Opaqueness of pp Collisions from 30 to 1500 GeV/c *

Alexander Wu Chao and Chen Ning Yang
Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11790
(Received 9 May 1973)

Assuming only the eikonal approximation and the approximate reality of the S matrix for elastic scattering we evaluate from experimental data the opaqueness of pp scattering from p_L = 30 to 1500 GeV/c. Parameters X and Y which characterize the shape of the function $1 - S(b)$ are defined and discussed.

INTRODUCTION

Recent CERN Intersecting Storage Rings (ISR) experiments[1] have refocused attention on the behavior of the total cross section and elastic differential cross section at very high energies. These results are especially interesting because of the earlier conjecture of Cheng and Wu[2] which seems to be remarkably confirmed. A phenomenological analysis of various data based on Cheng and Wu's picture has been made.[3] In this paper we make a less extensive phenomenological analysis, with emphasis on a description of high-energy pp collisions with as little theoretical prejudice as possible. Since the eikonal approximation and the nearly purely imaginary character of the scattering amplitudes both seem to be quite accurate, we shall adopt these assumptions but shall use no additional ones.

In the eikonal approximation[4]

$$\left(\frac{d\sigma}{dt}\right)_{\rm el} = \pi |a|^2 , \tag{1}$$

$$a = \langle 1 - S(b) \rangle , \tag{2}$$

where $\langle\ \rangle$ designates the Fourier transform from the two-dimensional space of the impact parameter $\vec{b}$ to the two-dimensional space of the momentum transfer $\vec{K}$ ($K^2 = |t|$):

$$\langle X \rangle = \frac{1}{2\pi} \iint X(\vec{b}) \exp(i\vec{K}\cdot\vec{b}) d^2b . \tag{3}$$

We shall also use the same symbol to designate the inverse Fourier transform. We neglect all spin-correlation effects.

The S matrix $S(b)$ will be written as

$$S = e^{-\Omega(b)} , \tag{4}$$

where Ω, the opaqueness (or blackness), will be assumed to be real. It is, of course, dependent on the incoming energy. (Ω must be almost purely imaginary for sufficiently large b, see Ref. 5, p. 357–358. But we neglect such contributions which are probably very small.)

In Sec. I we discuss the magnitude and shape of the opaqueness $\Omega(b)$ for the 10.8-on-10.8-GeV/c and the 26.8-on-26.8-GeV/c pp collisions. In Sec. II we discuss the mathematical range of the elasticity parameter and the slope parameter.

I. OPAQUENESS AT HIGH ENERGIES

It is easy to obtain[4] the opaqueness Ω from $(d\sigma/dt)_{\rm el}$ by using

$$\langle \Omega \rangle = a + \tfrac{1}{2} a \otimes a + \tfrac{1}{3} a \otimes a \otimes a + \cdots , \tag{5}$$

where $\otimes$ is the folding integral. The inverse of (5) is

$$a = \langle \Omega \rangle - \frac{1}{2!} \langle \Omega \rangle \otimes \langle \Omega \rangle + \frac{1}{3!} \langle \Omega \rangle \otimes \langle \Omega \rangle \otimes \langle \Omega \rangle - \cdots . \tag{6}$$

For the 10.8-on-10.8-GeV/c pp collision we use the unnormalized elastic data of Barbiellini *et al.*[6] and normalize with the total cross section $\sigma_T = 39.1 \pm 0.4$ mb estimated for this collision from the data in Ref. 1. This gives the following fit:

$$a(t=0) = 7.98 \pm 0.08 \ ({\rm GeV}/c)^{-2} ,$$

$$\frac{a}{a(t=0)} = (0.685 \pm 0.005) \exp[-(4.7 \pm 0.05)|t|] + [1 - (0.685 \pm 0.005)] \exp[-(9.0 \pm 0.5)|t|] \quad [|t| < 0.25 \ ({\rm GeV}/c)^2] . \tag{7}$$

Substitution into (5) gives the value of $\langle \Omega \rangle / \langle \Omega \rangle_{t=0}$ presented in Fig. 1. Also

$$\langle \Omega \rangle_{t=0} = 10.15 \pm 0.15 \ ({\rm GeV}/c)^{-2} . \tag{8}$$

We present the result this way because the error in $\langle \Omega \rangle_{t=0}$ is quite separate from that of $\langle \Omega \rangle / \langle \Omega \rangle_{t=0}$.

