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# Internal Conversion of Pseudoscalar Mesons into Lepton Pairs

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The Kroll-Wada formulation of the internal conversion of mesons into lepton pairs is modified by taking into account the correction due to the effect of exchange of leptons. Single- and double-pair decays, such as $\pi^0(K_L, \eta) \to e^+e^-\gamma$, $K_L(\eta) \to \mu^+\mu^-\gamma$, $\pi^0(K_L, \eta) \to e^+e^-e^+e^-$, and $K_L(\eta) \to \mu^+\mu^-\mu^+\mu^-$, are studied under the assumption of $CP$ invariance. A spectral analysis is made, stressing the importance of this exchange effect in the shape of the spectra. These kinds of experiments are proposed as a clue to obtain knowledge about the meson-photon-photon vertices.

## I. INTRODUCTION

Internal conversion of a neutral pion into one or two electron-positron pairs ($\pi^0 \to e^+e^-\gamma$ and $\pi^0 \to e^+e^-e^+e^-$) was first systematically studied by Kroll and Wada[1,2] in 1955. Their derivation was, however, based on the assumption that one could neglect the effect of exchange of electrons or positrons, which is really non-negligible as was pointed out by the present authors in Ref. 3; we treat the effect in this paper. Historically, the $\pi^0 \to e^+e^-e^+e^-$ decay was an interesting object in view of the determination of the intrinsic parity of the neutral pion, although it is a rare event in fact. Now that the parities of the mesons have been established, the decay processes of the internal-conversion type, meson $\to l\bar{l}\gamma$ and meson $\to l\bar{l}l\bar{l}$ with $l = e^-$ or $\mu^-$, are attractive in the following respect: The analysis of the spectra of such decay processes allows us to obtain knowledge about the form factors of the meson-$\gamma$-$\gamma$ vertices. From this we know the interaction of the photon and meson system. The momentum dependence of the form factors becomes important in the case of decays into muonic pairs, because there is large energy release. As will be shown in Secs. III and IV, owing to the propagators of the virtual photons, the momentum dependence of the vertices is neglected in the decays into electronic pairs, where there is small energy release.

Throughout this paper, invariance under $CP$ transformations is assumed and $K_L$ is regarded as a $CP$-odd eigenstate. In the actual measurement one could not, perhaps, explain the aspects of $K_L \to \mu^+\mu^-\mu^+\mu^-$ or $K_L \to e^+e^-e^+e^-$ decay only by taking the momentum dependence of the $CP$-conserving vertex into account. At the present status of experiments concerning the well-known $K_L \to \mu^+\mu^-$ puzzle, the vertices $K_{1,2}$-$\gamma$-$\gamma$ are considered as violating the $CP$ invariance.[4] The $CP$-conserving vertices alone induce a larger branching ratio of the $K_L \to \mu^+\mu^-$ decay than that of the actual measurement. Hence the efforts to obtain knowledge about the vertices of $K_{1,2}$-$\gamma$-$\gamma$ from the experiments on the $K_{L,S} \to l\bar{l}l\bar{l}$ decays lead us to obtain definite knowledge about the $K_L \to \mu^+\mu^-$ puzzle.

The deviation from the point interaction is seen by speculating about the spectra of the decay processes. So in this paper we stress the spectral analysis of the theoretical prediction. The correction of the Kroll-Wada formula by including the effect of exchange of leptons and antileptons takes a formidable aspect. In particular the integration of the matrix element over phase space is quite complicated. We evaluate this correction in the decay width as well as in the spectral shape. The correction is small in the decay width, but one cannot say anything about the exact shape of the spectra if one takes the Kroll-Wada term only. The term alone shows a sharp peak for small values of the four-momentum squared of lepton pairs. There is also, however, a broad plateau which cannot be explained with the Kroll-Wada term. This plateau was neglected in the analysis of spectra in Ref. 5. Each area represented by the two curves is equal. Up to now, nobody has taken up the latter broad shape. Therefore these aspects of exchange effect are important not only for merely modifying the Kroll-Wada formula, but also for obtaining exact knowledge about the meson-$\gamma$-$\gamma$ vertices. In such precise measurement as to decide the momentum dependence of the form factors or the ratio of $CP$-conserving and -nonconserving vertices it must play an important role.

## II. THE DECAYS OF PSEUDOSCALAR MESONS INTO TWO PHOTONS

The basic interaction of a pseudoscalar meson and two photons is taken as

$$H_I(x)=(f/4M)\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}(x)F^{\rho\sigma}(x)\phi(x)\,, \qquad (1)$$

where $F^{\mu\nu}(x)$ is an electromagnetic field tensor, and $\phi(x)$ is a meson field with $M$ its mass. $f$ is a coupling constant when the two photons are on the mass shell. But when the photons are off the mass shell, it depends on the momenta of the two photons in the momentum representation. Generally it depends on $(P_M-k_2)^2\equiv k_1^2$, $(P_M-k_1)^2\equiv k_2^2$, and $(k_1-k_2)^2$ (Fig. 1). This last, however, reduces to the linear combination of the first two:

$$(k_1-k_2)^2=2(k_1^2+k_2^2)-M^2\,.$$

Hence the form factor has the form

$$f=f(k_1^2,k_2^2)\,. \qquad (2)$$

With the interaction (1) we calculate the decay width of the meson$\to\gamma\gamma$ decay, taking into account the fact that there are identical particles in the final state,

$$\Gamma(\text{meson}\to\gamma\gamma)=|f(0,0)|^2M/16\pi\,. \qquad (3)$$

## III. SINGLE-PAIR CONVERSION OF PSEUDOSCALAR MESONS

This section treats the decays of the meson$\to l\bar{l}\gamma$ type. The processes were partly treated by one of the authors (T.M.),[6] so we briefly review the result and analyze the spectra. As the interaction between photons and leptons is given by conventional quantum electrodynamics, one easily obtains the matrix element of the meson$\to l\bar{l}\gamma$ decay (Fig. 2). Summing over the helicities of the final lepton pair, we have

$$\frac{d\Gamma(\text{meson}\to l\bar{l}\gamma)}{dx}=\frac{M\alpha}{12\pi^2}\frac{|f(x^2,0)|^2}{x}\left(1-\frac{x^2}{M}\right)^3 \times\left(1+\frac{2m^2}{x^2}\right)\left(1-\frac{4m^2}{x^2}\right)^{1/2}, \qquad (4)$$

where $m$ is the lepton mass, and $x^2=k'^2$. Here we note that the symmetry of the meson-$\gamma$-$\gamma$ vertex

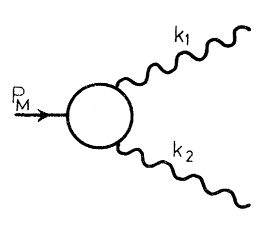


FIG. 1. Meson$\to\gamma\gamma$ decay.

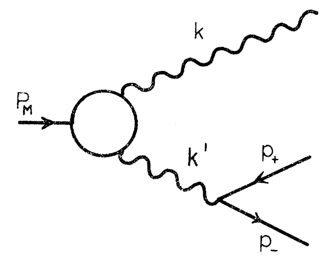


FIG. 2. Meson$\to l\bar{l}\gamma$ decay.

imposes $f(0,k'^2)=f(k'^2,0)$. The range of the variable $x$ is given by $2m\leq x\leq M$.

