider the simplest case where η and X are not mixed and have the same wave-function renormalization constants as π, K, and κ.

III. NO η-X MIXING

Using the Hamiltonian density, Eq. (1), we may compute the divergence of the current A_μ^8. It is given by

$$\partial_\mu A_\mu^8=[(\tfrac{2}{3})^{1/2}\epsilon_0-(\tfrac{1}{3})^{1/2}\epsilon_8]\pi_8+(\tfrac{2}{3})^{1/2}\epsilon_8\pi_0\,, \tag{16}$$

which means that we do not have simple PCAC as we did for the π and K mesons. Guided by the definitions of f_π, f_K, and f_κ in Eq. (2), we define f_η and f_X by

$$\partial_\mu A_\mu^8=\frac{1}{\sqrt{2}}f_\eta m_\eta^2\phi_\eta+\frac{1}{\sqrt{2}}f_X m_X^2\phi_X\,. \tag{17}$$

We shall assume in this section that the wave-function renormalization constants for the η and the X are the same as those for the π, K, and κ. We then have the relations

$$Z^{1/2}f_\eta m_\eta^2=\frac{2}{\sqrt{3}}\epsilon_0-(\tfrac{2}{3})^{1/2}\epsilon_8 \tag{18a}$$

and

$$Z^{1/2}f_X m_X^2=\frac{2}{\sqrt{3}}\epsilon_8\,. \tag{18b}$$

From Eqs. (4), (5), and (18), we see that

$$3f_\eta m_\eta^2=4f_K m_K^2-f_\pi m_\pi^2 \tag{19a}$$

and

$$f_X m_X^2=-\tfrac{2}{3}\sqrt{2}f_\kappa m_\kappa^2\,. \tag{19b}$$

Equation (19a) is clearly a modification of the usual Gell-Mann–Okubo mass formula, and reflects the fact that we have taken into account the SU(3) noninvariance of the vacuum. SU(3) invariance of the vacuum implies that

$$\langle\sigma_8\rangle=0\,, \tag{20}$$

in which case we would have $f_\kappa=0$ and $f_\pi=f_K=f_\eta$ [see Eqs. (9) and (23) below] under our assumption of equal wave-function renormalization constants. In this case, Eq. (19a) would reduce to the normal Gell-Mann–Okubo formula for the unmixed mass of the η.

In order to proceed further, we consider the two-point functions

$$i\int d^4x\langle 0|\,T\{\partial_\mu A_\mu^8(x)\phi_\eta(0)\}\,|0\rangle e^{-ip\cdot x} \tag{21a}$$

and

$$i\int d^4x\langle 0|\,T\{\partial_\mu A_\mu^8(x)\phi_X(0)\}|\,0\rangle e^{-ip\cdot x}\,. \tag{21b}$$

We integrate the two-point functions in Eq. (21) by parts (at zero momentum transfer), making use of the definition of $\partial_\mu A_\mu^8$ in Eqs. (16) and (17). Making use of our assumption of no η-X mixing we find

$$Z^{1/2}f_\eta=\frac{2}{\sqrt{3}}\langle\sigma_0\rangle-(\tfrac{2}{3})^{1/2}\langle\sigma_8\rangle \tag{22a}$$

and

$$Z^{1/2}f_X=\frac{2}{\sqrt{3}}\langle\sigma_8\rangle\,. \tag{22b}$$

Making use of Eqs. (9), we conclude that

$$3f_\eta=4f_K-f_\pi \tag{23a}$$

and

$$f_X=-\tfrac{2}{3}\sqrt{2}f_\kappa\,. \tag{23b}$$

Combining Eqs. (19) and (23), we have the relations

$$(4f_K-f_\pi)m_\eta^2=4f_K m_K^2-f_\pi m_\pi^2 \tag{24a}$$

and

$$m_\kappa^2 = m_X^2 . \tag{24b}$$

From (24a) we find

$$\frac{f_K}{f_\pi} = \frac{m_\eta^2 - m_\pi^2}{4(m_\eta^2 - m_K^2)} . \tag{25a}$$