For the 26.8-on-26.8-GeV/c pp collisions a similar procedure yields $a(t=0) = 8.83 \pm 0.12$ (GeV/c)$^{-2}$,

$$\frac{a}{a(t=0)} = (0.82 \pm 0.01) \exp[-(5.22 \pm 0.01)|t|] + [1 - (0.82 \pm 0.01)] \exp[-(12.15 \pm 0.15)|t|] \quad [|t| < 0.4 \ ({\rm GeV}/c)^2] , \tag{9}$$

and the plot of $\langle \Omega \rangle / \langle \Omega \rangle_{t=0}$ which is also exhibited in Fig. 1. For this energy

$$\langle\Omega\rangle_{t=0} = 11.4 \pm 0.2\ (\mathrm{GeV}/c)^{-2}. \quad (10)$$

The same analysis as above for pp collision at 29.7 GeV/c was already made in Ref. 4. The result is plotted also in Fig. 1. The value of $\langle\Omega\rangle_{t=0}$ is tabulated in Table I.

We notice the following facts:

(a) The opaqueness probably *decreases* slightly from $p_L = 29.7$ to 245 GeV/c, but *increases* from $p_L = 245$ to 1480 GeV/c. The over-all opaqueness in coordinate space,

$$\int\int \Omega(b) d^2b = 2\pi\langle\Omega\rangle_{t=0},$$

decreases by $(1.5 \pm 2)\%$ first and then increases by $(12.3 \pm 2.5)\%$.

(b) The shape of the opaqueness $\Omega(b)$ as a function of b *expands* from $p_L = 29.7$ to 245 GeV/c, but then does not change appreciably from $p_L = 245$ to 1480 GeV/c. (There are indications perhaps of a very slight expansion in this second energy region.)

It appears at first surprising that the increase of the slope parameter between $p_L = 245$ and 1480 GeV/c does not lead to any appreciable spatial expansion in $\Omega(b)$. Upon closer examination one finds that there is an opposite effect leading to a cancellation. With increasing energy the increasing total cross section leads to increasing importance of the higher order terms on the right-hand side of (5), which have smaller absolute values of the slope in K space.

We believe conclusions (a) and (b) to be quite firm, once one accepts the experimental data, since few assumptions have been made other than the validity of experimental data. We do not have any compelling reasons for this behavior of the magnitude and shape of the opaqueness $\Omega(b)$. We are further investigating this matter, especially considering the possibility of relinquishing the assumption that $S(b)$ is real.

In Fig. 2 we sketch the opaqueness $\Omega(b)$ vs b for all three energies. Since the errors on them are quite sensitively dependent on large K data we do not put error bars on these curves.

(c) It has been suggested[4,5] that

$$\langle\Omega\rangle = (\mathrm{constant})[F_1(K)]^2,$$

TABLE I. The opaqueness in momentum space evaluated at $t=0$ as a function of p_L.

p_L (GeV/c)	$\langle\Omega\rangle_{t=0}$ [(GeV/c)$^{-2}$]
29.7	10.3 ± 0.15
245	10.15 ± 0.15
1480	11.4 ± 0.2

where F_1 is the electric charge form factor of the proton. Figures 1 and 3 show that this suggestion is no longer good for the ISR data. Instead the formula

$$\langle\Omega\rangle = (\mathrm{constant})[G_E(K)]^2 \quad (11)$$

seems to be relatively good.