Combining Eqs. (3) and (4) we have

$$\frac{d\rho(\text{meson}\to l\bar{l}\gamma)}{dx}=\frac{2\alpha}{3\pi}\left|\frac{f(x^2,0)}{f(0,0)}\right|^2\frac{1}{x}\left(1-\frac{x^2}{M^2}\right)^3 \times\left(1+\frac{2m^2}{x^2}\right)\left(1-\frac{4m^2}{x^2}\right)^{1/2}, \qquad (5)$$

where the conversion rate $\rho$ is introduced as

$$\rho(\text{meson}\to l\bar{l}\gamma)=\frac{\Gamma(\text{meson}\to l\bar{l}\gamma)}{\Gamma(\text{meson}\to\gamma\gamma)}\,.$$

By putting

$$f(k'^2,0)=f(0,0)\,, \qquad (6)$$

namely, by adopting the assumption that the momentum dependence of the form factor be neglected, we can eliminate the coupling constant in Eq. (5). In such a case we plot the decay spectra $d\rho(\text{meson}\to l\bar{l}\gamma)/dx$ for the various decay processes of the single-pair conversion. In Figs. 3(a) and 3(b), we plot these results with the experimental values of Samios *et al.*[5] From the plot we immediately find that the contribution to the matrix element comes, for the most part, from small values of $x$. This, in turn, gives justification for neglecting the momentum dependence of the form factors for the decays into electron-positron pair conversion, because it is considered as a smooth function in the physical region and the electron mass is very small. The remarkable agreement of the experimental curve and the theoretical one indicates that the assumption (6) is quite reasonable.[7] For the muonic pair conversion there is not so definite a reason for neglecting the momentum dependence. The mass of the muon is not small, and one may not be convinced that the momentum dependence of the form factor could not play a role. In the muonic case there have been no experiments, unfortunately. So if the experiment were to be done and its values were to be on the

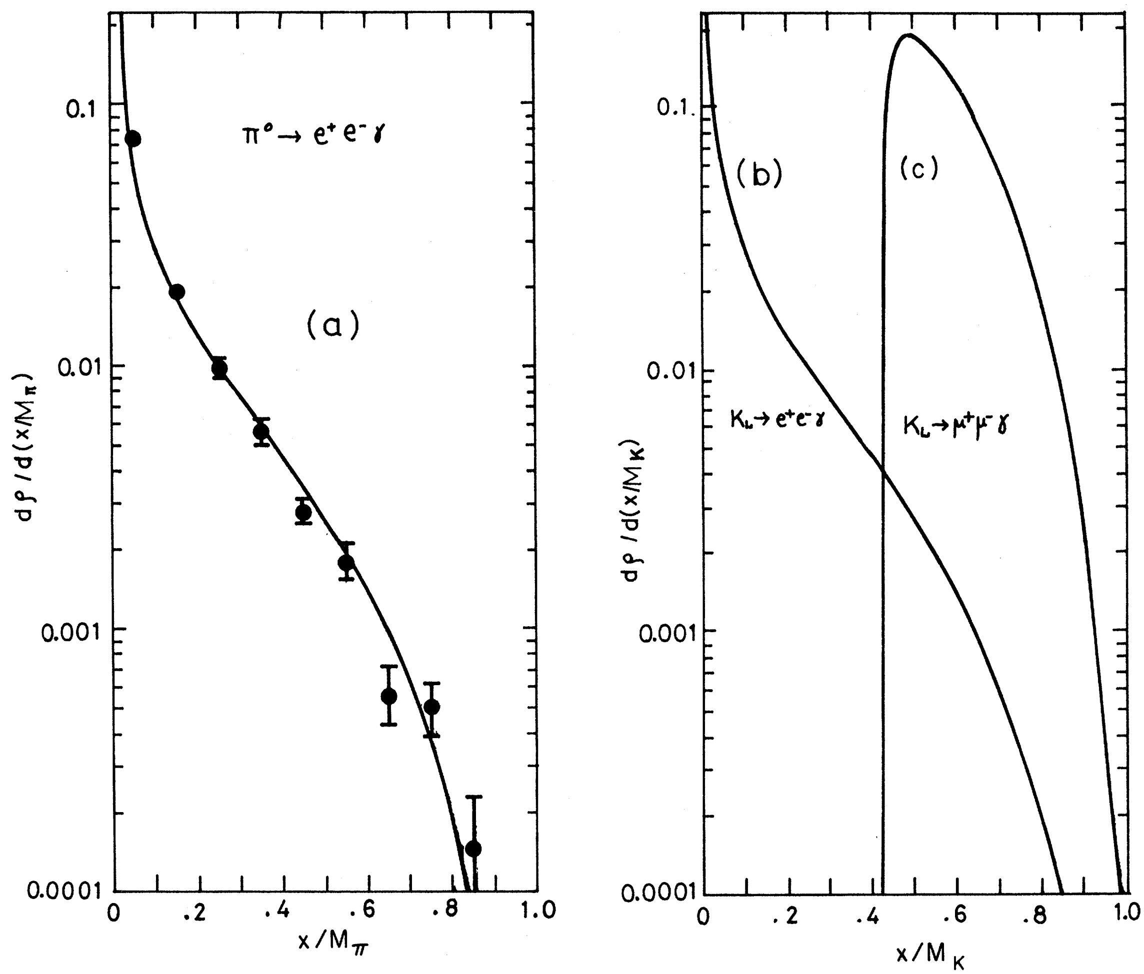


FIG. 3. Decay spectra with respect to the variable $x/M$ for the (a) $\pi^0 \to e^+e^-\gamma$, (b) $K_L \to e^+e^-\gamma$, and (c) $K_L \to \mu^+\mu^-\gamma$ decays. For the $K_L \to \mu^+\mu^-\gamma$ decay [(b)] the values multiplied by $10^2$ are plotted. Experimental values in (a) are taken from Refs. 5 and 7.

theoretical plot with (6) in Fig. 3(b), one could conclude that one has the justification for disregarding the momentum dependence. Otherwise, however, one must not assume Eq. (6) and one should estimate the form factor from the experimental spectra. As such, the experiments on meson $\to \mu^+\mu^-\gamma$ are desired urgently.

By integrating Eq. (5) using (6) we obtain

$$\rho(\text{meson} \to l\bar{l}\gamma) = \frac{2\alpha}{3\pi}\left\{\left[\ln\left(\frac{1+(1-a)^{1/2}}{1-(1-a)^{1/2}}\right)\right](1-\tfrac{9}{8}a^2+\tfrac{1}{8}a^3)+(1-a)^{1/2}(-\tfrac{7}{2}+\tfrac{13}{4}a+\tfrac{1}{4}a^2)\right\}, \quad (7)$$

where $a = 4m^2/M^2$. For $\pi^0$ decay we have

$$\frac{\Gamma(\pi^0 \to e^+e^-\gamma)}{\Gamma(\pi^0 \to \gamma\gamma)} = 0.0118,$$

which shows remarkable agreement with the experimental value.

For single-pair conversion of the $K_L$ meson, one would think that the vertex might be violated under $CP$ transformations. So one might add to the interaction (1)

$$H_I'(x) = (g/2M)F_{\mu\nu}(x)F^{\mu\nu}(x)\phi(x), \quad (8)$$

where $\phi(x)$ is a $K_1$ or $K_2$ field. The interaction $H_I(x)$ conserves $CP$ invariance for $K_2$ and violates it for $K_1$; the interaction $H_I'(x)$ conserves $CP$ invariance for $K_1$ and violates it for $K_2$. In the actual decay of $K_L$ and $K_S$, some complicated aspects appear. The mesons $K_L$ and $K_S$ do not form pure $CP$ eigenstates,

$$K_L = \epsilon K_1 + K_2$$

TABLE I. Numerical results of the conversion rates $\rho(\text{meson}\to l\bar{l}\gamma)=\Gamma(\text{meson}\to l\bar{l}\gamma)/\Gamma(\text{meson}\to\gamma\gamma)$ for various single-pair conversions.

| Decay modes | $\rho(\text{meson}\to l\bar{l}\gamma)$ |
|---|---|
| $\pi^0\to e^+e^-\gamma$ | $1.18\times10^{-2}$ |
| $K_{S,L}\to e^+e^-\gamma$ | $1.59\times10^{-2}$ |
| $K_{S,L}\to\mu^+\mu^-\gamma$ | $4.09\times10^{-4}$ |
| $\eta\to e^+e^-\gamma$ | $1.62\times10^{-2}$ |
| $\eta\to\mu^+\mu^-\gamma$ | $5.54\times10^{-4}$ |

and

$$K_S=K_1+\epsilon K_2\,,$$

with

$$\epsilon\simeq 2\times10^{-3}\times e^{-i\pi/4}$$

and the terms smaller than and equal to $\epsilon^2$ being neglected. There are two types of couplings for each of $K_{1,2}\to\gamma\gamma$:

$$K_1\xrightarrow{f_1,\,g_1}\gamma\gamma$$

and

$$K_2\xrightarrow{f_2,\,g_2}\gamma\gamma\,.$$

However, this complicated situation does not affect the final result that the decay spectrum is given by the form (5), with $f(k^2,0)$ now being replaced by the function of $f_{1,2}$ and $g_{1,2}$. In this case also the factor due to the form factor, $f(k^2,0)/f(0,0)$, is replaced by 1 if one disregards the momentum dependence of $f_{1,2}$ and $g_{1,2}$. The detailed derivation of this relation is seen in Ref. 6.

