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[26] For example, in the quark model we can write a $|\Delta S|=1$ tensor density in the form $(\bar{\mathcal{P}}\sigma_{\alpha\beta}\lambda)$. The corresponding vector current density $\partial_\beta[\bar{\mathcal{P}}(x)\sigma_{\alpha\beta}\lambda(x)]$ is first-class. A pseudotensor density is given by $(\bar{\mathcal{P}}\sigma_{\alpha\beta}\gamma_5\lambda)$, giving rise to the second-class current density $\partial_\beta[\bar{\mathcal{P}}(x)\sigma_{\alpha\beta}\gamma_5\lambda(x)]$. See Ref. 12.
[27] This has a χ^2 probability of 10^{-10}. Several heuristic attempts were made to include effects of correlations between A_ν, A_e, and A_p as well as influences of possible systematic errors in the data. Minima always occurred at essentially the same place in the parameter space, and in no case could we obtain a χ^2 probability better than 10^{-5}.
[28] It should be noted that we have used in this section only experimental information about Λ beta decay. Further constraints on exotic solutions like (9) could be obtained by assuming SU(3) symmetry and using additional information from other decays. Here we do not pursue this matter.

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Radiative Corrections to Deep-Inelastic Neutrino-Nucleon Scattering*

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A simple parton model is used to estimate the radiative corrections to neutrino-induced inclusive processes. An application of the resulting expressions to $\nu_\mu + p \to \mu^- + X$ at $E_\nu^{\rm LAB} = 100$ GeV shows that the muon spectrum is distorted by as much as 10% in some regions.

I. INTRODUCTION

The results from deep-inelastic, inclusive neutrino-nucleon scattering experiments which are in progress or planned for the near future will be an important input for current theoretical work. The effects of radiative corrections must be considered in interpreting these experimental results.[1]

Unfortunately, it is impossible to calculate the radiative corrections to an inclusive process which is controlled by unspecified dynamics. There are two reasons for this. First, the long-wavelength photons are sensitive to changes in the large-scale distribution of electric charges and currents. This information is not available unless the general features of the hadronic final state are specified. Second, the short-wavelength photons are sensitive to details of the current distribution in the interaction region. Again, this information is not available in the absence of a theory for the basic interaction. Thus, in order to estimate radiative corrections, we need a model which specifies the electromagnetic currents in some detail. We will use the parton model.[2]

In this model, the nucleon target is to be viewed as a collection of weakly bound, relatively light point particles. The neutrino is assumed to have a weak interaction with one of these target partons. In the deep-inelastic region, this parton gets a large acceleration, and the leptonic system suffers a large reaction. The other partons are assumed to receive accelerations much smaller than that of the leptonic system or the struck parton.

Classical intuition suggests that the charges which are accelerated the most will make the major contribution to the radiative correction. Thus, we will consider only contributions where the photon is attached to the struck parton or the outgoing muon, and we will sum over the partons incoherently as usual.

This is analogous to the usual practice of calculating radiative corrections by considering only the proton in the target which is struck and then summing incoherently over the protons in the target. This restriction of the number of Feynman graphs is gauge-invariant so long as we ignore the interactions between the partons.

For the purposes of this calculation, we will assume further that the final-state interactions which "dress" the outgoing parton give a jet of outgoing physical particles which have the same charge and essentially the same momentum as the parton. The very long-wavelength photons will not be sensitive to the difference between a single particle and a jet of particles with the same charge and with small average momentum transverse to the jet direction.[3] The short-wavelength photons, which see better, are coupled most strongly to the region of the primary violent interaction of the bare particles rather than to the relatively smooth current distributions of the final state. This primary interaction to which the high-energy photons are most sensitive is taken to be a pointlike Fermi interaction between the leptons and the parton.

This model is very crude. We stress that the results which it gives should be considered semiquantitatively at most. The approximations of the model are probably reasonable only for the very long- and the very short-wavelength photons. However, it is these regions of the integration over photon momentum which are most important. Thus, we expect to reproduce the gross features of the radiative corrections correctly.

The situation is somewhat simpler in electroproduction. There it is possible to separate out the radiative corrections to the electron line in a gauge-invariant way. The problem of radiative corrections to the photon-parton interaction in electroproduction has not been faced.

In Sec. II, we calculate the basic cross sections. Sections III, IV, and V calculate the contributions from the self-energy, vertex, and bremsstrahlung graphs, respectively. In Sec. VI, these results are combined and numerical results for $\nu p \to \mu^- +$ anything and $\bar{\nu} p \to \mu^+ +$ anything at $E_\nu = 100$ GeV are given. When considered as a function of muon energy at fixed lab angle, the cross section is typically decreased by about 10% at large muon energies and increased by about 10% at small muon energies by the radiative corrections.

II. BASIC CROSS SECTION

The calculation will be carried out in the following way: First, we assume that the partons are quarks and gluons. The gluons are assumed to have no weak or electromagnetic interactions. The small size of $\sin^2\theta_{\text{Cabibbo}}$ will allow us to neglect the λ and $\bar{\lambda}$ quarks. Thus, we are interested in the $\sigma(\nu\mathcal{P})$, $\sigma(\nu\mathfrak{N})$, $\sigma(\nu\bar{\mathcal{P}})$, $\sigma(\nu\bar{\mathfrak{N}})$, $\sigma(\bar{\nu}\mathcal{P})$, $\sigma(\bar{\nu}\mathfrak{N})$, $\sigma(\bar{\nu}\bar{\mathcal{P}})$, $\sigma(\bar{\nu}\bar{\mathfrak{N}})$ neutrino-quark cross sections. Charge conservation and the spectrum of quark charges give

$$\sigma(\nu\mathcal{P}) = \sigma(\nu\bar{\mathfrak{N}}) = \sigma(\bar{\nu}\mathfrak{N}) = \sigma(\bar{\nu}\bar{\mathcal{P}}) = 0 .$$

We will assume CP invariance and get

$$\sigma(\nu\mathfrak{N}) = \sigma(\bar{\nu}\bar{\mathfrak{N}})$$

and

$$\sigma(\nu\bar{\mathcal{P}}) = \sigma(\bar{\nu}\mathcal{P})$$

for the spin-averaged cross sections. Note that these relations hold even with a final-state photon whose polarization has been summed over. Thus, we need only calculate two cross sections:

$$\sigma_{\text{I}} = \sigma(\nu\mathfrak{N})$$

and

$$\sigma_{\text{II}} = \sigma(\nu\bar{\mathcal{P}}) .$$

We will describe the calculation of case I only. Case II is very similar and we quote only the final results.

After calculating the cross sections, we let the parton momentum p_1 go to xP_1, where P_1 is the target nucleon momentum, multiply each of the cross sections by the parton distribution function $F(x)$ appropriate to that kind of parton, integrate over x, and sum over parton types. (In electroproduction,

$$\nu W_2(x) = x \sum_{\text{parton types}} f_i F_i(x) ,$$

f_i being the charge of a parton of type i.)