To estimate f_K/f_π, we use average masses for the π and K. (Numerically we use $m_\pi^2 = 0.0190\,\mathrm{GeV}^2$, $m_K^2 = 0.2458\,\mathrm{GeV}^2$, $m_\eta^2 = 0.3012\,\mathrm{GeV}^2$.) We find from Eq. (25a) that

$$f_K/f_\pi = 1.274 . \tag{25b}$$

If we combine this with the fact that our theoretical assumptions force us to use the exact SU(3) value for $f_+(0)$ (see Sec. II), then we make the theoretical prediction

$$\frac{f_K}{\sqrt{2}f_+(0)f_\pi} = 1.274 . \tag{26a}$$

This is in excellent agreement with the experimental value[6]

$$\frac{f_K}{\sqrt{2}f_+(0)f_\pi} = 1.27 \pm 0.03 . \tag{26b}$$

From Sec. II we have the result

$$f_K = f_\pi + f_\kappa , \tag{10}$$

so that from Eq. (25b) we find

$$f_\kappa/f_\pi = 0.274 , \tag{27}$$

which is in good agreement with an independent estimate[7] (0.30) we have made of this quantity. Combining Eq. (25a) with the theoretical relation (11)

$$\frac{f_K}{f_\pi} = \frac{m_\kappa^2 - m_\pi^2}{m_\kappa^2 - m_K^2} , \tag{11}$$

we are able to predict

$$m_\kappa = 1030\,\mathrm{MeV} . \tag{28a}$$

In addition, we conclude from Eq. (24b) that

$$m_X = m_\kappa = 1030\,\mathrm{MeV}. \tag{28b}$$

We know from experiment[8] that

$$m_X = 960\,\mathrm{MeV} , \tag{29}$$

while an analysis of K-π scattering data[8] indicates that the most likely interpretation of the data suggests a broad S-wave (or κ) resonance in the vicinity of 1100 MeV.

In summary, then, this naive model makes quite reasonable experimental predictions. The model is, however, wrong because of its incorrect prediction of the X mass. The next logical step in improving the model is to take account of η-X mixing.

IV. η-X MIXING

In order to construct a more realistic model, we must take account of η-X mixing. We suppose that the physical states $|\eta\rangle$ and $|X\rangle$ are linear combinations of pure octet and singlet states $|\eta_8\rangle$ and $|X_0\rangle$:

$$\begin{aligned} |\eta\rangle &= \cos\theta|\eta_8\rangle - \sin\theta|X_0\rangle , \\ |X\rangle &= \cos\theta|X_0\rangle + \sin\theta|\eta_8\rangle , \end{aligned} \tag{30}$$

where θ is the mixing angle.

We assume, as in the previous section, that the wave-function renormalization constants for the η and X are equal to those for the π, K, and κ. We integrate the two-point functions [Eq. (21)] by parts (at zero momentum transfer), making use of the definition of $\partial_\mu A_\mu^8$ in Eq. (16). In this section, however, we take account of η-X mixing. We find that the inverse propagator (at zero momentum transfer), which is the mass-squared matrix M^2, satisfies

$$M^2\begin{bmatrix} (\frac{2}{3})^{1/2}\langle\sigma_0\rangle - (\frac{1}{3})^{1/2}\langle\sigma_8\rangle \\ (\frac{2}{3})^{1/2}\langle\sigma_8\rangle \end{bmatrix} = \begin{bmatrix} (\frac{2}{3})^{1/2}\epsilon_0 - (\frac{1}{3})^{1/2}\epsilon_8 \\ (\frac{2}{3})^{1/2}\epsilon_8 \end{bmatrix} . \tag{31}$$