Hence the experiment on the decays of meson $\to l\bar{l}\gamma$ is suitable for determining the form factors. Table I gives the numerical result of Eq. (7). We note here that the decay rate of $K_L\to\mu^+\mu^-\gamma$ is much larger than that of $K_L\to\mu^+\mu^-$. In the latter case, Clark *et al.*[8] showed

$$\frac{\Gamma(K_L\to\mu^+\mu^-)}{\Gamma(K_L\to\text{all})}\lesssim1.8\times10^{-9}$$

(90% confidence level).[9]

On the other hand, our calculation shows

$$\frac{\Gamma(K_L\to\mu^+\mu^-\gamma)}{\Gamma(K_L\to\text{all})}=2.0\times10^{-7}$$

using the world average of $\Gamma(K_L\to\gamma\gamma)$.[10]

## IV. DOUBLE-PAIR CONVERSION OF A PSEUDOSCALAR MESON

The decay matrix element $\mathfrak{M}$ of the meson $\to l\bar{l}l\bar{l}$ decay is calculated by conventional quantum electrodynamics with the interaction (1). The element $\mathfrak{M}$, is divided into two parts $\mathfrak{M}_1$ and $\mathfrak{M}_2$; the former comes from Fig. 4(a), the latter from Fig. 4(b). The detailed calculation is stated in Appendix B from which we have

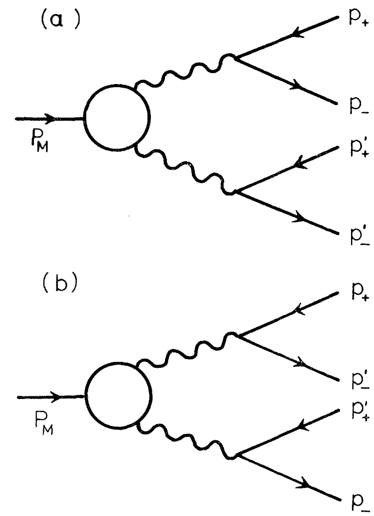


FIG. 4. Meson$\to l\bar{l}l\bar{l}$ decay. $\mathfrak{M}_1$ comes from the diagram (a), and $\mathfrak{M}_2$ from (b).

$$\begin{aligned}\mathfrak{M}_1&=\frac{2f}{M}\epsilon_{\mu\nu\rho\sigma}\frac{(p_++p_-)^\nu(p'_++p'_-)^\sigma}{(p_++p_-)^2(p'_++p')^2}\\&\quad\times\bar{u}(p_-)\gamma^\mu v(p_+)\bar{u}(p'_-)\gamma^\rho v(p'_+)\,,\\ \mathfrak{M}_2&=-\frac{2f}{M}\epsilon_{\mu\nu\rho\sigma}\frac{(p_++p'_-)^\nu(p'_++p_-)^\sigma}{(p_++p'_-)^2(p'_++p_-)^2}\\&\quad\times\bar{u}(p'_-)\gamma^\mu v(p_+)\bar{u}(p_-)\gamma^\rho v(p'_+)\,.\end{aligned}\tag{9}$$

The decay width is obtained by

$$\Gamma=\int|\mathfrak{M}|^2d\Phi\,,$$

where $d\Phi$ is the phase-space volume element. The phase-space integration is performed in Appendix A 2.

Using Eq. (9) $\Gamma$ is divided into three parts:

$$\begin{aligned}&\Gamma=\Gamma_1+\Gamma_2+\Gamma_{12}\,,\\&\Gamma_{1,2}=\int|\mathfrak{M}_{1,2}|^2d\Phi\,,\\&\Gamma_{12}=\int(\mathfrak{M}_1\mathfrak{M}_2^*+\mathfrak{M}_2\mathfrak{M}_1^*)\,d\Phi\,.\end{aligned}\tag{10}$$

We must note here that $|\mathfrak{M}_2|^2$ integrated over the whole phase space is identical with $|\mathfrak{M}_1|^2$ integrated over the whole domain:

$$\Gamma_1=\Gamma_2\,.$$

Here we introduce the conversion rate by the relations

$$\rho = \Gamma/\Gamma(\text{meson} \to \gamma\gamma),$$

$$\rho_i = \Gamma_i/\Gamma(\text{meson} \to \gamma\gamma).$$

Then we find

$$\rho = 2\rho_1 + \rho_{12}. \qquad (11)$$

Combining with Eq. (3), we can calculate $\rho$ and $\rho_i$. The first term on the right-hand side is

$$2\rho_1 = \frac{1}{\pi}\left(\frac{\alpha}{4\pi}\right)^2 \int_{2m}^{M-2m} dx_1 \int_{2m}^{M-x_1} dx_2 \int_{-\eta_1}^{\eta_1} dy_1 \int_{-\eta_2}^{\eta_2} dy_2 \int d\phi \left|\frac{f(x_1^2, x_2^2)}{f(0,0)}\right|^2 \left[1 - \frac{2(x_1^2 + x_2^2)}{M^2} + \frac{(x_1^2 - x_2^2)^2}{M^4}\right]^{3/2}$$
$$\times \left\{\left[\frac{1}{x_1 x_2} + \left(\frac{y_1^2}{x_1} + \frac{4m^2}{x_1^3}\right)\left(\frac{y_2^2}{x_2} + \frac{4m^2}{x_2^3}\right)\right]\sin^2\phi + \left[\frac{y_1^2 + y_2^2}{x_1 x_2} + \frac{4m^2(x_1^2 + x_2^2)}{x_1^3 x_2^3}\right]\cos^2\phi\right\}, \qquad (12)$$

where the definition of the new variables is mentioned in Appendixes A and B, and

$$\eta_{1,2} = [1 - (2m/x_{1,2})^2]^{1/2}.$$

In the case of the $K_L \to \mu^+\mu^- e^+ e^-$ decay, the situation becomes simpler. We have $\Gamma_{12} = 0$, and the terms $4m^2/x_2^3$ and $4m^2(x_1^2 + x_2^2)/x_1^3 x_2^3$ of the integrand of (12) are replaced by $4m'^2/x_2^3$ and $4mm'(x_1^2 + x_2^2)/x_1^3 x_2^3$, respectively, where $x_2^2$ is the four-momentum squared for the muonic pair, with $m'$ its mass.