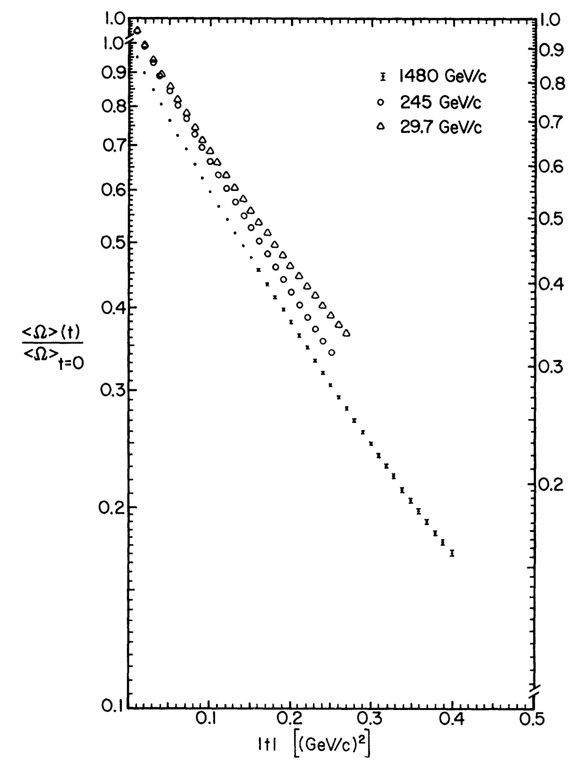

FIG. 1. The normalized opaqueness $\langle\Omega\rangle$ in momentum space for three different momenta: $p_L = 29.7$, 245, and 1480 GeV/c. They are normalized to unity at the origin. The triangles, open circles, and black dots represent the opaqueness for 29.7, 245, and 1480 GeV/c, respectively. Error bars are shown only for the last curve. The magnitude of the omitted error bars are comparable to those shown. The logarithmic scale on the left side is for the 1480-GeV/c case while that on the right-hand side is for the two lower energies. The fit used for the 29.7-GeV/c data is

$$a(t) = [7.89 \pm 0.08] \times \{(0.71 \pm 0.01)\exp[-(7.36 \pm 0.1)|t|] + [(1-(0.71 \pm 0.01)] \times \exp[-(2.36 \pm 0.1)|t|]\}\ (\mathrm{GeV}/c)^{-2}$$

which is very good for $|t| \le 0.3\ (\mathrm{GeV}/c)^2$. The experimental data for $(d\sigma/dt)/(d\sigma/dt)_{t=0}$ at 29.7 GeV/c are those of Edelstein *et al.*, Phys. Rev. D 5, 1073 (1972), normalized by $\sigma_{tot} = 38.6 \pm 0.4$ mb. The experimental data for $(d\sigma/dt)/(d\sigma/dt)_{t=0}$ at 245 and 1480 GeV/c are those of Barbiellini *et al.* (Ref. 6) normalized by $\sigma_{tot} = 39.1 \pm 0.4$ mb and 43.2 ± 0.6 mb, respectively.

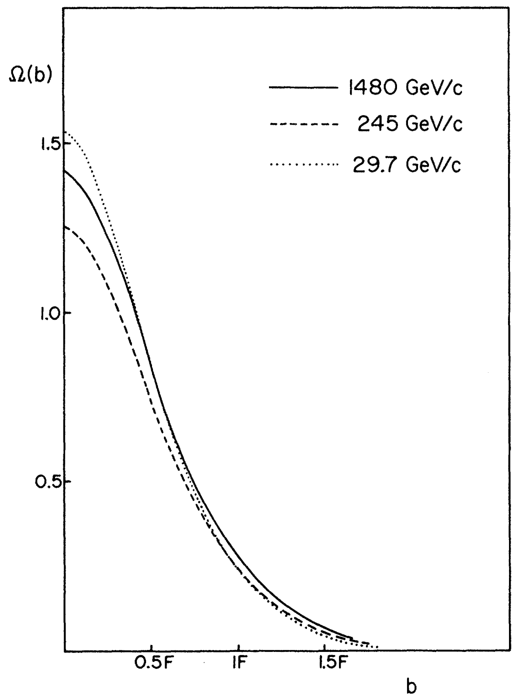

FIG. 2. The opaqueness [in units of $(\mathrm{GeV}/c)^{-2}$] in coordinate space for three different momenta: $p_L=29.7$, 245, 1480 GeV/c. The value of $\Omega(b=0)$ is quite sensitive to $a(t)$ at large $|t|$, therefore these curves are not accurate.

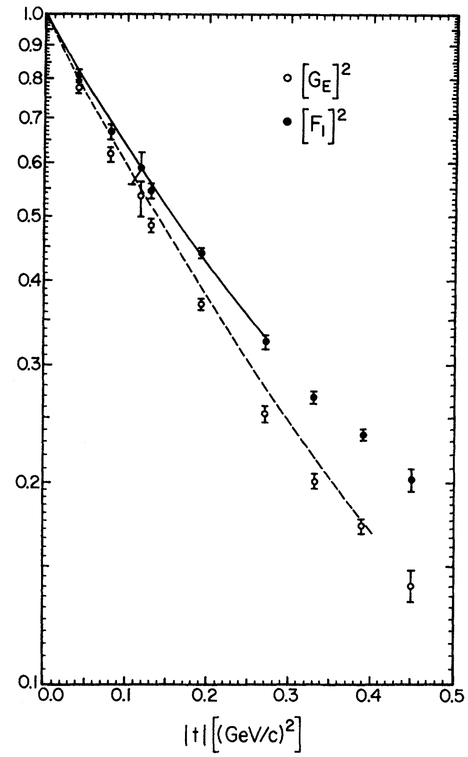

FIG. 3. The proton form factors $[F_1(t)]^2$ and $[G_E(t)]^2$ normalized to unity at $t=0$. Also drawn for comparison are two curves copied from Fig. 1. The solid curve is $\langle\Omega\rangle/\langle\Omega\rangle_{t=0}$ for 29.7 GeV/c and the dashed curve the same for 1480 GeV/c. Data for the form factors are taken from L. E. Price *et al.*, Phys. Rev. D 4, 45 (1971).