From Eq. (30) we see that in the η_8-X_0 basis

$$M^2 = \begin{bmatrix} m_\eta^2\cos^2\theta + m_X^2\sin^2\theta & (m_X^2 - m_\eta^2)\sin\theta\cos\theta \\ (m_X^2 - m_\eta^2)\sin\theta\cos\theta & m_X^2\cos^2\theta + m_\eta^2\sin^2\theta \end{bmatrix} . \tag{32}$$

From Eqs. (31) and (32), together with the definitions of ϵ_0, ϵ_8, $\langle\sigma_0\rangle$ and $\langle\sigma_8\rangle$ given in Sec. II, we find that the mixing angle θ satisfies

$$\begin{aligned} \tan\theta &= \frac{\alpha - c m_\eta^2}{\beta - d m_\eta^2} , \\ \tan\theta &= -\frac{\beta - d m_X^2}{\alpha - c m_X^2} . \end{aligned} \tag{33}$$

c, d, α, and β are given by

$$\begin{aligned} c &= \frac{1}{3\sqrt{2}}(4f_K - f_\pi) , \\ d &= -\tfrac{2}{3}(f_K - f_\pi) , \\ \alpha &= \frac{1}{3\sqrt{2}}(4f_K m_K^2 - f_\pi m_\pi^2) , \\ \beta &= -\tfrac{2}{3}(f_K m_K^2 - f_\pi m_\pi^2) . \end{aligned} \tag{34}$$

It is possible to eliminate the mixing angle from Eq. (33) and determine f_K/f_π. There are two solutions for f_K/f_π, and two corresponding solutions for the mixing angle θ. The smaller of the two solutions for f_K/f_π is

$$f_K/f_\pi = 1.434 , \tag{35a}$$

corresponding to a mixing angle

$$\theta = 3.5^\circ . \tag{36}$$

(We reject the larger solution, $f_K/f_\pi = 1.78$.)

This mixing angle is small compared with the normal Gell-Mann–Okubo quadratic mixing angle of 10.4°. It corresponds to an unmixed mass of 551 MeV for the η which is very close to the physical mass of 549MeV.

From Eq. (11), we find that the value $f_K/f_\pi = 1.434$ implies that the κ mass is

$$m_\kappa = 875\,\text{MeV} . \tag{37}$$

The results expressed in Eqs. (35)–(37) are quite sensitive to the input values of the masses. Average masses for the π and K masses were used. In order to compare with other authors, we used the following numerical values for the squared masses: $m_\pi^2 = 0.0190$, $m_K^2 = 0.2458$, $m_\eta^2 = 0.3012$, $m_X^2 = 0.9168$, in units of GeV^2. In terms of the parameters a and b [see Eq. (15)] we have

$$\begin{aligned} a &= -0.921 , \\ b &= -0.224 . \end{aligned} \tag{38}$$

The results we obtain in this section are very close to those obtained by Carruthers and Haymaker,[9] who examine this problem in detail using an SU(3) σ model. They obtain

$$\begin{aligned} m_\kappa &= 910\,\text{MeV} , \\ \theta &= 2.4^\circ , \\ a &= -0.919 , \\ b &= -0.210 . \end{aligned} \tag{39}$$

The slightly different values they obtain for these quantities can presumably be ascribed to the slightly different input values they use for the π, K, η, and X masses. Although not explicitly stated in their paper, they obtain a value of $f_K/f_\pi (1.40)$ ver close to our result.

Analyses of the K-π scattering data[8] indicate an alternative, less likely, interpretation to the one described in Sec. III, namely, a sharp S-wave, $I = \frac{1}{2}$ (κ) resonance at about 890MeV. The theoretical prediction Eq. (37) is in good agreement with this result and, by inference, supports the value of $f_K/f_\pi = 1.434$ obtained in this section [Eq. (35a)].