Following the same reasoning mentioned in Sec. III, we can disregard the momentum dependence of the form factors for the conversion into electron-positron pairs. Then Eq. (12) just reduces to the Kroll-Wada formula in the case of the $\pi^0 \to e^+e^-e^+e^-$ decay. As was stressed in Ref. 3, Kroll and Wada neglected the cross term $\rho_{12}$. This term expresses the effect of exchange of leptons or antileptons. It is obtained in Eq. (A6) in Appendix B. Using a computer, we numerically calculate each term of Eq. (11). In Table II we list each term of Eq. (11) for various double-conversion processes with the momentum dependence of the form factor being neglected. As is seen immediately from Table II, the contribution of the cross term is small by a factor $\frac{1}{10}$ or so compared with the direct (Kroll-Wada) term. Notwithstanding its smallness, this correction term is important,[11] because the main purpose of measuring the decays of the type of internal conversion is to try to understand the meson-$\gamma$-$\gamma$ vertices. For example, the measurement of the $K_L \to \mu^+\mu^- e^+ e^-$ and/or $K_L \to e^+e^-e^+e^-$ decay informs us of the $K_L$-$\gamma$-$\gamma$ vertex: how much its vertex is violated under the $CP$ transformations. This in turn influences the rate of the $K_L \to \mu^+\mu^-$ decay. In such precise determination of the vertex, the correction term should not be disregarded. The discussion about the $CP$-conserving and -nonconserving interaction is not so simple in the case of double-pair conversion as in the single-pair conversion of Sec. III. This attractive problem is set aside until the future.

TABLE II. Numerical results of the conversion rates for various double-pair conversions.

| Decay modes | $2\rho_1$ | $\rho_{12}$ | $\rho$ |
|---|---|---|---|
| $\pi^0 \to e^+e^-e^+e^-$ | $3.46\times10^{-5}$ | $-0.18\times10^{-5}$ | $3.28\times10^{-5}$ |
| $K_L \to e^+e^-e^+e^-$ | $6.26\times10^{-5}$ | $-0.35\times10^{-5}$ | $5.89\times10^{-5}$ |
| $K_L \to e^+e^-\mu^+\mu^-$ | $1.42\times10^{-6}$ | 0 | $1.42\times10^{-6}$ |
| $K_L \to \mu^+\mu^-\mu^+\mu^-$ | $0.997\times10^{-9}$ | $-0.051\times10^{-9}$ | $0.946\times10^{-9}$ |
| $\eta \to e^+e^-e^+e^-$ | $6.50\times10^{-5}$ | $-0.36\times10^{-5}$ | $6.14\times10^{-5}$ |
| $\eta \to e^+e^-\mu^+\mu^-$ | $1.99\times10^{-6}$ | 0 | $1.99\times10^{-6}$ |
| $\eta \to \mu^+\mu^-\mu^+\mu^-$ | $6.73\times10^{-9}$ | $-0.50\times10^{-9}$ | $6.23\times10^{-9}$ |

Henceforth we take the form factor to be an on-shell coupling constant and calculate the spectra of the meson $\to l\bar{l}l\bar{l}$ decay. This is done by fixing a special variable and integrating over the remaining variables in Eq. (10). We must note here that although the contribution of $|\mathfrak{M}_1|^2$ and $|\mathfrak{M}_2|^2$ to the decay width is the same, their contribution to the decay spectrum is quite different as long as the remaining variables are integrated. The Kroll-Wada term only is insufficient to analyze the spectra. In Figs. 5–7, we plot the decay spectra with respect to the variables $x_1/M$, $y_1$, and $\phi$ for various decay processes. Each contribution of $|\mathfrak{M}_1|^2$, $|\mathfrak{M}_2|^2$, and $\mathfrak{M}_1\mathfrak{M}_2^* + \mathfrak{M}_2\mathfrak{M}_1^*$ is also plotted, as well as the total decay spectra which can be measured. Remarkable is the fact that the $|\mathfrak{M}_1|^2$ contribution shows a sharp peak for small $x_1$, and the $|\mathfrak{M}_2|^2$ contribution shows a rather broad plateau. This apparent difference is due to the phase-space integration, though the area of the spectrum is just the same in each term. The interference between these two is what the contribution of $\mathfrak{M}_1\mathfrak{M}_2^* + \mathfrak{M}_2\mathfrak{M}_1^*$ shows. Adding the three contributions we have the resultant decay spectrum.

There has been an experiment concerning the $\pi^0 \to e^+e^-e^+e^-$ decay. Samios *et al.*[5] analyzed the decay spectrum. But, unfortunately, their analysis is quite misleading. They assumed the condition

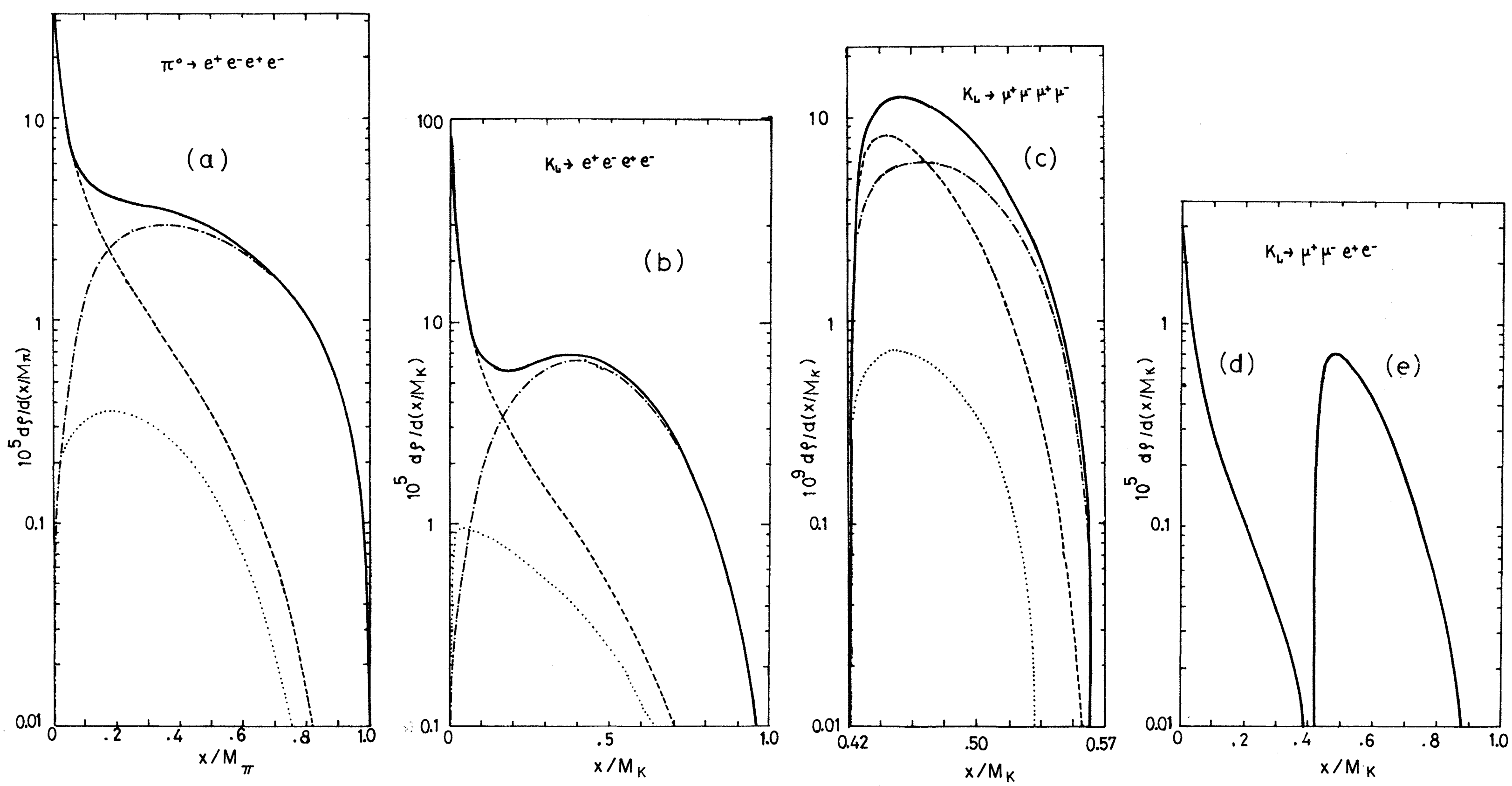


FIG. 5. $x$ distribution for the meson$\rightarrow l\bar{l}l\bar{l}$ decays. In the $K_L \rightarrow \mu^+\mu^-e^+e^-$ decay the distribution is plotted in two ways: taking $x$ as the momentum of $e^+e^-$ (d) or $\mu^+\mu^-$ (e). The contributions from $|\mathfrak{M}_1|^2$, $|\mathfrak{M}_2|^2$, and $\mathfrak{M}_1\mathfrak{M}_2^* + \mathfrak{M}_2\mathfrak{M}_1^*$ are represented by broken lines, dashed-dotted lines, and dotted lines, respectively. The solid lines represent the total conversion distribution.