II. RANGE OF SOME PARAMETERS

Three of the most important experimental parameters are the total cross section σ_T, the total elastic cross section σ_{el}, and the slope parameter:

$$B=\left|\frac{d}{dt}\ln\left(\frac{d\sigma}{dt}\right)\right| \quad \text{at} \quad t=0\,. \tag{12}$$

All three are of the dimension (length)^2. We define the dimensionless ratios

$$X=\frac{\sigma_{el}}{\sigma_T}, \quad Y=\frac{\sigma_T}{16\pi B}\,.$$

Experimental values of X and Y are listed in Table II for various energies. The close equality of X and Y is a reflection of the empirical fact that $\ln d\sigma/dt$ has almost a linear dependence on t. (A strict linear dependence means that $1-S$ is Gaussian in b. See the Gaussian model in Table III.)

Under the assumption that $\Omega(b)=\text{real}\geq 0$ one has

$$\sigma_T=4\pi\int_0^\infty (1-S)b\,db\,, \tag{13}$$

$$\sigma_{el}=2\pi\int_0^\infty (1-S)^2 b\,db\,, \tag{14}$$

$$B=\int_0^\infty (1-S)b^3 db\left[2\int_0^\infty (1-S)b\,db\right]^{-1}. \tag{15}$$

Thus

$$B=\tfrac{1}{2}[\text{average of } b^2 \text{ with weight } (1-S)]\,. \tag{16}$$

Table III lists for some models the values of these parameters. Notice that X and Y are range-independent. I.e., they are not changed by the transformation $S(b)\to S(cb)$ where $c=\text{constant}$. X and Y therefore are parameters characteristic of the *shape* of the function $1-S(b)$ vs b. If one increases $1-S(b)$ by a uniform factor α, X and Y both increase by the same factor α. Thus, qualitatively, transparent scatterings are indicated by small values of X and Y and opaque scatterings are indicated by large values of X and Y. Also "compact" scatterings, as in a gray-disk model, or a Gaussian model, are associated with large ratios Y/X while "noncompact" scatterings, as in a two-tiered-platform model with a large and low lower tier, are associated with a small ratio Y/X.

These qualitative features are indicated in Fig. 4. We notice that pp scattering in the $p_L\cong 5-1500$-

TABLE II. X and Y as functions of p_L. The data are taken from (and interpolated): *NN* and *ND* Interactions-Berkeley Compilation (1970), Ref. 1; S. P. Denisov *et al.*, Phys. Lett. 36B, 415 (1971); V. Bartenev *et al.*, report, 1972 (unpublished); and G. G. Beznogikh *et al.*, Phys. Lett. 39B, 411 (1972).

p_L (GeV/c)	X	Y
2.8	0.42	0.39
4.0	0.318	0.312
5.0	0.302	0.283
5.5	0.297	0.270
6.0	0.294	0.258
7.0	0.281	0.254
8.0	0.278	0.246
9.0	0.275	0.240
11.0	0.270	0.230
15	0.254	0.210
20	0.238	0.198
25	0.226	0.191
30	0.220	0.187
55	0.196	0.174
100	0.178	0.172
200	0.175	0.170
300	0.173	0.170
500	0.172	0.169
1200	0.176	0.170
1500	0.176	0.169

GeV/c region is relatively transparent. In fact the transmission coefficient (of the amplitude) for even a head-on collision is still sizable at $p_L = 1500$ GeV/c:

$$S^{-\Omega(0)}=e^{-1.4}=0.25\,.$$

It is also quite compact, resembling very much a Gaussian, as Fig. 1 indicates.

Equations (13) to (15) and the condition $0 \leqslant S \leqslant 1$ show that not all of the X-Y plane is allowed. We shall assume $S(b)$ to be piecewise differentiable. The allowed region is shown in Fig. 4.

The boundaries BC and OC are obvious.