However, this value of f_K/f_π is higher than one would expect. In fact, remembering that our basic theoretical assumptions force us to use $\sqrt{2} f_+(0) = 1$, we find

$$\frac{f_K}{\sqrt{2} f_+(0) f_\pi} = 1.434 , \tag{35b}$$

in complete disagreement with the experimental value[6]

$$\frac{f_K}{\sqrt{2} f_+(0) f_\pi} = 1.27 \pm 0.03 . \tag{26b}$$

In summary, the assumptions made in this section have led to the following predictions:

(1) $m_\kappa = 875\,\text{MeV}$, which is not in conflict with present data on K-π scattering;

(2) a mixing angle $\theta = 3.5^\circ$, indicating that the η is very weakly mixed with the X;

(3) a prediction for f_K/f_π which cannot be reconciled with experiment given our basic assumptions in Sec. II.

We must, therefore, reject the hypothesis made at the beginning of this section, that the wave-function renormalization constants for the η and X are equal to those for the π, K, and κ. Equally we must reject any models appearing in the literature which explicitly or implicitly lead to equality of all wave-function constants.

V. UNEQUAL RENORMALIZATION CONSTANTS

In order to improve upon the predictions made in Secs. III and IV, we must assume now that the wave-function renormalization constants are not all equal. In Sec. II, we indicated that there was good evidence for the equality $Z_\pi = Z_K = Z_\kappa$. The most reasonable and simple assumption we can make about nonequality of renormalization constants is that all octet renormalization constants Z_π, Z_K, Z_κ, and Z_{η_8} are equal but different from the singlet renormalization constant Z_{X_0}.

The mass-squared matrix M^2 describing η-X mixing may be written as usual in the form shown in Eq. (32). However, Eq. (31) is modified to

$$M_0^2 \begin{bmatrix} (\frac{2}{3})^{1/2} \langle \sigma_0 \rangle - (\frac{1}{3})^{1/2} \langle \sigma_8 \rangle \\ (\frac{2}{3})^{1/2} \langle \sigma_8 \rangle \end{bmatrix} = \begin{bmatrix} (\frac{2}{3})^{1/2} \epsilon_0 - (\frac{1}{3})^{1/2} \epsilon_8 \\ (\frac{2}{3})^{1/2} \epsilon_8 \end{bmatrix} , \tag{40}$$

where

$$M_0^2 = \begin{bmatrix} m_\eta^2 \cos^2\theta + m_X^2 \sin^2\theta & Z_0^{-1/2} (m_X^2 - m_\eta^2) \sin\theta \cos\theta \\ Z_0^{-1/2} (m_X^2 - m_\eta^2) \sin\theta \cos\theta & Z_0^{-1} (m_\eta^2 \sin^2\theta + m_X^2 \cos^2\theta) \end{bmatrix} , \tag{41}$$

and

$$Z_0 = Z_\pi / Z_{X_0} . \tag{42}$$

This means that Eq. (33) is modified to

$$\tan\theta = Z_0^{-1/2}\frac{\alpha - cm_\eta^2}{\beta - Z_0^{-1}dm_\eta^2}, \tag{43a}$$

$$\tan\theta = -Z_0^{1/2}\frac{\beta - Z_0^{-1}dm_X^2}{\alpha - cm_X^2}, \tag{43b}$$

where α, β, c, and d are defined in Eq. (34).

It is not possible to do anything with these two equations other than obtain a relation between the three unknowns f_K/f_π, Z_0, and θ. In order to make some predictions, we must put in further information. The most reasonable input would seem to be the mixing angle θ or, equivalently, the unmixed mass of the η. The usual formula for the unmixed mass of the $\eta(m_0)$ is the Gell-Mann–Okubo formula,

$$3m_0^2 = 4m_K^2 - m_\pi^2 \,. \tag{44}$$

However, as we pointed out in Sec. III, this formula assumes that the vacuum is SU(3)-invariant, i.e., $\langle\sigma_8\rangle = 0$. In Sec. IV, it was observed that the mixing angle θ was quite small and quite different from the prediction of the Gell-Mann–Okubo formula. This, presumably, reflects the fact that we are taking account of SU(3) noninvariance of the vacuum in this paper.