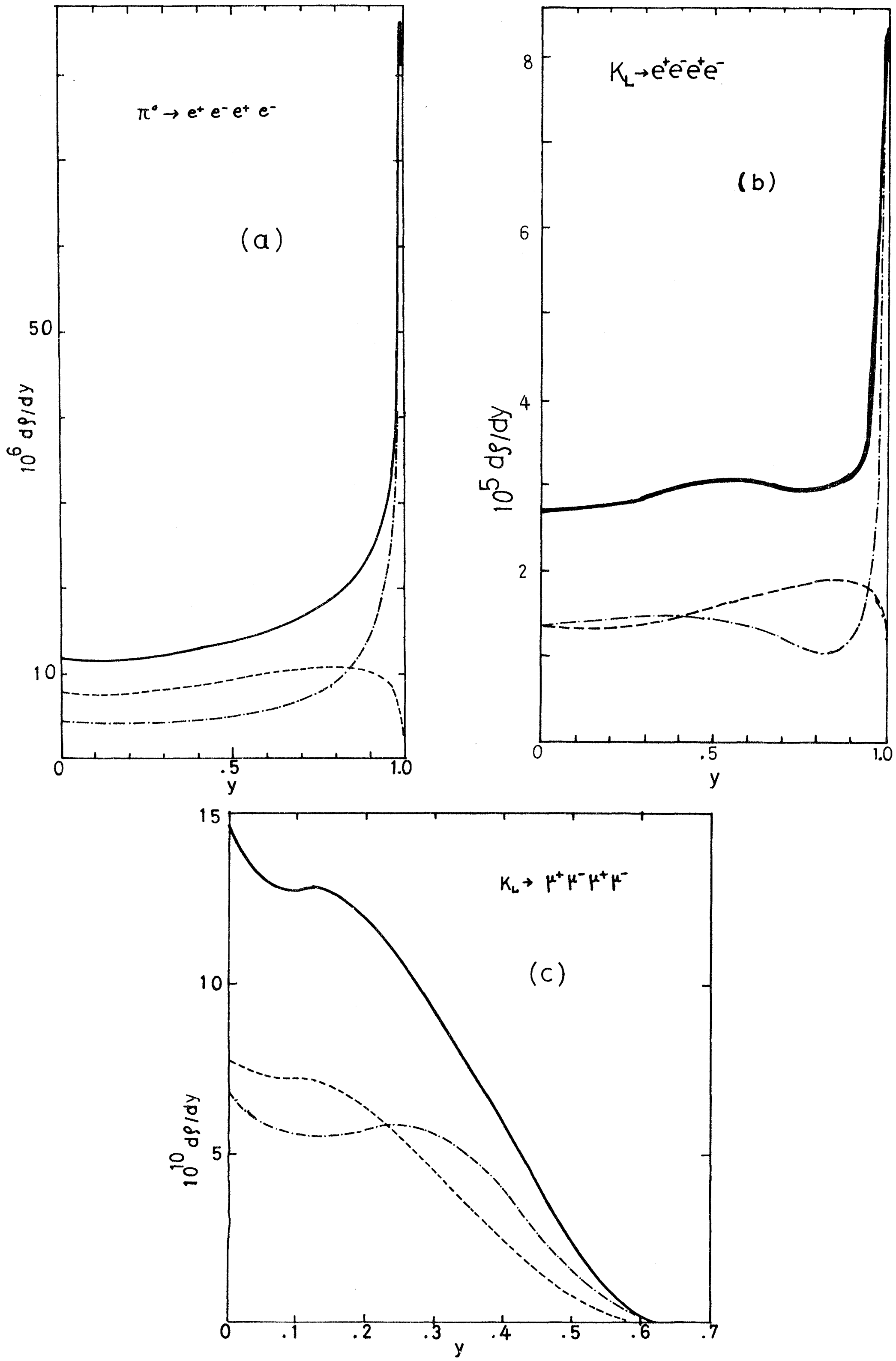


FIG. 6. $y$-distribution for the meson$\to l\bar{l}l\bar{l}$ decays. Broken lines and dashed-dotted lines represent the contributions $|\mathfrak{M}_1|^2$ and $|\mathfrak{M}_2|^2$, respectively. Solid lines represent the total distribution.