We consider the scattering of a muon neutrino ν_μ of momentum k_1 off an $\mathfrak{N}$ quark of momentum p_1, mass m_1, and charge $fe = \frac{1}{3}e$. The final state has a μ^- of momentum k_2, mass m_μ, and charge e, and a $\mathcal{P}$ quark of momentum p_2, mass m_2, and charge $f'e = (f-1)e = -\frac{2}{3}e$. Photons, real or virtual, have momentum k. The graphs which contribute are shown in Fig. 1. The graphs of Fig. 1(a) contribute a cross section[4]

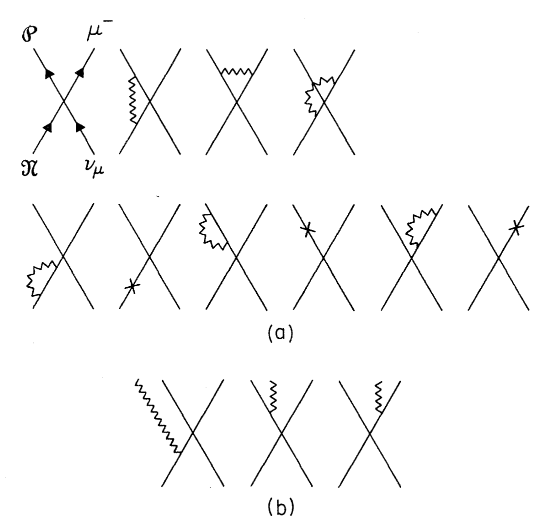

FIG. 1. Graphs contributing to $\nu_\mu\mathfrak{N} \to \mu^-\mathcal{P}$ with radiative corrections.

$$k_{20}\frac{d\Sigma_E}{d^3k_2}=\int_0^1 dx\,F(x)k_{20}\frac{d\sigma_E}{d^3k_2}\,, \tag{1}$$

with the quark cross section

$$k_{20}\frac{d\sigma_E}{d^3k_2}=\frac{1}{2\pi^2}\,\delta((k_1+p_1-k_2)^2-m_2{}^2)\times\theta(k_{10}+p_{10}-k_{20})\frac{m^4M}{k_1\cdot p_1}\,,$$
$$m^4\equiv m_\nu m_\mu m_1 m_2\,; \tag{2}$$

$M\equiv$absolute square of the matrix element for the first three sets of graphs averaged over initial spins, summed over final spins, and evaluated at $p_2=\Delta\equiv k_1+p_1-k_2$. This contains an $m_\nu{}^{-1}$ which cancels the m_ν in m^4 after which we take $m_\nu\to 0$.

The bremsstrahlung graphs of Fig. 1(b) contribute a cross section

$$k_{20}\frac{d\Sigma_{\rm IE}}{d^3k_2}=\int_0^1 dx\,F(x)k_{20}\frac{d\sigma_{\rm IE}}{d^3k_2}\,, \tag{3}$$

with

$$k_{20}\frac{d\sigma_{\rm IE}}{d^3k_2}=\frac{1}{(2\pi)^5}\int\frac{d^3k}{k_0}\,\delta((\Delta-k)^2-m_2{}^2)\times\theta(\Delta_0-k_0)\frac{m^4N}{k_1\cdot p_1}\,, \tag{4}$$

and $N\equiv$absolute square of the matrix element for the bremsstrahlung processes appropriately summed and averaged over spins and evaluated at $p_2=\Delta-k$.

With these preliminaries out of the way, we proceed with the purpose of this section which is to calculate M_0, the contribution to M from the graph of Fig. 2:

$$\mathfrak{M}_0=\frac{G}{\sqrt{2}}\bar u(p_2)\gamma_\lambda(1-\gamma_5)u(p_1)\,\bar u(k_2)\gamma^\lambda(1-\gamma_5)u(k_1)\,. \tag{5}$$

We square this, sum over initial- and final-state spins, and divide by 2 for the average over quark spins. (There is no dividing by 2 for the neutrino since only one spin state contributes.) The result is

$$M_0=\frac{G^2}{2}\frac{1}{m^4}\,8k_1\cdot p_1 k_2\cdot p_2\,. \tag{6}$$

III. SELF-ENERGY CORRECTIONS

In this section we consider the contribution from the graphs of Fig. 3. The contribution which they make to M will be $2\,\mathrm{Re}\mathfrak{M}_0^*(\mathfrak{M}_1+\mathfrak{M}_2+\mathfrak{M}_3)$. We get $\mathfrak{M}_1$ from $\mathfrak{M}_0$ by the replacement

$$u(p_1)\to\frac{\not p_1+m_1}{p_1{}^2-m_1{}^2}\,[f^2\Sigma(p_1)-\delta m_1]u(p_1)\,, \tag{7}$$

with

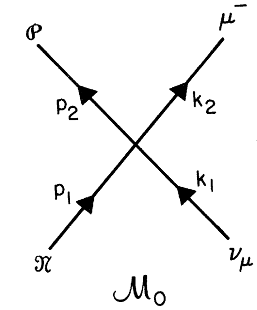

FIG. 2. Feynman graph for the uncorrected process $\nu_\mu\mathfrak{N}\to\mu^-\mathcal{P}$.

$$\Sigma(p)\equiv-\frac{i\alpha}{4\pi^3}\int d^4k\,\frac{\gamma^\mu(\not p-\not k+m)\gamma_\mu}{k^2[(p-k)^2-m^2]}\,.$$

$\Sigma(p)$ is calculated by the regularization procedure

$$\Sigma(p)=\frac{-i\alpha}{4\pi^3}\int d^4k\,\frac{\gamma^\mu(\not p-\not k+m)\gamma_\mu}{k^2[(p-k)^2-m^2]}\to\lim_{\substack{\Lambda\to\infty\\ \lambda\to 0}}\left[\frac{-i\alpha}{4\pi^3}\int d^4k\,\frac{\gamma^\mu(\not p-\not k+m)\gamma_\mu}{(k^2-\lambda^2)[(p-k)^2-m^2]}-\frac{-i\alpha}{4\pi^3}\int d^4k\,\frac{\gamma^\mu(\not p-\not k+m)\gamma_\mu}{(k^2-\Lambda^2)[(p-k)^2-m^2]}\right].$$

The calculation must be carried out for $p^2\neq m^2$. Only after $\Sigma(p)$ is inserted between the spinor and the propagator do we take $p^2=m^2$. As is usual we write

$$\Sigma(p)=A+B(\not p-m)+C(\not p-m)^2\,.$$

A and B are numbers independent of p. C is a 4×4 matrix finite at $\Lambda\to\infty$ and $p^2\to m^2$. Thus, between a propagator $1/(\not p-m)$ and a spinor $u(p)$, the C term will not contribute. A standard calculation gives

$$A=\frac{\alpha}{\pi}\frac{3m}{2}\left(\ln\frac{\Lambda}{m}+\tfrac{1}{4}\right),$$
$$B=-\frac{\alpha}{2\pi}\left(\tfrac{9}{4}+\ln\frac{\Lambda}{m}+\ln\frac{\lambda^2}{m^2}\right).$$

The contribution from A is canceled by taking $\delta m=A$.

The contribution from the B term appears as $B(\not p-m)^{-1}(\not p-m)u(p)$, which is undefined. This

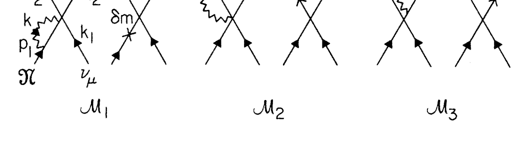

FIG. 3. The self-energy graphs.