If we are to be *consistent* in considering the SU(3)-noninvariance of the vacuum, then we must use the correct formula for the unmixed mass of the η which takes this into account. This formula was given in Sec. III. From Eq. (24b) we find that the correct formula for the unmixed mass is given by

$$m_0^2 = \frac{4f_K m_K^2 - f_\pi m_\pi^2}{4f_K - f_\pi}. \tag{45}$$

Now the mixing angle θ satisfies

$$\tan^2\theta = \frac{m_0^2 - m_\eta^2}{m_X^2 - m_0^2}, \tag{46}$$

where m_0^2 is given in Eq. (45). We therefore have

$$\tan^2\theta = -\frac{(4f_K m_K^2 - f_\pi m_\pi^2) - (4f_K - f_\pi)m_\eta^2}{(4f_K m_K^2 - f_\pi m_\pi^2) - (4f_K - f_\pi)m_X^2}. \tag{47}$$

From Eqs. (43) we have

$$\tan^2\theta = -\frac{\alpha - cm_\eta^2}{\beta - Z_0^{-1}dm_\eta^2}\cdot\frac{\beta - Z_0^{-1}dm_X^2}{\alpha - cm_X^2} \tag{48a}$$

$$= -\frac{(4f_K m_K^2 - f_\pi m_\pi^2) - (4f_K - f_\pi)m_\eta^2}{(f_K m_K^2 - f_\pi m_\pi^2) - Z_0^{-1}(f_K - f_\pi)m_\eta^2} \times\frac{(f_K m_K^2 - f_\pi m_\pi^2) - Z_0^{-1}(f_K - f_\pi)m_X^2}{(4f_K m_K^2 - f_\pi m_\pi^2) - (4f_K - f_\pi)m_X^2}, \tag{48b}$$

using the definitions of α, β, c, and d given in Eq. (34).

It is clear that if we equate the expressions (47) and (48b) for $\tan^2\theta$, we must conclude either

$$\text{(a)}\quad Z_0^{-1}m_\eta^2 = Z_0^{-1}m_X^2, \tag{49}$$

which means that Z_0 is infinite, since $m_\eta^2 \neq m_X^2$, or

$$\text{(b)}\quad 4f_K m_K^2 - f_\pi m_\pi^2 = (4f_K - f_\pi)m_\eta^2. \tag{50}$$

If Z_0 is infinite, we conclude from Eqs. (40) and (41) that

$$m_0^2 = \frac{4f_K m_K^2 - f_\pi m_\pi^2}{4f_K - f_\pi},$$

which is what we assumed for the form of m_0^2 in Eq. (45). In order to obtain a nontrivial result, we must turn to alternative (b) above. We find from Eq. (50) that

$$\frac{f_K}{f_\pi} = \frac{m_\eta^2 - m_\pi^2}{4(m_\eta^2 - m_K^2)}, \tag{51}$$

which is exactly the solution, Eq. (25a), we arrived at in Sec. III. From Eqs. (43) or (47), we deduce the remarkable fact that θ vanishes. From (43b) we find that

$$m_\kappa^2 = Z_0^{-1}m_X^2, \tag{52}$$

where m_κ is found from Eqs. (11) and (51) to be given by

$$m_\kappa = 1030 \text{ MeV}, \tag{28}$$

just as in Sec. III. We conclude, therefore, that

$$Z_0 = 0.87. \tag{53}$$

In summary, the hypotheses made in this section lead to the following predictions:

(1) The wave-function renormalization constant for the singlet pseudoscalar meson is slightly different from the octet renormalization constants ($Z_{X_0}/Z_\pi = Z_0^{-1} = 1.15$).

(2) The ratio $f_K/[\sqrt{2}\, f_+(0) f_\pi]$ is predicted to be 1.274, in good agreement with the experimental value[6] 1.27 ± 0.03.