To prove that OA is a boundary we use the follow-

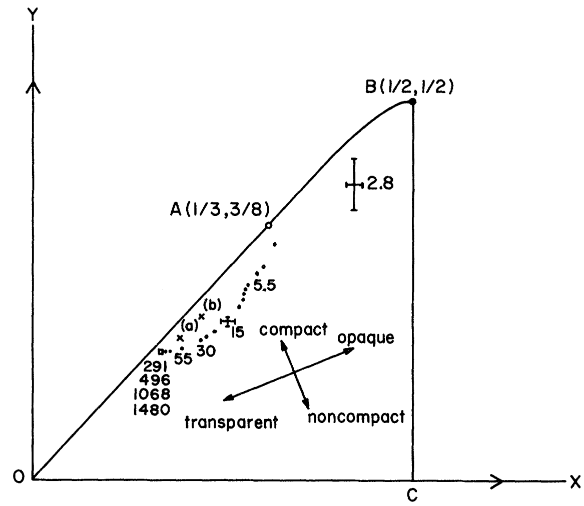

FIG. 4. The allowed region of the dimensionless variables X and Y defined in the text, OA, OC, BC are straight lines, and AB has a parametric form given in the text. Qualitative features "opaque," "transparent," "compact," and "noncompact" refer to the shape of $1-S(b)$. Also drawn in this figure are experimental values from Table II. Representative error bars have been drawn for $p_L = 2.8$ and 15 GeV/c. The square labeled by $p_L = 291$, 496, 1068, 1480 GeV/c is the point corresponding to the four ISR data in Ref. 1. The crosses labeled by a and b are calculated for collisions in which $\sigma_T = 50$ and 60 mb, respectively, under the assumption that the *shape* of $\Omega(b)$ remains the same as that for $p_L = 1480$ GeV/c, at which $\sigma_T = 43$ mb.

TABLE III. X and Y for various models.

Model	$1-S$	σ_T	σ_{el}	$16\pi B$	X	Y
gray disk	α if $b<R$, 0 if $b>R$ $(0\le\alpha\le1)$	$2\pi\alpha R^2$	$\pi\alpha^2R^2$	$4\pi R^2$	$\frac{1}{2}\alpha$	$\frac{1}{2}\alpha$
Gaussian	$\alpha e^{-b^2/R^2}$ $(0\le\alpha\le1)$	$2\pi\alpha R^2$	$\frac{1}{2}\pi\alpha^2R^2$	$8\pi R^2$	$\frac{1}{4}\alpha$	$\frac{1}{4}\alpha$
two-tiered platform	α if $b<R$, $\alpha\beta$ if $R<b<(1+\gamma)^{1/2}R$, 0 if $b>(1+\gamma)^{1/2}R$ $(0\le\alpha\le1,\ 0\le\gamma,\ 0\le\beta\le1)$	$2\pi\alpha(1+\gamma\beta)R^2$	$\pi\alpha^2(1+\gamma\beta^2)R^2$	$4\pi\frac{1+2\beta\gamma+\beta\gamma^2}{1+\beta\gamma}R^2$	$\frac{1}{2}\alpha\frac{1+\gamma\beta^2}{1+\gamma\beta}$	$\frac{1}{2}\alpha\frac{(1+\beta\gamma)^2}{1+2\beta\gamma+\beta\gamma^2}$
truncated parabola	α if $b<\gamma R$, $\alpha(b^2/R^2-1)(\gamma^2-1)^{-1}$ if $\gamma R<b<R$, 0 if $b>R$ $(0\le\alpha\le1,\ 0\le\gamma\le1)$	$\pi\alpha(1+\gamma^2)R^2$	$\pi\alpha^2(\frac{1}{3}+\frac{2}{3}\gamma^2)R^2$	$\frac{8}{3}\pi\frac{1+\gamma^2+\gamma^4}{1+\gamma^2}R^2$	$\frac{1}{3}\alpha\frac{1+2\gamma^2}{1+\gamma^2}$	$\frac{3}{8}\alpha\frac{(1+\gamma^2)^2}{1+\gamma^2+\gamma^4}$

ing transformations:

$$b^2=\frac{\sigma_T}{2\pi}z, \quad \int_0^{b^2}(1-S)d(b^2)=\frac{\sigma_T}{2\pi}\xi\,. \tag{17}$$

Then

$$X=\tfrac{1}{2}\int_0^1 \frac{d\xi}{(dz/d\xi)}\,,$$

$$Y=\frac{1}{4\int_0^1 z\,d\xi} = \frac{1}{4\int_0^1 (1-\xi)(dz/d\xi)d\xi}\,, \tag{18}$$

$$1-S=\frac{d\xi}{dz}\,. \tag{19}$$

Using Schwarz's inequality one obtains from (18)

$$XY^{-1}\geq 2\left[\int_0^1 (1-\xi)^{1/2}d\xi\right]^2=\tfrac{8}{9}\,. \tag{20}$$

This shows that all points are to the right of the line OA. This is the same restriction as one obtains at high energies from the MacDowell-Martin bound,[7] if one neglects the real part of the near forward amplitude and the spin dependence of B.