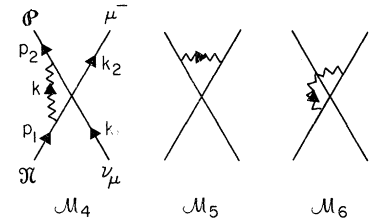

FIG. 4. The vertex graphs.

is resolved, as usual, by identifying the wave-function renormalization in this order and taking $B(\not{p}-m)^{-1}(\not{p}-m)u(p)=\frac{1}{2}Bu(p)$. Thus, the contribution to M from the self-energy graphs is

$$-\frac{\alpha}{\pi}\left[f^2\left(\tfrac{9}{8}+\tfrac{1}{2}\ln\frac{\Lambda}{m_1}+\ln\frac{\lambda}{m_1}\right)+f'^2\left(\tfrac{9}{8}+\tfrac{1}{2}\ln\frac{\Lambda}{m_2}+\ln\frac{\lambda}{m_2}\right)\right.$$
$$\left.+\left(\tfrac{9}{8}+\tfrac{1}{2}\ln\frac{\Lambda}{m_\mu}+\ln\frac{\lambda}{m_\mu}\right)\right]M_0\,. \quad (8)$$

IV. VERTEX CORRECTIONS

In this section, we consider the graphs of Fig. 4. These contribute to M in the combination

$$2\,\mathrm{Re}\mathfrak{M}_0^*(\mathfrak{M}_4+\mathfrak{M}_5+\mathfrak{M}_6)\,.$$

We will sketch the treatment of $\mathfrak{M}_5$; $\mathfrak{M}_4$ and $\mathfrak{M}_6$ are very similar.

$$\mathfrak{M}_5=\frac{G}{\sqrt{2}}\frac{-i\alpha f'}{4\pi^3}\int d^4k\frac{1}{k^2}\frac{1}{(p_2-k)^2-m_2{}^2}\frac{1}{(k_2+k)^2-m_\mu{}^2}$$
$$\times\bar{u}(p_2)\gamma^\mu(\not{p}_2-\not{k}+m_2)\gamma_\lambda(1-\gamma_5)u(p_1)\times\bar{u}(k_2)\gamma_\mu(\not{k}_2+\not{k}+m_\mu)\gamma^\lambda(1-\gamma_5)u(k_1)\,. \quad (9)$$

Its contribution to M is

$$M_5=\mathrm{Re}\frac{-i\alpha f'}{4\pi^3}\int d^4k\frac{1}{k^2}\frac{1}{(p_2-k)^2-m_2{}^2}\frac{1}{(k_2+k)^2-m_\mu{}^2}T_5\,,$$

with

$$T_5\equiv\frac{G^2}{2}\frac{1}{m^4}\tfrac{1}{4}\mathrm{Tr}[\gamma_\nu(1-\gamma_5)(\not{p}_2+m_2)\gamma^\mu(\not{p}_2-\not{k}+m_2)\gamma_\lambda(1-\gamma_5)(\not{p}_1+m_1)]$$
$$\times\tfrac{1}{4}\mathrm{Tr}[\gamma^\nu(1-\gamma_5)(\not{k}_2+m_\mu)\gamma_\mu(\not{k}_2+\not{k}+m_\mu)\gamma^\lambda(1-\gamma_5)\not{k}_1]\,.$$

M_5 contains both infrared and ultraviolet divergences. We regulate by taking

$$\frac{1}{k^2}\to\lim_{\substack{\Lambda\to\infty\\ \lambda\to 0}}\left(\frac{1}{k^2-\lambda^2}-\frac{1}{k^2-\Lambda^2}\right)$$

as before. The use of Feynman parameters gives

$$M_5=\mathrm{Re}\frac{-i\alpha f'}{4\pi^3}\int_0^1\int_0^1 dz_1dz_2\delta(1-z_1-z_2)\int_0^1\int_0^1\int_0^1 dx_1dx_2dx_3\delta(1-x_1-x_2-x_3)\int d^4k\frac{T_5'}{(k^2-D_5{}^2-x_1\lambda^2+i\epsilon)^3}-(\lambda\to\Lambda)\,,$$

with

$$D_5\equiv(x_2+x_3)C_5\,,\qquad C_5\equiv(z_1p_2-z_2k_2)\,,$$

and

$$T_5'\equiv 8k_2\cdot p_2M_0-8k^2M_0-8D_5{}^2M_0+\frac{32}{m^4}k_1\cdot p_1(m_\mu{}^2D_5\cdot p_2-m_2{}^2D_5\cdot k_2+2D_5\cdot p_2k_2\cdot p_2-2D_5\cdot k_2k_2\cdot p_2)$$
$$\equiv A+k^2B\,.$$

After doing the k integration, we find that

$$M_5=-\frac{\alpha}{\pi}\mathrm{Re}\tfrac{1}{4}f'\int_0^1\int_0^1 dz_1dz_2\delta(1-z_1-z_2)\int_0^1\int_0^1\int_0^1 dx_1dx_2dx_3\delta(1-x_1-x_2-x_3)$$
$$\times\left[\frac{A}{(1-x_1)^2C_5{}^2+x_1\lambda^2-i\epsilon}-2B\ln\left|\frac{(1-x_1)^2C_5{}^2+x_1\Lambda^2-i\epsilon}{(1-x_1)^2C_5{}^2+x_1\lambda^2-i\epsilon}\right|\right].$$

We now carry out the x_2 and x_3 integrals and change variables to $x\equiv 1-x_1$. A can be separated as $A=A_0+xA_1+x^2A_2$. The result is

$$M_5=-\frac{\alpha}{\pi}\tfrac{1}{4}f'\,\mathrm{Re}\int_0^1\int_0^1 dz_1dz_2\delta(1-z_1-z_2)\left[\frac{A_1+\frac{1}{2}A_2}{C_5{}^2-i\epsilon}-\tfrac{1}{2}A_0\frac{1}{C_5{}^2-i\epsilon}\ln\frac{\lambda^2}{C_5{}^2-i\epsilon}-B\left(\ln\frac{\Lambda^2}{m_2m_\mu}-\tfrac{1}{2}\right)+B\ln\left|\frac{C_5{}^2-i\epsilon}{m_2m_\mu}\right|\right].$$

At this point, we take advantage of the fact that we are interested in a kinematic region in which $k_A \cdot k_B \gg m_A m_B$, with k_A and k_B typical momenta. Thus, we drop terms with masses and identify

$$B=-8M_0\,,$$

$$A_0=8k_2\cdot p_2 M_0\,,$$

$$A_1+\tfrac{1}{2}A_2 \cong -8(z_1+z_2)k_2\cdot p_2 M_0 - 4C_5{}^2 M_0\,,$$

and

$$\frac{A_1+\frac{1}{2}A_2}{C_5{}^2}=\frac{-8(z_1+z_2)k_2\cdot p_2 M_0}{C_5{}^2}-4M_0\,.$$

After integration the first term goes like $M_0 \ln(k_2\cdot p_2/m_2 m_\mu)$, while the second is, of course, just $\sim M_0$. We will make the approximation of dropping terms which are of order one relative to terms of order $\ln(k_A\cdot k_B/m_A m_B)$.

The result is

$$M_5=-\frac{\alpha}{\pi}f'\,\mathrm{Re}\int_0^1\int_0^1 dz_1 dz_2\,\delta(1-z_1-z_2)M_0\left[2\ln\frac{\Lambda^2}{m_\mu m_2}-2\ln\left|\frac{C_5{}^2-i\epsilon}{m_\mu m_2}\right|-\frac{k_2\cdot p_2}{C_5{}^2-i\epsilon}\ln\frac{\lambda^2}{C_5{}^2-i\epsilon}-\frac{2k_2\cdot p_2}{C_5{}^2-i\epsilon}\right].$$