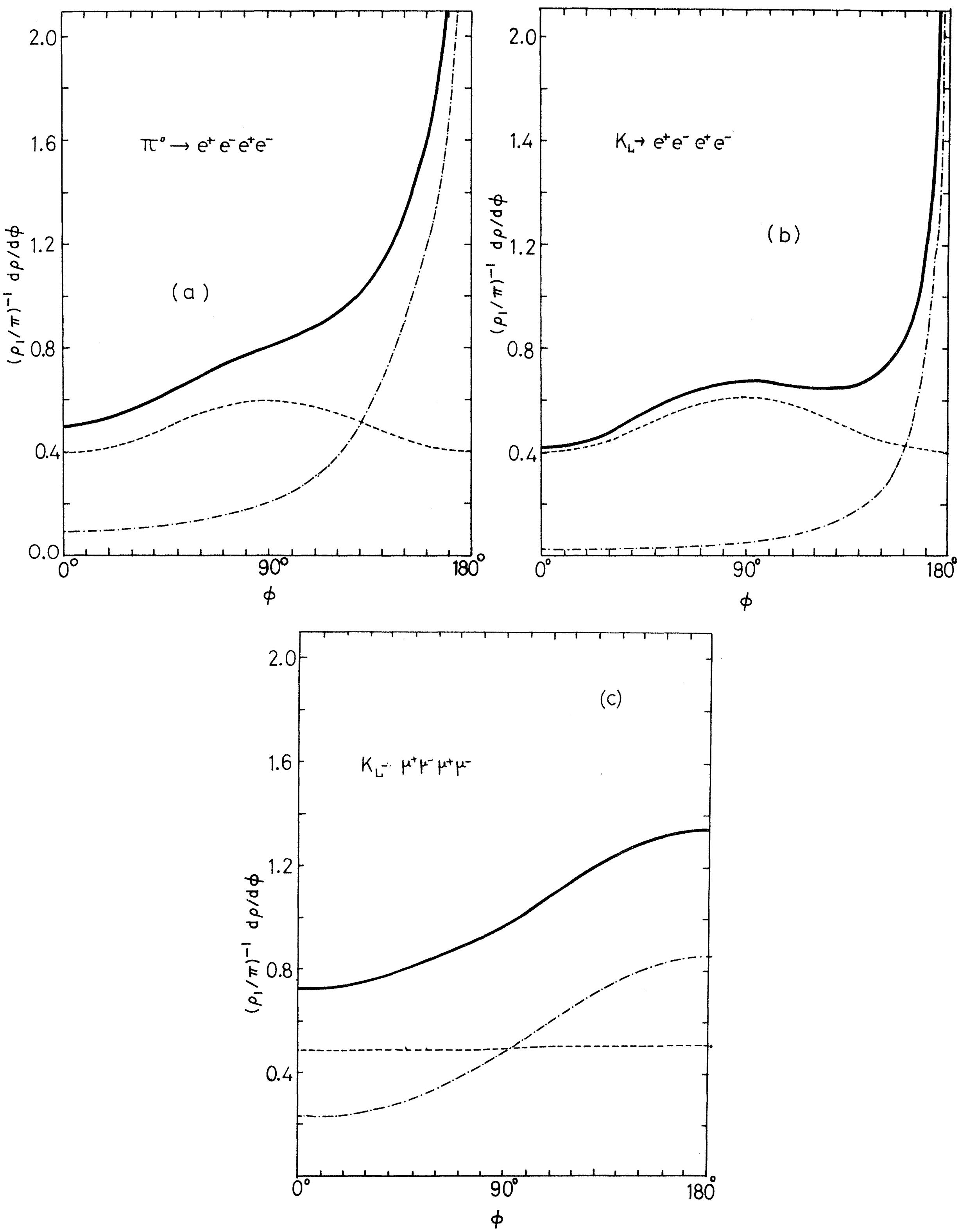


FIG. 7. Angular distribution for the meson$\rightarrow l\bar{l}l\bar{l}$ decays. $\phi$ is the angle between $\vec{q}$ and $\vec{q}\,'$. Solid, broken, and dashed-dotted lines represent the same things as in Fig. 6.

$$|\mathfrak{M}_1|^2 > 10|\mathfrak{M}_2|^2 + (\mathfrak{M}_1\mathfrak{M}_2^* + \mathfrak{M}_2\mathfrak{M}_1^*) . \tag{13}$$

In the case where this condition did not hold, the pairing which gave the larger value for the matrix element was used. By such a procedure they analyzed the spectra only with the Kroll-Wada term. But, evidently, this analysis was valid only for small $x_1$. Except in that region, one could not say anything about the spectrum. Now we cannot employ their results of the decay spectrum. We do hope that some experiments of this kind will be done in the near future and analyzed, taking the above points well into account.

We now discuss the form factors. If the experimental curves lie on the theoretical ones shown in Figs. 5–7, we can take the form factor to be an on-shell coupling constant:

$$f(k_1^2, k_2^2) = f(0, 0) .$$

This would be the case for the electronic case,[11] because near the threshold $d\rho/dx_1$ shows a sharp peak. Moreover, we have the fact that the case of the $\pi^0$-$\gamma$-$\gamma$ vertex with one off-shell photon allows us to have approximately a constant form factor. In muonic conversion nothing could be said about the form factor. There has been no experiment in this case. The threshold value of $x_1$ is large (equal to $2m_\mu$), so it might happen that one could not replace the form factor by a coupling constant. Hence the spectral analysis of the decays determines the form factors, or in other words, it determines the meson-$\gamma$-$\gamma$ interaction.

## V. CONCLUSION

The experiments on the decays of the meson $\rightarrow l\bar{l}l\bar{l}$ type are important in the following respects.

(a) In the electronic double-pair conversion of $\pi^0 \rightarrow e^+e^-e^+e^-$ and $\eta \rightarrow e^+e^-e^+e^-$, we can check the interaction of the form (1). Here we have good reason to disregard the momentum dependence of the form factor.

(b) In the muonic double-pair conversion of $\eta \rightarrow \mu^+\mu^-\mu^+\mu^-$ and $\eta \rightarrow \mu^+\mu^-e^+e^-$, we can estimate the momentum dependence of the form factor.

(c) The decays of $K_L \rightarrow e^+e^-e^+e^-$, $K_L \rightarrow \mu^+\mu^-\mu^+\mu^-$, and $K_L \rightarrow \mu^+\mu^-e^+e^-$ are most attractive. If the invariance under the $CP$ transformations holds, the momentum dependence of the form factor for $K_L \rightarrow \mu^+\mu^-\mu^+\mu^-$ and $K_L \rightarrow \mu^+\mu^-e^+e^-$ can be estimated from the experiments. From the form factor we can calculate the decay width of $K_L \rightarrow \mu^+\mu^-$ via the two-photon intermediate state. Hence we have definite knowledge of $\Gamma(K_L \rightarrow \mu^+\mu^-)$. At the present time the estimate is done theoretically with respect to the lower bound. If the theory were to be violated under the $CP$ transformations, especially when the $K_{1,2}$-$\gamma$-$\gamma$ vertex does not conserve $CP$ invariance, the most interesting experiment would be that on $K_L \rightarrow e^+e^-e^+e^-$.

Disregarding the momentum dependence, there are four coupling constants:

$$H_{I\,1,2}(x) = \left(\frac{f_{1,2}}{4M}\right)\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}(x)F^{\rho\sigma}(x)\phi_{K_{1,2}}(x),$$

$$H_{I\,1,2}(x) = \left(\frac{g_{1,2}}{2M}\right)F^{\mu\nu}(x)F_{\mu\nu}(x)\phi_{K_{1,2}}(x) .$$

By analyzing the spectra of the $K_L \rightarrow e^+e^-e^+e^-$ decay with the four parameters $f_{1,2}$ and $g_{1,2}$, we know how much the $CP$-conserving and -nonconserving interactions are mixed. This determination of the $K_{1,2}$-$\gamma$-$\gamma$ vertices allows us to obtain an estimate of $\Gamma(K_L \rightarrow \mu^+\mu^-)$.

The detailed analysis of the case where the vertices are violated under the $CP$ transformations is quite an interesting object and will appear separately. Anyway the experiments on these single- and double-conversion decays are needed quite urgently.

## APPENDIX A: PHASE-SPACE INTEGRATION

In Appendix A we treat the phase-space integration for the decays of (a) meson $\rightarrow l\bar{l}\gamma$ and (b) meson $\rightarrow l\bar{l}l\bar{l}$. In the whole calculation we take the meson rest frame $P_M = (M, \vec{0})$.

### 1. Three-Body Decay (Fig. 2)

The phase-space volume element $d\Phi$ is given by

$$d\Phi = \frac{m^2 d^3p_+ d^3p_- d^3k}{(2\pi)^8 4Mp_+^0 p_-^0 k^0}(2\pi)^4\delta^{(4)}(p_+ + p_- + k - P_M)$$

$$= \frac{m^2 d^4p_+ d^3p_-}{(2\pi)^5 4Mp_+^0 p_-^0 k^0}\delta(p_+^0 + p_-^0 + k^0 - M) .$$

We transform the variables $\vec{p}_+$, and $\vec{p}_-$ into $|\vec{q}|$, $\lambda$, $\phi$, and $\vec{P}$ by the following relations:

$$\vec{p}_+ = \vec{q} + \tfrac{1}{2}(1+\lambda)\vec{P},$$

$$\vec{p}_- = -\vec{q} + \tfrac{1}{2}(1-\lambda)\vec{P},$$

$$\vec{q}\cdot\vec{P} = 0,$$

and $\phi$ is the azimuthal angle of $\vec{q}$. Then we immediately obtain the relation

$$d^3p_+ d^3p_- = \tfrac{1}{2}|\vec{q}|\,|\vec{P}|\,d|\vec{q}|\,d\lambda\, d\phi\, d^3P$$

and moreover we have

$$q\,dq\,\delta(p_+^0 + p_-^0 + k^0 - M) = \frac{p_+^0 p_-^0}{P^0} ,$$

where

$$P^0 = p_+^0 + p_-^0 .$$

The range of the variable $\lambda$ is obtained by the requirement $|\vec{q}|^2 \geqslant 0$, where by a straightforward calculation

$$|\vec{q}|^2 = \tfrac{1}{4} P^2 \left(1 - \lambda^2 \frac{|\vec{P}|^2}{P^{02}}\right) - m^2 .$$

We have finally

$$d\Phi = \frac{m^2 |\vec{P}|^2 d|\vec{P}| d\lambda d\phi d\Omega}{8(2\pi)^2 M P^0},$$

with $d\Omega$ the solid-angle differential around the $\vec{P}$ vector.