That AB is a boundary can be obtained by a variational calculation for minimizing (XY^{-n}), where $n\geq 1$. One obtains the minimum

$$XY^{-n}\geq(\tfrac{1}{2}-\tfrac{1}{6}y_0)(2+\tfrac{2}{3}y_0{}^2)^n\,, \tag{21}$$

where y_0 is related to n by

$$n=-\frac{1}{2}+\frac{1}{2y_0}+\frac{2}{3-y_0} \quad (0<y_0\leq 1)\,. \tag{22}$$

The minimum (21) is realized by the truncated parabola model of Table II at

$$\gamma=\frac{1-y_0}{1+y_0} \text{ and } \alpha=1\,.$$

The envelope of the boundary curves (21) is AB, which is parametrically

$$X=\tfrac{1}{2}-\tfrac{1}{6}y_0\,,$$
$$Y=(2+\tfrac{2}{3}y_0{}^2)^{-1}, \quad 0\leq y_0\leq 1\,. \tag{23}$$

Are all points inside and on the curve $OABCO$ realized by some model for which $0\leq S\leq 1$?

The answer is yes. But the line OC can only be realized if the integral for B in (15) is divergent. Also the line BC can only be realized by a model where $S=0$ or 1 everywhere. (I.e., for a collection of black rings with or without a central black disk.) If B is assumed to be convergent and $1-S(b)$ is assumed to be nonincreasing with increasing b, then the lines OC and BC cannot be realized except for the point B. In such a case only the open region $OABCO$, the open curve OAB and the point B can be realized.

We shall not give the detailed proof of these statements. Suffice it to mention the following two observations which are helpful: (a) Starting from any given model, replacing $1-S$ by $\alpha(1-S)$ leads to a new model at αX, αY. (b) Starting from any model one can add a very large but very transparent wing. If the area of the wing is A and the value of $1-S$ on the wing is ϵ, then fixing ϵA^2 but making $A\to\infty$ would lead to no change in σ_T and σ_{el} but possible finite addition to B.

ACKNOWLEDGMENT

We would like to thank Dr. Chris Quigg for discussions and for information about experimental data.

*Supported in part by NSF Grant No. GP32998X.

[1] U. Amaldi, R. Biancastelli, C. Bosio, G. Matthiae, J. V. Allaby, W. Bartel, G. Cocconi, A. N. Diddens, R. W. Dobinson, and A. M. Wetherell, Phys. Lett. 44B, 112 (1973); S. R. Amendolia, G. Bellettini, P. L. Braccini, C. Bradaschia, R. Castaldi, V. Cavasinni, C. Cerri, T. Del Prete, L. Foa, P. Giromini, P. Laurelli, A. Menzione, L. Ristori, G. Sanguinetti, M. Valdata, G. Finocchiaro, P. Grannis, D. Green, R. Mustard, and R. Thun, Phys. Lett. 44B, 119 (1973).

[2] Hung Cheng and Tai Tsun Wu, Phys. Rev. Lett. 24, 1456 (1970), and references quoted therein.

[3] Hung Cheng, James K. Walker, and Tai Tsun Wu, report (unpublished).

[4] T. T. Chou and Chen Ning Yang, Phys. Rev. 170, 1591 (1968).

[5] T. T. Chou and Chen Ning Yang, in *Proceedings of the Second International Conference on High Energy Physics and Nuclear Structure, Rehovoth, Israel, 1967*, edited by G. Alexander (North-Holland, Amsterdam, The Netherlands, 1967).

[6] G. Barbiellini, M. Bozzo, P. Darriulat, G. Diambrini Palazzi, G. DeZorzi, A. Fainberg, M. I. Ferrero, M. Holder, A. McFarland, G. Maderni, S. Orito, J. Pilcher, C. Rubbia, A. Santroni, G. Sette, A. Staude, P. Strolin, and K. Tittel, Phys. Lett. 39B, 663 (1972).

[7] S. W. MacDowell and A. Martin, Physics (N.Y.) 135B, 960 (1964).