These integrals must be evaluated with great care. The correct procedure has been given by Yennie, Frautschi, and Suura.[5] The result is

$$M_5=-\frac{\alpha}{\pi}f'\left[2\ln\frac{\Lambda^2}{m_\mu m_2}+\int_0^1\int_0^1 dz_1 dz_2\,\delta(1-z_1-z_2)\frac{k_2\cdot p_2}{\overline{C}_5{}^2}\ln\frac{\lambda^2}{\overline{C}_5{}^2}\right]M_0\,, \tag{10a}$$

with $\overline{C}_5 \equiv z_1 p_2 + z_2 k_2$. Similarly,

$$M_4=-\frac{\alpha}{\pi}ff'\left[-\tfrac{1}{2}\ln\frac{\Lambda^2}{m_1 m_2}-\tfrac{3}{2}\ln\frac{p_1\cdot p_2}{m_1 m_2}-\int_0^1\int_0^1 dz_1 dz_2\,\delta(1-z_1-z_2)\frac{p_1\cdot p_2}{C_4{}^2}\ln\frac{\lambda^2}{C_4{}^2}\right]M_0\,, \tag{10b}$$

$$M_6=-\frac{\alpha}{\pi}f\left[-\tfrac{1}{2}\ln\frac{\Lambda^2}{m_\mu m_1}-\tfrac{3}{2}\ln\frac{k_2\cdot p_1}{m_\mu m_2}-\int_0^1\int_0^1 dz_1 dz_2\,\delta(1-z_1-z_2)\frac{k_2\cdot p_1}{C_6{}^2}\ln\frac{\lambda^2}{C_6{}^2}\right]M_0\,, \tag{10c}$$

Terms of order $(\alpha/\pi)1M_0$ have been dropped.

V. BREMSSTRAHLUNG CONTRIBUTION

In this section we consider the contribution from the graphs of Fig. 5:

$$\mathfrak{N}_1=\frac{G}{\sqrt{2}}\frac{ef}{(p_1-k)^2-m_1{}^2}\,\overline{u}(p_2)\gamma_\lambda(1-\gamma_5)(2\epsilon\cdot p_1-\not k\not\epsilon)u(p_1)\times\overline{u}(k_2)\gamma^\lambda(1-\gamma_5)u(k_1)\,, \tag{11a}$$

$$\mathfrak{N}_2=\frac{G}{\sqrt{2}}\frac{ef'}{(p_2+k)^2-m_2{}^2}\,\overline{u}(p_2)(2\epsilon\cdot p_2+\not\epsilon\not k)\gamma_\lambda(1-\gamma_5)u(p_1)\times\overline{u}(k_2)\gamma^\lambda(1-\gamma_5)u(k_1)\,, \tag{11b}$$

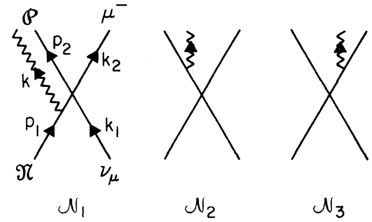

FIG. 5. The bremsstrahlung graphs.

$$\mathfrak{N}_3=\frac{G}{\sqrt{2}}\frac{e}{(k_2+k)^2-m_\mu{}^2}\,\overline{u}(p_2)\gamma_\lambda(1-\gamma_5)u(p_1)\times\overline{u}(k_2)(2\epsilon\cdot k_2+\not\epsilon\not k)\gamma^\lambda(1-\gamma_5)u(k_1)\,. \tag{11c}$$

At $\epsilon=k$, $\mathfrak{N}_1+\mathfrak{N}_2+\mathfrak{N}_3=0$ demonstrating the gauge invariance of the graphs.

Squaring this amplitude, summing, and averaging properly over all spins gives

$$m^4 N=-2\pi\alpha\left[\frac{f^2 U_1}{(k^2-2k\cdot p_1)^2}+\frac{f'^2 U_2}{(k^2+2k\cdot p_2)^2}+\frac{U_3}{(k^2+2k\cdot k_2)^2}+\frac{2ff'U_4}{(k^2-2k\cdot p_1)(k^2+2k\cdot p_2)}+\frac{2fU_5}{(k^2-2k\cdot p_1)(k^2+2k\cdot k_2)}+\frac{2f'U_6}{(k^2+2k\cdot p_2)(k^2+2k\cdot k_2)}\right].$$

This is to be evaluated at $k^2=\lambda^2$ and $p_2=\Delta-k$. In the denominators, the dot products are of order λ when k_0 is of order λ. Thus, since $k^2=\lambda^2$, we can drop the k^2 in the denominators and also in the U's.

The result is

$$m^4N=-2\pi\alpha\left[\frac{f^2U_1}{4(k\cdot p_1)^2}+\frac{f'^2U_2}{4(k\cdot\Delta)^2}+\frac{U_3}{4(k\cdot k_2)^2}-\frac{2ff'U_4}{4k\cdot p_1k\cdot\Delta}-\frac{2fU_5}{4k\cdot p_1k\cdot k_2}+\frac{2f'U_6}{4k\cdot\Delta k\cdot k_2}\right],$$

with (from now on $M_0=M_0|_{p_2=\Delta}$)

$$U_1=8m_1^2M_0+\bar U_1,\quad U_4=8\Delta\cdot p_1M_0+\bar U_4,$$
$$U_2=8m_2^2M_0+\bar U_2,\quad U_5=8k_2\cdot p_1M_0+\bar U_5,$$
$$U_3=8m_\mu^2M_0+\bar U_3,\quad U_6=8\Delta\cdot k_2M_0+\bar U_6.$$

The $\bar U$'s have at least one power of k and thus give infrared-finite contributions. Complete expressions will be quoted in Sec. VI. This identification of the infrared-divergent and infrared-finite parts suggests that we write

$$k_{20}\frac{d\sigma_{IE}}{d^3k_2}=k_{20}\frac{d\sigma_{IE}^{IR}}{d^3k_2}+k_{20}\frac{d\sigma_{IE}^F}{d^3k_2},\tag{12}$$

where σ^{IR} is computed with

$$m^4N=m^4N^{IR},$$

$$m^4N^{IR}\equiv-4\pi\alpha M_0\left[\frac{f^2m_1^2}{(k\cdot p_1)^2}+\frac{f'^2m_2^2}{(k\cdot\Delta)^2}+\frac{m_\mu^2}{(k\cdot k_2)^2}-\frac{2ff'\Delta\cdot p_1}{k\cdot\Delta k\cdot p_1}-\frac{2fk_2\cdot p_1}{k\cdot k_2k\cdot p_1}+\frac{2f'\Delta\cdot k_2}{k\cdot\Delta k\cdot k_2}\right],$$

and σ^F is computed using the $\bar U$'s.

Now we must extract the infrared divergence from

$$k_{20}\frac{d\Sigma_{IE}^{IR}}{d^3k_2}\equiv\int_0^1dx\,F(x)k_{20}\frac{d\sigma_{IE}^{IR}}{d^3k_2},\tag{13}$$

where $k_{20}d\sigma_{IE}^{IR}/d^3k_2$ is now evaluated with $p_1=xP_1$, $m_1=xM_H$, and M_H is the mass of the nucleon:

$$k_{20}\frac{d\Sigma_{IE}^{IR}}{d^3k_2}=\int_0^1dx\,F(x)\frac{1}{(2\pi)^5}\times\int\frac{d^3k}{k_0}\delta[(\Delta-k)^2-m_2^2]\theta(\Delta_0-k_0)\frac{m^4N^{IR}}{k_1\cdot p_1},\tag{14}$$

where $k_0=+(|\vec k|^2+\lambda^2)^{1/2}$. We begin by splitting up the $\int d|\vec k|$:

$$\int\frac{d^3k}{k_0}=\int\frac{d|\vec k|d\Omega_{\hat k}|\vec k|^2}{k_0}=\left[\int_0^\epsilon d|\vec k|+\int_\epsilon^\infty d|\vec k|\right]\int d\Omega_{\hat k}\frac{|\vec k|^2}{k_0},$$

with $\lambda\ll\epsilon$, but $\epsilon\to0$ after $\lambda\to0$. Thus, in the second integral, we take $\lambda=0$ with no problem. Then, $k_{20}d\Sigma_{IE}^{IR}/d^3k_2=I_1+I_2$, with

$$I_1=\int_0^1dx\,F(x)\frac{1}{(2\pi)^5}\int_{\substack{|\vec k|<\epsilon\\k^2=\lambda^2}}\frac{d^3k}{k_0}\delta[(\Delta-k)^2-m_2^2]\times\theta(\Delta_0-k_0)\frac{m^4N^{IR}}{k_1\cdot p_1},$$

$$I_2=\int_0^1dx\,F(x)\frac{1}{(2\pi)^5}\int_{\substack{|\vec k|>\epsilon\\k^2=0}}\frac{d^3k}{k_0}\delta[(\Delta-k)^2-m_2^2]\times\theta(\Delta_0-k_0)\frac{m^4N^{IR}}{k_1\cdot p_1}.$$

I_1 is evaluated by using the δ function to do the x integration. We can then set $k=0$ except in the denominators of N^{IR}, parametrize these denominators, and carry out the $\int_{|\vec k|<\epsilon;k^2=\lambda^2}d^3k$.