(3) The ratio f_κ/f_π is predicted to be 0.274, in good agreement with independent[7] theoretical predictions of 0.30.

(4) The mass of the κ is predicted to be 1030 MeV, in reasonable agreement with analyses of the K-π scattering data.[8]

(5) The η-X mixing angle is predicted to be zero.

The last prediction is at first sight a puzzling one, in view of the fact that one is used to the idea of a mixing angle of about 10°. It should be remembered, however, that the idea of a mixing angle arises only from a desire to make the Gell-

Mann–Okubo formula exact. In fact, it was postulated[10] that the deviation from the Gell-Mann–Okubo formula observed for the pseudoscalar octet was due to mixing between the η meson and a heavier unitary singlet pseudoscalar meson *before the discovery of the X meson.* In this sense, one could regard the idea of nonzero mixing as a theoretical prejudice rather than a theoretical necessity.

Recently, many authors,[11] using chiral Lagrangian techniques, have found a mixing angle θ very close to zero. Their estimates of θ have varied slightly by one or two degrees only because of differences in the input masses of the pseudoscalar mesons, i.e., whether the charged, neutral, or average masses of the π and K were used. (It should be noted, however, that some authors seem to assume equal renormalization constants for the pseudoscalar nonet. If this is so, then from the arguments presented in Secs. II and IV, a consistent treatment of their work would lead to an unacceptable prediction for f_K/f_π.)

One could regard this calculation as more exact because it does not depend on the actual numerical values of the masses and, therefore, a logical extension of some of the previous calculations as far as the estimate of θ is concerned.

It was noted by Carruthers and Haymaker[9] in their work that the η seemed more purely octet than expected, and they ascribed this to higher-order corrections. (We would prefer to say that the η is pure octet when one takes account of *both* octet breaking in the interaction *and* octet breaking of the vacuum.) In this connection, it is interesting to note that, in a recent paper, Gürsey and Serdaroǧlu[12] used a nonlinear realization of SU(3) $\times$ SU(3) for pseudoscalar mesons, and obtained a mass formula for the pseudoscalar mesons different from the Gell-Mann–Okubo formula. In their work, the symmetry-breaking term for the masses contained not only an octet part, but also a part transforming as the $\underline{27}$ representation of SU(3). The formula they obtained was

$$m_\eta^{\,2} = \tfrac{4}{3} m_K^{\,2} - m_\pi^{\,2}, \tag{54}$$

which leads to a mass of 550 MeV for the η using the average squared masses for $m_\pi^{\,2}$ and $m_K^{\,2}$ quoted in Secs. III and IV. It is reassuring to find that different calculations using different chiral-Lagrangian techniques all predict a mixing angle θ very close to zero.

Let us finally turn our attention to the main problem[13] that seems to confront a theory which predicts zero η-X mixing—or for that matter any of the numerous theories that predict a very small mixing angle. This problem is the enhancement of the $\eta \to 2\gamma$ rate compared with the SU(3) prediction of this rate from the $\pi^0 \to 2\gamma$ rate. Traditionally, this enhancement is explained by invoking η-X mixing, despite the fact that the absolute decay rate for $X \to 2\gamma$ is not known *and* the mixing angle is not known from experiment. However, other explanations can be envisaged.

(1) There may be a substantial violation of SU(3) for $P \to 2\gamma$ decays. A possible mechanism for this could be given as follows: In the presence of electromagnetism, the neutral PCAC equations contain a normal term [see Eqs. (2) and (17)] together with an anomalous term[14] which is mainly responsible for $\pi^0 \to 2\gamma$ decay. One may apply the Sutherland-Veltman[15] theorem to the "normal" terms in $\partial_\mu A_\mu^3$ and $\partial_\mu A_\mu^8$, concluding that the normal terms lead to vanishing amplitudes as the meson masses go to zero. However, when one extrapolates from zero mass to m_η, it is quite possible that one may obtain a significant contribution[16] from the normal term to the decay rate $\eta \to 2\gamma$. This may be contrasted with the decay rate $\pi^0 \to 2\gamma$, where the contribution of the normal term seems to be insignificant compared with the anomalous contribution. In this way, the considerable mass difference between the π and the η could lead to a significant violation of the SU(3) prediction.