### 2. Four-Body Decay [Figs. 4(a) and 4(b)]

In this case we transform the integration variables $\vec{p}_+$, $\vec{p}_-$, $\vec{p}_+'$, and $\vec{p}_-'$ by the following relations:

$$\begin{aligned}
\vec{p}_+ &= \vec{q} + \tfrac{1}{2}(1+\lambda)\vec{P},\\
\vec{p}_+' &= \vec{q}' + \tfrac{1}{2}(1+\mu)\vec{P}',\\
\vec{p}_- &= -\vec{q} + \tfrac{1}{2}(1-\lambda)\vec{P},\\
\vec{p}_-' &= -\vec{q}' + \tfrac{1}{2}(1-\mu)\vec{P}',\\
\vec{q}\cdot\vec{P} &= 0,\\
\vec{q}'\cdot\vec{P}' &= 0.
\end{aligned} \tag{A1}$$

Then we immediately obtain as before

$$\begin{aligned}
d^3p_+ d^3p_- &= \tfrac{1}{2}|\vec{q}|\,|\vec{P}|\,d|\vec{q}|\,d\lambda d\phi d^3P,\\
d^3p_+' d^3p_-' &= \tfrac{1}{2}|\vec{q}'|\,|\vec{P}'|\,d|\vec{q}'|\,d\lambda d\phi d^3P'.
\end{aligned}$$

The phase-space volume element $d\Phi$ is calculated using the conventional technique:

$$\begin{aligned}
d\Phi &= \frac{m^4}{2M(2\pi)^{12}} \frac{d^3p_+ d^3p_- d^3p_+' d^3p_-'}{4p_+^0 p_-^0 p_+'^0 p_-'^0} (2\pi)^4 \delta^{(4)}(p_+ + p_- + p_+' + p_-' - P_M)\\
&= \frac{m^4}{8M(2\pi)^8} d^4K\, d^4K' \delta^{(4)}(K+K'-P_M) \frac{d^3p_+ d^3p_-}{p_+^0 p_-^0} \delta^{(4)}(p_+ + p_- - K) \frac{d^3p_+' d^3p_-'}{p_+'^0 p_-'^0} \delta^{(4)}(p_+' + p_-' - K'),
\end{aligned}$$

where the factor $\frac{1}{4}$ is added because there are two sets of identical particles in the application to the final $e^+e^-e^+e^-$ or $\mu^+\mu^-\mu^+\mu^-$ state.

$$\begin{aligned}
I_1 &= \int \frac{d^3p_+ d^3p_-}{p_+^0 p_-^0} \delta^{(4)}(p_+ + p_- - K)\\
&= \tfrac{1}{2}\int \frac{|\vec{q}|\,|\vec{P}|\,d|\vec{q}|\,d\lambda d\phi d^3P}{p_+^0 p_-^0} \delta^{(4)}(p_+ + p_- - K)\\
&= \frac{\pi|\vec{K}|}{K^0} \int d\lambda ,
\end{aligned}$$

with $\vec{K} = \vec{p}_+ + \vec{p}_- = \vec{P}$ and $K^0 = p_+^0 + p_-^0 = P^0$.

$$\begin{aligned}
I_2 &= \int \frac{d^3p_+' d^3p_-'}{p_+'^0 p_-'^0} \delta^{(4)}(p_+' + p_-' - K')\\
&= \frac{\pi|\vec{K}'|}{K'^0} \int d\mu ,
\end{aligned}$$

with $\vec{K}' = \vec{p}_+' + \vec{p}_-' = \vec{P}' = -\vec{P}$ and $K'^0 = p_+'^0 + p_-'^0 = P'^0$. Moreover, we have

$$\begin{aligned}
\int d^4K\, d^4K' \delta^{(4)}(K+K'-P_M) \cdots &= \int dx_1^2\, dx_2^2\, d^4K\, d^4K' \delta^{(4)}(K+K'-P_M) \delta(x_1^2 - K^2)\delta(x_2^2 - K'^2) \cdots\\
&= \pi \int dx_1^2\, dx_2^2 \frac{|\vec{P}|}{M} \cdots ,
\end{aligned}$$

where

$$|\vec{P}| = \frac{1}{2M}[x_1^4 + x_2^4 + M^4 - 2x_1^2x_2^2 - 2x_1^2M^2 - 2x_2^2M^2]^{1/2}. \tag{A2}$$

Hence we have

$$I = \int d\Phi\, |\mathfrak{M}|^2 = \frac{m^4\pi}{32M^2(2\pi)^8} \int dx_1^2\, dx_2^2 d\lambda\, d\mu\, d\phi\, d\phi' \frac{|\vec{P}|^3}{P^0 P'^0} |\mathfrak{M}|^2 . \tag{A3}$$

The range of the variables $\lambda$ and $\mu$ is determined by

$|\vec{q}|^2 \geqslant 0$ and $|\vec{q}'|^2 \geqslant 0$, where

$$|\vec{q}|^2 = \tfrac{1}{4}x_1{}^2\left(1-\frac{|\vec{P}|^2}{P^{02}}\lambda^2\right) - m^2\,, \tag{A4}$$

$$|\vec{q}\,'|^2 = \tfrac{1}{4}x_2{}^2\left(1-\frac{|\vec{P}|^2}{P'^{02}}\mu^2\right) - m^2\,. \tag{A5}$$

Of course when one applies to the $K_L(\eta)\to e^+e^-\mu^+\mu^-$ decay, one may remove the $\frac{1}{4}$ factor and replace the $m^2$ in Eqs. (A3)–(A5), for example, by the respective $m_e{}^2$ or $m_\mu{}^2$.

## APPENDIX B: MATRIX ELEMENT FOR THE MESON $\to l\bar{l}l\bar{l}$ DECAY

The basic interaction we take for the meson-$\gamma$-$\gamma$ vertex is

$$H_I(x) = (f/4M)\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}(x)F^{\rho\sigma}(x)\phi(x)\,. \tag{B1}$$

With this interaction we write down the matrix element, which is divided into two parts. Each of them corresponds to the contribution from the diagrams of Figs. 3(a) and 3(b), respectively.

$$\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2\,,$$

$$\mathfrak{M}_1 = \frac{2f}{M}\epsilon_{\mu\nu\rho\sigma}\frac{(p_+ + p_-)^\nu(p_+' + p_-')^\sigma}{(p_+ + p_-)^2(p_+' + p_-')^2}\,\bar{u}(p_-)\gamma^\mu v(p_+)\cdot\bar{u}(p_-')\gamma^\rho v(p_+')\,,$$

$$\mathfrak{M}_2 = -\frac{2f}{M}\epsilon_{\mu\nu\rho\sigma}\frac{(p_+ + p_-')^\nu(p_+' + p_-)^\sigma}{(p_+ + p_-')^2(p_+' + p_-)^2}\,\bar{u}(p_-')\gamma^\mu v(p_+)\cdot\bar{u}(p_-)\gamma^\rho v(p_+')\,.$$