For I_2 we have

$$I_2=\int_0^1dx\frac{F(x)}{(2\pi)^5}\frac{1}{(\Delta^2-m_2^2)}\int d\Omega_{\hat k}\int_\epsilon^{\Delta_0}dk_0\delta\left(\frac{\Delta^2-m_2^2}{2(\Delta^0-\hat k\cdot\vec\Delta)}-k_0\right)\frac{k_0^2m^4N^{IR}}{k_1\cdot p_1}$$
$$=\int_0^1dx\frac{F(x)}{(2\pi)^5}\frac{1}{(\Delta^2-m_2^2)}\int d\Omega_{\hat k}\frac{k_0^2m^4N^{IR}}{k_1\cdot p_1}\int_\epsilon^{\Delta^0}dk_0\delta\left(\frac{\Delta^2-m_2^2}{2(\Delta^0-\hat k\cdot\vec\Delta)}-k_0\right),$$

since $k_0^2m^4N^{IR}/k_1\cdot p_1$ is independent of k_0. The evaluation of I_2 is begun with a further split: $I_2=I_{2A}+I_{2B}$, where

$$I_{2A}=\int_0^1dx\frac{1}{(2\pi)^5}\frac{1}{\Delta^2-m_2^2}\int d\Omega_{\hat k}\left[\frac{k_0^2m^4N^{IR}}{k_1\cdot p_1}-\frac{k_0^2m^4N^{IR}}{k_1\cdot p_1}\bigg|_{x=x_+}\right]\int_\epsilon^{\Delta^0}dk_0\delta\left(\frac{\Delta^2-m_2^2}{2(\Delta^0-\hat k\cdot\vec\Delta)}-k_0\right),$$

$$I_{2B}=\int_0^1dx\frac{1}{(2\pi)^5}\frac{1}{\Delta^2-m_2^2}\int d\Omega_{\hat k}\frac{k_0^2m^4N^{IR}}{k_1\cdot p_1}\bigg|_{x=x_+}\int_\epsilon^{\Delta^0}dk_0\delta\left(\frac{\Delta^2-m_2^2}{2(\Delta^0-\hat k\cdot\vec\Delta)}-k_0\right).$$

x_+ is the positive root of $\Delta^2-m_2^2=0$. (Recall Δ depends on x through $p_1=xP_1$.) In I_{2A}, we can take $\epsilon=0$, do the $\int dk_0$ calculation, and get

$$I_{2A}=\int_0^1dx\frac{\theta(\Delta^2-m_2^2)}{(2\pi)^5(\Delta^2-m_2^2)}\int d\Omega_{\hat k}\left[\frac{k_0^2m^4N^{IR}}{k_1\cdot p_1}-\frac{k_0^2m^4N^{IR}}{k_1\cdot p_1}\bigg|_{x=x_+}\right],$$

which is an invariant (although not manifestly so as written). I_{2B} is evaluated by parametrizing the denominators in N^{IR} and carrying out the $\int d\Omega$ and $\int dk_0$. The calculation is best done in the rest frame of the resulting parametrized four vector.

Even with this simplification, the calculation is rather messy. However, in combining I_{2B} with I_1, the dependence on ϵ cancels out, as it must, and the result is covariant.

Considerable simplification results from dropping terms of order one relative to logs. The parametric integrals are handled using the techniques of Ref. 5. We also drop terms which have an extra $1/q\cdot P_1$ as they are small in the deep-inelastic region. The result is

$$k_{20}\frac{d\Sigma_{IE}^{IR}}{d^3k_2}=\frac{\alpha}{\pi}\frac{F(x_+)M_0}{2\pi^2 M_H{}^2(x_+-x_-)k_1\cdot p_1}\bigg|_{x=x_+}(A+B+C)\,, \tag{15a}$$

$$A=\int_{x_+}^1 dx\frac{1}{x-x_+}\left[\frac{M_0F(x)}{k_1\cdot p_1}\bigg|_{x=x_+}\right]^{-1}\left\{\frac{M_0F(x)}{k_1\cdot p_1}\left[ff'\ln\frac{(\Delta\cdot p_1)^2}{m_1{}^2\Delta^2}+f\ln\frac{(k_2\cdot p_1)^2}{m_1{}^2m_\mu{}^2}-f'\ln\frac{(\Delta\cdot k_2)^2}{m_\mu{}^2\Delta^2}\right]-(x\to x_+)\right\}, \tag{15b}$$

$$B=f^2\ln\frac{\lambda}{m_1}+f'^2\ln\frac{\lambda}{m_2}+\ln\frac{\lambda}{m_\mu}-\int_0^1\int_0^1 dz_1dz_2\delta(1-z_1-z_2)\left(ff'\frac{\Delta\cdot p_1}{C_4{}^2}\ln\frac{\lambda^2}{C_4{}^2}+f\frac{k_2\cdot p_1}{C_6{}^2}\ln\frac{\lambda^2}{C_6{}^2}-f'\frac{\Delta\cdot k_2}{\bar{C}_5{}^2}\ln\frac{\lambda^2}{\bar{C}_5{}^2}\right), \tag{15c}$$

$$\begin{aligned}C=&-f^2\left[-\ln\frac{2\Delta\cdot p_1}{m_1m_2}+\ln(1-x_+)+\ln\frac{2q\cdot P_1}{M_H{}^2}+\ln\frac{M_H{}^2}{m_1m_2}\right]\\&-f'^2\left[\ln(1-x_+)+\ln\frac{2q\cdot P_1}{M_H{}^2}+\ln\frac{M_H{}^2}{m_2{}^2}\right]-\left[-\ln\frac{2\Delta\cdot k_2}{m_2m_\mu}+\ln(1-x_+)+\ln\frac{2q\cdot P_1}{M_H{}^2}+\ln\frac{M_H{}^2}{m_2m_\mu}\right]\\&-ff'\left\{\ln\frac{2\Delta\cdot p_1}{m_2m_1}\left[\tfrac{3}{2}\ln\frac{2\Delta\cdot p_1}{m_2m_1}-2\ln\frac{2q\cdot P_1}{M_H{}^2}-2\ln(1-x_+)+\ln\frac{m_1m_2}{M_H{}^2}+2\ln\frac{m_2}{M_H}\right]-\tfrac{1}{2}\ln^2\frac{m_2}{m_1}\right\}\\&-f\left\{\ln\frac{2k_2\cdot p_1}{m_\mu m_1}\left[\ln\frac{2\Delta\cdot k_2}{m_2m_\mu}+\ln\frac{2\Delta\cdot p_1}{m_2m_1}-2\ln\frac{2q\cdot P_1}{M_H{}^2}+\ln\frac{m_1m_\mu}{M_H{}^2}-2\ln(1-x_+)+2\ln\frac{m_2}{M_H}\right]+\ln\frac{\Delta\cdot k_2}{\Delta\cdot p_1}\ln\frac{m_1}{m_\mu}+\tfrac{1}{2}\ln^2\frac{\Delta\cdot k_2}{\Delta\cdot p_1}\right\}\\&+f'\left\{\ln\frac{2\Delta\cdot k_2}{m_2m_\mu}\left[\tfrac{3}{2}\ln\frac{2\Delta\cdot k_2}{m_2m_\mu}-2\ln\frac{2q\cdot P_1}{M_H{}^2}-2\ln(1-x_+)+\ln\frac{m_\mu m_2}{M_H{}^2}+2\ln\frac{m_2}{M_H}\right]-\tfrac{1}{2}\ln^2\frac{m_2}{m_\mu}\right\}.\end{aligned} \tag{15d}$$