(2) In addition to conventional electromagnetic interactions, there may be a term in the Hamiltonian density, Eq. (1), proportional to σ_3 which would violate isospin invariance. The existence of such a term has been used by Coleman and Glashow[17] in order to calculate electromagnetic mass differences. Such a term has also been used in certain theories[18] which attempt to determine the Cabibbo angle. Oakes's theory, in particular, uses such a term not only to determine the Cabibbo angle, but also to determine[19] the decay rate $\eta \to 3\pi$ (without invoking η - X mixing), to a good approximation. The existence of such a direct isospin-violating term may also be used to enhance the $\eta \to 2\gamma$ decay rate above the usual SU(3) prediction.

(3) A combination of the above mechanisms may be responsible for enhancement of $\eta \to 2\gamma$ above the usual SU(3) prediction.

VI. SUMMARY AND DISCUSSION

Certain properties of the scalar and pseudoscalar mesons have been investigated assuming the $(3, \bar{3}) + (\bar{3}, 3)$ model for chiral-symmetry breaking. In particular, the approach of Glashow and Weinberg has been followed.

In order to obtain reasonable agreement with experiment, we found that the constraint of equal wave-function renormalization constants for both the singlet and octet pseudoscalar interpolating fields had to be dropped. In order to make predictions, an assumption was made about the unmixed mass of the η taking into account both octet breaking of the interactions and octet breaking of the vacuum.

From this assumption, it was concluded that predictions for f_K/f_π and m_K could be made in good agreement with experiment, while the prediction for f_κ/f_π was in good agreement with an independent theoretical estimate for this quantity. The η-X mixing angle was predicted to be zero, and it was noted that various authors have recently predicted a mixing angle close to zero.

It was argued that the main problem to be faced by a zero mixing angle at the present time was the decay rate $\eta \to 2\gamma$. Mechanisms other than η-X mixing were conjectured for this enhancement.

The reason for the complete decoupling of the X from the other pseudoscalar mesons (a decoupling which is hinted at in the works of other authors) is not answered here. In the SU(3)-symmetric limit such a decoupling would occur, but it is not clear why it should occur when SU(3) symmetry is broken. It is quite possible, of course, that this conclusion is incorrect (a) because the model we assumed is too simple and therefore wrong or (b) because the assumption made for the form of the unmixed mass of the η is incorrect.

Support for the conclusion of zero mixing would follow if reliable calculations of $\eta \to 2\gamma$ and $\eta \to 3\pi$ could be carried out, perhaps along the lines suggested. Such calculations, however, have proved notoriously difficult in the past.

It is sometimes thought that the ninth pseudoscalar meson is the $E(1422)$, which may have the same quantum numbers as the η (see Ref. 8). This would in no way change our conclusion that the η is decoupled from the singlet. The only change would be in the value of the renormalization constant for the "X" ($Z_\pi/Z_E \approx 1.4$, which is interestingly close to the value of $\sqrt{2}$ suggested by Glashow[1] on quite different grounds).

An additional complication would arise if *both* the E and the X had the same quantum numbers as the η. The existence of ten pseudoscalar mesons seems theoretically unpalatable but could be possible.

It is hoped that in the near future experimental and theoretical work will clarify the situation. Certainly it seems an important matter to measure the absolute decay rate $X \to 2\gamma$, and to determine the quantum numbers of the X and the E with certainty.

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*Permanent address: Theoretical Physics, Research School of Physical Sciences, The Australian National University, Canberra, Australia.

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