We calculate $|\mathfrak{M}|^2$, summing over the helicities of the final leptons and antileptons. In this calculation there appears a trace of the product of eight $\gamma$'s, which induces 105 terms. So it is very complicated. After all calculations, we have

$$\begin{aligned}
|\mathfrak{M}_1|^2 = {}& \frac{4|f|^2}{m^4M^2}\left(\frac{1}{(p_+ + p_-)^2(p_+'p_-')^2}\right)^2 \epsilon_{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}(p_+ + p_-)^\nu(p_+' + p_-')^\sigma(p_+ + p_-)^\beta(p_+' + p_-')^\delta \\
&\times[(m^2 + p_+p_-)(m^2 + p_+'p_-')g^{\mu\alpha}g^{\rho\gamma} - (m^2 + p_+p_-)g^{\mu\alpha}(p_+'^{\gamma}p_-'^{\rho} + p_+'^{\rho}p_-'^{\gamma}) \\
&\quad - (m^2 + p_+'p_-')g^{\rho\gamma}(p_+^\alpha p_-^\mu + p_+^\mu p_-^\alpha) + (p_+^\alpha p_-^\mu + p_+^\mu p_-^\alpha)(p_+'^{\gamma}p_-'^{\rho} + p_+'^{\rho}p_-'^{\gamma})]\,,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{M}_1\mathfrak{M}_2^* + \mathfrak{M}_2\mathfrak{M}_1^* = {}& -\frac{4|f|^2}{m^4M^2}\,\frac{1}{(p_+ + p_-)^2(p_+' + p_-')^2(p_+ + p_-')^2(p_+' + p_-)^2} \\
&\times \epsilon_{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}g^{\rho\gamma}(p_+ + p_-)^\nu(p_+' + p_-')^\sigma(p_+ + p_-')^\beta(p_+' + p_-)^\delta \\
&\times[(m^2 + p_+p_-)(p_+'^{\mu}p_-'^{\alpha} - p_+'^{\alpha}p_-'^{\mu}) + (m^2 + p_+'p_-')(p_+^\mu p_-^\alpha - p_+^\alpha p_-^\mu) - (m^2 - p_-p_+')(p_+^\mu p_-'^{\alpha} + p_+^\alpha p_-'^{\mu}) \\
&\quad + (m^2 - p_+p_+')(p_-^\mu p_-'^{\alpha} + p_-^\alpha p_-'^{\mu}) - (m^2 + p_+'p_-)(p_+^\mu p_-'^{\alpha} - p_+^\alpha p_-'^{\mu}) \\
&\quad + (m^2 + p_+p_-')(p_-^\mu p_+'^{\alpha} - p_-^\alpha p_+'^{\mu})]\,.
\end{aligned}$$

$|\mathfrak{M}_2|^2$ is given by the replacement of $p_- \leftrightarrow p_-'$ in $|\mathfrak{M}_1|^2$.

The spectra are not studied with respect to the original variables of $p_+$, $p_-$, $p_+'$, and $p_-'$, but with respect to the new variables which are obtained by transforming the above variables by the relations (A1). So we write down the matrix element in view of these new variables.

$$|\mathfrak{M}_1|^2 = \left(\frac{2|f|}{m^2x_1{}^2x_2{}^2}\right)^2\vec{P}^2\left[\tfrac{1}{2}x_1{}^2x_2{}^2 - x_2{}^2\vec{q}^{\,2} - x_1{}^2\vec{q}\,'^2 + \frac{4(\vec{P}\cdot\vec{q}\times\vec{q}\,')^2}{P^2}\right],$$

$$\begin{aligned}
\mathfrak{M}_1\mathfrak{M}_2^* + \mathfrak{M}_2\mathfrak{M}_1^* = {}& -\left(\frac{2|f|}{m^2x_1x_2}\right)^2\frac{\vec{P}^2}{(p_+ + p_-')^2(p_+' + p_-)^2}\{(m^2 + p_+p_-)(\vec{q}\,'^2 - \vec{q}\cdot\vec{q}\,') + (m^2 + p_+'p_-')(\vec{q}^{\,2} - \vec{q}\cdot\vec{q}\,') \\
&+ \tfrac{1}{2}(m^2 - p_-p_-')[(\lambda + \mu - 2)\vec{q}\cdot\vec{q}\,' + (\mu + 1)\vec{q}^{\,2} + (\lambda + 1)\vec{q}\,'^2] \\
&+ \tfrac{1}{2}(m^2 - p_+p_+')[-(\lambda + \mu + 2)\vec{q}\cdot\vec{q}\,' - (\mu - 1)\vec{q}^{\,2} - (\lambda - 1)\vec{q}\,'^2] \\
&+ \tfrac{1}{2}(m^2 + p_-p_+')[(-\lambda + \mu - 2)\vec{q}\circ\vec{q}\,' - (\mu - 1)\vec{q}^{\,2} + (\lambda + 1)\vec{q}\,'^2] \\
&+ \tfrac{1}{2}(m^2 + p_+p_-')[(\lambda - \mu - 2)\vec{q}\circ\vec{q}\,' + (\mu + 1)\vec{q}^{\,2} - (\lambda - 1)\vec{q}\,'^2]\}\,,
\end{aligned} \tag{B2}$$

$$|\mathfrak{M}_2|^2=\left(\frac{2|f|}{m^2(p_++p'_-)^2(p'_++p_-)^2}\right)^2\{\tfrac{1}{2}(m^2+p_+p'_-)(m^2+p'_+p_-)[4(\vec{q}-\vec{q}\,')^2+(\lambda+\mu)^2\vec{P}^2]$$
$$-\tfrac{1}{2}(m^2+p_-p'_+)[4(\vec{q}\times\vec{q}\,')^2+\vec{P}^2((1+\mu)\vec{q}-(1+\lambda)\vec{q}\,')^2]$$
$$-\tfrac{1}{2}(m^2+p_+p'_-)[4(\vec{q}\times\vec{q}\,')^2+\vec{P}^2((1-\mu)\vec{q}-(1-\lambda)\vec{q}\,')^2]+4(\vec{P}\cdot\vec{q}\times\vec{q}\,')^2\},$$

where $x_1^2=P^2$, $x_2^2=P'^2$, and

$$p_+p'_-=\vec{q}\cdot\vec{q}\,'+\tfrac{1}{4}\left[(PP')\left(1-\frac{\vec{P}^2}{P^0P'^0}\lambda\mu\right)+\left(\frac{\lambda}{P^0}-\frac{\mu}{P'^0}\right)\vec{P}^2M\right],$$

$$p_-p'_+=\vec{q}\cdot\vec{q}\,'+\tfrac{1}{4}\left[(PP')\left(1-\frac{\vec{P}^2}{P^0P'^0}\lambda\mu\right)-\left(\frac{\lambda}{P^0}-\frac{\mu}{P'^0}\right)\vec{P}^2M\right],$$

$$p_+p_-=\tfrac{1}{2}x_1^2-m^2,$$

$$p'_+p'_-=\tfrac{1}{2}x_2^2-m^2.$$

$\vec{P}^2$ is given by Eq. (A2). Finally we transform the variables $\lambda$ and $\mu$ into $y_1$ and $y_2$ as follows:

$$y_1=\frac{|\vec{P}|}{P^0}\lambda,\quad y_2=\frac{|\vec{P}|}{P'^0}\mu.$$

With respect to these variables ($x$'s, $y$'s) and $\phi$ (the angle between $\vec{q}$ and $\vec{q}\,'$), we analyze the spectra. Note that although $|\mathfrak{M}_1|^2$ and $|\mathfrak{M}_2|^2$ contribute equally in the decay width, i.e.,

$$\int|\mathfrak{M}_1|^2d\Phi=\int|\mathfrak{M}_2|^2d\Phi,$$

their contribution to the shape of the spectra in these variables defined previously is quite different. For example, the contribution from $|\mathfrak{M}_1|^2$ shows a sharp peak at a small value of $x_1$, and that of $|\mathfrak{M}_2|^2$ shows rather a broad plateau in $x_1$, with the same area.

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$$\frac{\Gamma(K_L\to\mu^+\mu^-)}{\Gamma(K_L\to\text{all})}\approx(1.0\pm0.5)\times10^{-8}.$$

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$$f(x_1^2,x_2^2)=f(0,0)\,[1+a(x_1^2+x_2^2)/M^2],$$

the decay width is reduced to the form $\Gamma=\Gamma_0(1+0.118a)$. The value of $a$ is quite small. For example, the latest experiment of Devons *et al.* shows $a=0.01$. Also the radiative correction is considered as small as 1% or so by the analogy of the radiative correction to $\pi^0\to e^+e^-\gamma$ [K. O. Mikaelian and J. Smith, Phys. Rev. D 5, 1763 (1972)].