B and C are both to be considered evaluated at $x=x_+$:

$$x_\pm=-\frac{q\cdot P_1}{M_H{}^2}\pm\left[\left(\frac{q\cdot P_1}{M_H{}^2}\right)^2-\frac{q^2-m_2{}^2}{M_H{}^2}\right]^{1/2},\qquad \lim_{\text{Bj}}x_+=\frac{-q^2}{2q\cdot P_1}\,.$$

VI. RESULTS

In this section we will combine our results from Secs. I–V. Recall that the graphs of Fig. 1(a) give

$$\begin{aligned}k_{20}\frac{d\Sigma_E}{d^3k_2}&=\int_0^1 dx\,F(x)\frac{1}{2\pi^2}\,\delta(\Delta^2-m_2{}^2)\theta(\Delta_0)\frac{m^4M}{k_1\cdot p_1}\\&=\frac{F(x_+)}{2\pi^2M_H{}^2(x_+-x_-)}\frac{m^4M}{k_1\cdot p_1}\bigg|_{x=x_+}.\end{aligned}$$

The contributions to M from the basic, self-energy, and vertex graphs have been given in Secs. II, III, and IV. The bremsstrahlung graphs give

$$k_{20}\frac{d\Sigma_{IE}}{d^3k_2}=k_{20}\frac{d\Sigma_{IE}^{IR}}{d^3k_2}+k_{20}\frac{d\Sigma_{IE}^{F}}{d^3k_2}\,.$$

We can now make the gratifying observation that the λ-dependent terms in Σ_{IE}^{IR} cancel those in Σ_E. Not so gratifying, however, is the fact that the Λ-dependent terms do not cancel. Since these terms are the same in case II as we just obtained in case I, we can interpret them as a renormalization of G by writing

$$\begin{aligned}\frac{G^2}{2}\left(1+2\frac{\alpha}{\pi}\ln\frac{\Lambda}{M_H}+\cdots\right)&\cong\tfrac{1}{2}\left[G\left(1+\frac{\alpha}{\pi}\ln\frac{\Lambda}{M_H}\right)\right]^2\\&\quad\times(1+\cdots)\\&=\frac{G'^2}{2}(1+\cdots)\,,\end{aligned}$$

with

$$G'\equiv G\left(1+\frac{\alpha}{\pi}\ln\frac{\Lambda}{M_H}\right). \tag{16}$$

G' is interpreted as the renormalized weak coupling constant to be identified with the observed coupling constant in a reaction such as β decay, where the Λ-dependent parts of the radiative correction are the same. (This identification may not be justified since the model we use is not actually applicable to a low-energy process such as β decay.)

We are also troubled by the explicit appearance of m_2, the quark mass. Its presence reflects the uncertainties and ambiguities of the parton model such as neglecting the transverse momenta of the partons. Deep-inelastic ep scattering suggests that $m_2 << M_H$. However, we cannot put $m_2 = 0$ because it appears essentially as $\ln(m_2/M_H)$. We take $m_2 = 0.3$ GeV and hope for the best.

Finally, we have for case I or II

$$k_{20}\frac{d\Sigma}{d^3k_2} = \left[\frac{G'^2F(x)M_0}{4\pi^2 M_H{}^2(x_+ - x_-)k_1\cdot p_1}\right]_{x=x_+}\left[1 + \frac{\alpha}{\pi}(A+B)\right] + k_{20}\frac{d\Sigma^F_{IE}}{d^3k_2}, \tag{17}$$

with

$$M_{0\mathrm{I}} = 8k_1\cdot p_1 k_2\cdot\Delta, \qquad M_{0\mathrm{II}} = 8k_1\cdot\Delta k_2\cdot p_1,$$

$$A_{\mathrm{I}} = \int_{x_+}^1 dx\frac{1}{x-x_+}\frac{1}{x_+ s F(x_+)}\left\{(xs-\Delta^2)F(x)\left[f_{\mathrm{I}}f'_{\mathrm{I}}\ln\frac{\nu^2}{\Delta^2} + f_{\mathrm{I}}\ln\frac{u^2}{M_H{}^2m_\mu{}^2} - f'_{\mathrm{I}}\ln\frac{(xs-\Delta^2)^2}{m_\mu{}^2\Delta^2}\right] - [x\to x_+]\right\},$$

$$A_{\mathrm{II}} = \int_{x_+}^1 dx\frac{1}{x-x_+}\frac{1}{(x_+u - m_2{}^2)F(x_+)}\left\{(xu-\Delta^2)F(x)\left[f_{\mathrm{II}}f'_{\mathrm{II}}\ln\frac{\nu^2}{\Delta^2} - f_{\mathrm{II}}\ln\frac{u^2}{M_H{}^2m_\mu{}^2} + f'_{\mathrm{II}}\ln\frac{(xs-\Delta^2)^2}{m_\mu{}^2\Delta^2}\right] - [x\to x_+]\right\}.$$

$$\begin{aligned}
B_{\mathrm{I}} = {} & -f_{\mathrm{I}}{}^2[-\tfrac{3}{2}\ln x_+ + \ln(1-x_+)] - f_{\mathrm{I}}'^2\left[\ln(1-x_+) + \ln\frac{2\nu}{M_H}\right] - \left[-\ln\frac{s}{2M_H\nu} - \ln x_+ + \ln(1-x_+)\right] \\
& - f_{\mathrm{I}}f'_{\mathrm{I}}\left\{\tfrac{1}{2}\ln x_+ - \tfrac{1}{2}\ln^2\frac{x_+M_H}{m_2} + \ln\frac{2\nu}{m_2}\left[-\tfrac{1}{2}\ln\frac{2\nu}{m_2} + \ln x_+ - 2\ln(1-x_+) + \ln\frac{m_2}{M_H} - \tfrac{3}{2}\right]\right\} \\
& - f'_{\mathrm{I}}\left\{-\ln\frac{x_+s}{m_2m_\mu}\left[\tfrac{3}{2}\ln\frac{x_+s}{m_2m_\mu} - 2\ln\frac{2\nu}{M_H} - 2\ln(1-x_+) + \ln\frac{m_\mu m_2}{M_H{}^2} + 2\ln\frac{m_2}{M_H}\right]\right\} \\
& - f_{\mathrm{I}}\left\{\tfrac{1}{2}\ln^2\frac{s}{2M_H\nu} + \ln\frac{s}{2M_H\nu}\ln\frac{x_+M_H}{m_\mu} + \tfrac{1}{2}\ln x_+ + \ln\frac{-u}{m_\mu M_H}\left[\ln\frac{s}{2M_H\nu} + 2\ln x_+ - 2\ln(1-x_+) - \tfrac{3}{2}\right]\right\},
\end{aligned}$$

$$\begin{aligned}
B_{\mathrm{II}} = {} & -f_{\mathrm{II}}{}^2[-\tfrac{3}{2}\ln x_+ + \ln(1-x_+)] - f_{\mathrm{II}}'^2\left[\ln(1-x_+) + \ln\frac{2\nu}{M_H}\right] - \left[-\ln\frac{s}{2M_H\nu} - \ln x_+ + \ln(1-x_+)\right] \\
& - f_{\mathrm{II}}f'_{\mathrm{II}}\left\{+\tfrac{1}{2}\ln x_+ - \tfrac{1}{2}\ln^2\frac{x_+M_H}{m_2} + \ln\frac{2\nu}{m_2}\left[-\tfrac{1}{2}\ln\frac{2\nu}{m_2} + \ln x_+ - 2\ln(1-x_+) + \ln\frac{m_2}{M_H} - \tfrac{3}{2}\right]\right\} \\
& - f'_{\mathrm{II}}\left\{\ln\frac{x_+s}{m_2m_\mu}\left[\tfrac{3}{2}\ln\frac{x_+s}{m_2m_\mu} - 2\ln\frac{2\nu}{M_H} - 2\ln(1-x_+) + \ln\frac{m_\mu m_2}{M_H{}^2} + 2\ln\frac{m_2}{M_H} - \tfrac{3}{2}\right]\right\} \\
& - f_{\mathrm{II}}\left\{-\tfrac{1}{2}\ln^2\frac{s}{2M_H\nu} - \ln\frac{s}{2M_H\nu}\ln\frac{x_+M_H}{m_\mu} - 2\ln x_+ - \ln\frac{-u}{m_\mu M_H}\left[\ln\frac{s}{2M_H\nu} + 2\ln x_+ - 2\ln(1-x_+)\right]\right\},
\end{aligned}$$

$$s = (k_1 + P_1)^2, \qquad \nu = \frac{1}{M_H}q\cdot P_1, \qquad u = (P_1 - k_2)^2,$$

$$f_{\mathrm{I}} = \tfrac{1}{3}, \quad f'_{\mathrm{I}} = f_{\mathrm{I}} - 1, \quad f_{\mathrm{II}} = -\tfrac{2}{3}, \quad f'_{\mathrm{II}} = f_{\mathrm{II}} + 1, \quad p_1 = xP_1,$$

$$k_{20}\frac{d\Sigma^F_{IE}}{d^3k_2} = \int_0^1 dx\,F(x)\frac{1}{(2\pi)^5}\int\frac{d^3k}{k_0}\delta[(\Delta-k)^2 - m_2{}^2]\theta(\Delta_0 - k_0)\frac{m^4N^F}{k_1\cdot p_1},$$

$$m^4N^F = -2\pi\alpha\left[\frac{f^2\overline{U}_1}{4(k\cdot p_1)^2} + \frac{f'^2\overline{U}_2}{4(k\cdot\Delta)^2} + \frac{\overline{U}_3}{4(k\cdot k_2)^2} - \frac{2ff'\overline{U}_4}{4k\cdot p_1k\cdot\Delta} - \frac{2f\overline{U}_5}{4k\cdot p_1k\cdot k_2} + \frac{2f'\overline{U}_6}{4k\cdot\Delta k\cdot k_2}\right],$$

$$\frac{2}{G'^2}\overline{U}_{1\mathrm{I}} = -64m_1{}^2k_1\cdot p_1k\cdot k_2 - 64m_1{}^2k\cdot k_1k_2\cdot\Delta + 64m_1{}^2k\cdot k_1k\cdot k_2 - 64k\cdot k_1k\cdot p_1k_2\cdot\Delta + 64k\cdot k_1k\cdot p_1k\cdot k_2,$$

$$\frac{2}{G'^2}\overline{U}_{2\mathrm{I}} = -64k\cdot\Delta k\cdot k_2k_1\cdot p_1,$$

$$\frac{2}{G'^2}\overline{U}_{3\mathrm{I}} = -64m_\mu{}^2k_1\cdot p_1k\cdot k_2 + 64m_\mu{}^2k\cdot\Delta k_1\cdot p_1 - 64k\cdot k_2k\cdot\Delta k_1\cdot p_1,$$

$$\frac{2}{G'^2}\overline{U}_{4\mathrm{I}} = -32k\cdot k_2\Delta\cdot p_1k_1\cdot p_1 - 32k\cdot p_1\Delta\cdot k_2k_1\cdot p_1 - 32k\cdot\Delta k_1\cdot p_1k_2\cdot p_1 - 32(\Delta\cdot k_2 - k\cdot k_2)(k\cdot k_1\Delta\cdot p_1 - k\cdot p_1\Delta\cdot k_1 + k\cdot\Delta k_1\cdot p_1),$$

$$\frac{2}{G'^2}\overline{U}_{5\mathrm{I}} = -64k\cdot k_2 k_1\cdot p_1 k_2\cdot p_1 + 32k\cdot\Delta k_1\cdot p_1 k_2\cdot p_1 - 32k\cdot k_2\Delta\cdot p_1 k_1\cdot p_1$$
$$+ 32k\cdot p_1\Delta\cdot k_2 k_1\cdot p_1 - 32(\Delta\cdot k_2 - k\cdot k_2)(k\cdot k_1 k_2\cdot p_1 - k\cdot p_1 k_1\cdot k_2 + k\cdot k_2 k_1\cdot p_1)\,,$$

$$\frac{2}{G'^2}\overline{U}_{6\mathrm{I}} = -64k\cdot k_2\Delta\cdot k_2 k_1\cdot p_1 - 32m_\mu{}^2 k\cdot\Delta k_1\cdot p_1 + 32k_1\cdot p_1(2k\cdot\Delta\Delta\cdot k_2 - k\cdot k_2 m_2{}^2)\,,$$

$$\frac{2}{G'^2}\overline{U}_{1\mathrm{II}} = -64m_1{}^2 k\cdot k_1 k_2\cdot p_1 - 64m_1{}^2 k\cdot k_2\Delta\cdot k_1 + 64m_1{}^2 k\cdot k_2 k\cdot k_1 - 64k\cdot k_2 k\cdot p_1\Delta\cdot k_1 + 64k\cdot k_2 k\cdot p_1 k\cdot k_1\,,$$

$$\frac{2}{G'^2}\overline{U}_{2\mathrm{II}} = -64k\cdot k_1 k\cdot\Delta k_2\cdot p_1\,,$$

$$\frac{2}{G'^2}\overline{U}_{3\mathrm{II}} = -64m_\mu{}^2 k\cdot k_1 k_2\cdot p_1 + 64m_\mu{}^2 k\cdot p_1\Delta\cdot k_1 - 64m_\mu{}^2 k\cdot p_1 k\cdot k_1 - 64k\cdot k_2 k\cdot p_1 k_1\cdot\Delta + 64k\cdot k_2 k\cdot p_1 k\cdot k_1\,,$$

$$\frac{2}{G'^2}\overline{U}_{4\mathrm{II}} = -32k\cdot k_1\Delta\cdot p_1 k_2\cdot p_1 - 32k\cdot p_1\Delta\cdot k_1 k_2\cdot p_1 - 32k\cdot\Delta k_1\cdot p_1 k_2\cdot p_1 - 32(\Delta\cdot k_1 - k\cdot k_1)(k\cdot\Delta k_2\cdot p_1 - k\cdot p_1\Delta\cdot k_2 + k\cdot k_2\Delta\cdot p_1)\,,$$

$$\frac{2}{G'^2}\overline{U}_{5\mathrm{II}} = 64k\cdot k_1 k_2\cdot p_1 k_2\cdot p_1 + 64k\cdot k_2 k\cdot p_1\Delta\cdot k_1 - 64k\cdot k_1 k\cdot k_2 k\cdot p_1$$
$$-32(\Delta\cdot k_1 - k\cdot k_1)(2k\cdot p_1 k_2\cdot p_1 - m_1{}^2 k\cdot k_2 - 2k\cdot k_2 k_2\cdot p_1 + m_\mu{}^2 k\cdot p_1)\,,$$

$$\frac{2}{G'^2}\overline{U}_{6\mathrm{II}} = 32k\cdot k_1\Delta\cdot k_2 k_2\cdot p_1 + 32k\cdot k_2\Delta\cdot k_1 k_2\cdot p_1 + 32k\cdot\Delta k_1\cdot k_2 k_2\cdot p_1 - 32(\Delta\cdot k_1 - k\cdot k_1)(k\cdot p_1\Delta\cdot k_2 - k\cdot k_2\Delta\cdot p_1 + k\cdot\Delta k_2\cdot p_1)\,.$$

At this point it should be noted that the corrections which arise from the bremsstrahlung graphs depend upon the form of the parton distribution function $F(x)$. The result is particularly sensitive to the small x region.

To get the radiative corrections to an actual cross section such as νp, we must use the cross section (case I or II) and an $F(x)$ appropriate for each kind of quark ($\mathcal{P}, \mathfrak{N}, \overline{\mathcal{P}}, \overline{\mathfrak{N}}$) and then sum over quarks in the target.

As an example, we have worked out νp and $\bar{\nu} p$ at $E_\nu = 100$ GeV in the lab. We have used distribution functions from Kuti and Weisskopf[6] and have done the integrations for A and Σ_{IE}^F numerically. Kuti and Weisskopf give

$$F(\overline{\mathcal{P}}\text{ in }p) = F(\overline{\mathfrak{N}}\text{ in }p) = F(\overline{\mathcal{P}}\text{ in }n) = F(\overline{\mathfrak{N}}\text{ in }n) = \frac{1}{3}\frac{(1-x)^{7/2}}{x}\,,$$

$$F(\mathcal{P}\text{ in }p) = F(\mathfrak{N}\text{ in }n) = \frac{1}{3}\frac{(1-x)^{7/2}}{x} + \frac{105}{48}\frac{(1-x)^3}{\sqrt{x}}\,,$$

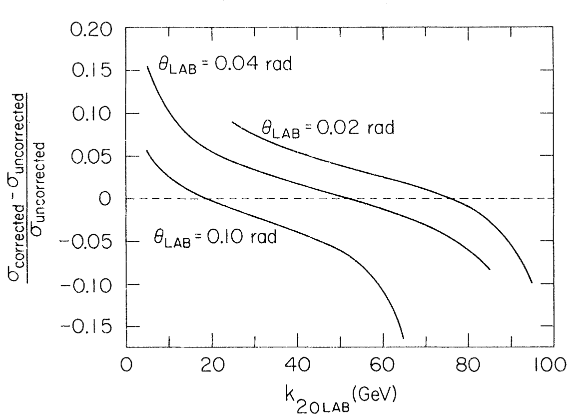

FIG. 6. Plot of $(\sigma_{\text{corrected}} - \sigma_{\text{uncorrected}})/\sigma_{\text{uncorrected}}$ for various lab angles and energies of the outgoing muon in the reaction $\nu_\mu p \to \mu^- X$ at an incident neutrino energy of the 100 GeV in the lab. [$\sigma_{\text{corrected}}$ is $k_{20}\, d\Sigma/d^3k_2$ of Eq. (17).]

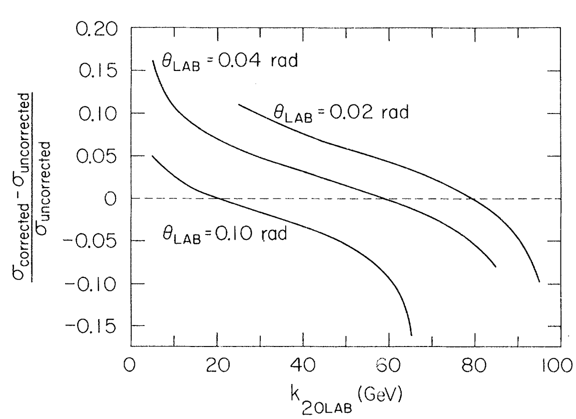

FIG. 7. Plot of $(\sigma_{\text{corrected}} - \sigma_{\text{uncorrected}})/\sigma_{\text{uncorrected}}$ for various lab angles and energies of the outgoing muon in the reaction $\bar{\nu}_\mu p \to \mu^+ X$ at an incident neutrino energy of 100 GeV in the lab. [$\sigma_{\text{corrected}}$ is $k_{20}\, d\Sigma/d^3k_2$ of Eq. (17).]

$$F(\mathfrak{N} \text{ in } p) = F(\mathcal{P} \text{ in } n) = \frac{1}{3}\frac{(1-x)^{7/2}}{x} + \frac{105}{96}\frac{(1-x)^3}{\sqrt{x}}.$$

Typical results can be seen in Figs. 6 and 7.

These curves show features typical of radiative corrections in other processes,[7] which is not surprising since it is primarily a classical effect. At fixed lab angle and fixed incident neutrino energy, the spectrum of the muons is decreased by about 10% at the high end and increased by about 10% at the low end. We note again that the approximations of the calculation are valid only in the scaling region with all momenta dot products much bigger than the corresponding masses.

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Asymptotic Multiplicity Distributions and Analog of the Central-Limit Theorem*

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Under the assumption of gentle behavior of higher cumulants or correlation moments, we discuss how the multiplicity distributions approach the Gaussian (normal) or approximately Gaussian distribution at high energy. This is an analog of the central-limit theorem. A detailed comparison with experiment is made based on this formalism and shows that such an approach may be useful. It is pointed out that if the 2-prong inelastic cross section in the pp reaction is identified with the lower end point of the multiplicity distribution, then a deviation from the Gaussian form is necessary at the present energy. The asymptotic relation $(1/\sqrt{2\pi})\sigma_{\text{inel}}/\sigma_m = \gamma = \langle(n - \langle n\rangle)^2\rangle^{1/2}$ is well satisfied by experimental data, where σ_m and γ stand for the maximum of the topological cross sections and the width of the limiting Gaussian form, respectively. If the ratio of the width γ and the modal multiplicity m approaches a nonvanishing value at infinite energy, then we obtain a scaling of the distribution function, the scaling function being of approximately Gaussian form with the scaling variable n/m.

I. INTRODUCTION

For a long time, the Poisson distribution has been a favorite model of physicists for describing the high-energy multiplicity distribution. Recent experiments,[1] however, indicate a departure from it by exhibiting nonvanishing correlation moments. It has been pointed out, in fact, that the asymptotic multiplicity distribution seems to approach a normal distribution[2-4] as energy increases. Such a phenomenon resembles the central-limit theorem in statistics and was proved by Haldane some time ago in the case of a continuous distribution on the interval $(-\infty, \infty)$. The assumption that leads to this result is that the higher cumulants do not grow too fast, a condition which is met by multiperipheral models,[5] field-theoretical models,[6] and a gas model.[7]

In this article, we elaborate on the Haldane theorem and present it in a form suitable for analyzing experimental data. In Sec. II, the central-limit theorem is exhibited for the Poisson distribution so as to be useful for discussions of the later sections. Section III presents the definition of various moments and their relationships. In Sec. IV, we prove the Haldane theorem for a discrete