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Comment on Broken Scale Invariance

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Broken scale invariance is briefly reviewed. A constraint on the equal-time commutator between the trace of the energy-momentum tensor and symmetry-breaking terms is discussed. A sum rule relating matrix elements of chiral-symmetry-breaking terms with their dimensions is derived.

Much has been written[1] over the past few years about a theory in which the energy density operator is given by

$$\theta_{00}=\overline{\theta}_{00}+\epsilon\delta+\epsilon(u_0+cu_8). \tag{1}$$

In this relation, $\overline{\theta}_{00}$ is SU(3)×SU(3)- and dilatation-invariant, δ is a c number which breaks dilatation invariance but preserves SU(3)×SU(3) symmetry, and u_0 and u_8 are scalar densities which break both dilatation and SU(3)×SU(3) invariance and transform according to a $(3,3^*)+(3,3^*)$ representation of SU(3)×SU(3).

Equation (1) is supposed to describe a world in which dilatation and SU(3)×SU(3) are broken spontaneously, leading in the SU(3)×SU(3) limit ($\epsilon=0$) to an octet of massless pseudoscalar mesons and a massless scalar meson. The scalar meson is usually referred to as the dilaton.

In spite of the fact that there are several good reviews on the subject, there are still some points which, in our opinion, have not been sufficiently discussed in the literature. The purpose of this note is twofold: We shall first make a few comments on these points, and then we shall derive a sum rule relating matrix elements of u_0 and u_8 with the dimension d_u associated with these operators. The dimension of an operator $w(x)$, having a unique dimension d_w, is defined through the equal-time commutator

$$[D_0(t), w(x)]_{x_0=t}=i(d_w-x^\mu\partial_\mu)w(x), \tag{2}$$

where

$$D_0(t)=\int d^3x\,x^\mu\theta_{\mu 0}(x) \tag{3}$$

is the generator of dilatations.

Assuming that the u's have a unique dimension, d_u, from Eqs. (1)–(3) it follows that

$$\theta^\mu{}_\mu=(4-d_u)\epsilon(u_0+cu_8)+4\delta. \tag{4}$$

The most general form of the energy-momentum tensor compatible with Eqs. (1) and (4) is

$$\theta^{\mu\nu}=\overline{\theta}^{\mu\nu}+[(1-\tfrac{1}{3}d)u+\delta]g^{\mu\nu}+\tfrac{1}{3}dug^{\mu 0}g^{\nu 0}, \tag{5}$$

where u stands for $\epsilon(u_0+cu_8)$.

Obviously $\overline{\theta}^{\mu\nu}$ is not a tensor, and although the dimension of $\overline{\theta}^{00}$ is 4 in all Lorentz frames, when boosted it becomes a mixture of pieces of dimension 4 and d_u. The reason is that in a non-scale-invariant world the generators of the Lorentz group do not commute with the generator of dilatations.

On the other hand, since the u's are scalar operators, their dimension must be preserved by the boosts. Otherwise, the dimension would be frame-dependent and the whole scheme would then be meaningless. Let us apply an infinitesimal boost, along the ith spatial direction, to

$$[D(0), u_a(0)]=d_u u_a(0),\quad a=0,1,\ldots,8 \tag{6}$$

namely,

$$[M_{0i},[D(0),u_a(0)]]=0. \tag{7}$$

Using the Jacobi identity together with Eq. (2) and the fact that $\theta_{\mu\nu}$ is conserved, we obtain

$$\int d^3x\,x^i[\theta^\mu{}_\mu(x), u_a(0)]_{x_0=0}=0. \tag{8}$$

Equation (8) means that terms proportional to first derivatives of δ functions in the equal-time commutator between the trace of the energy-momentum tensor and u_a, for $a=0,1,\ldots,8$, must vanish. This constraint is the necessary and sufficient condition for d_u to be frame-independent.

We shall now derive a sum rule relating matrix elements of u_0 and u_8 with d_u. In order to do this, let us consider matrix elements of θ^{ii}, given by Eq. (5), between identical, single-particle states at rest. We obtain, after subtracting the vacuum expectation value,

$$\langle\alpha(p)|\theta^{ii}|\alpha(p)\rangle|_{\vec{p}=0}=\langle\alpha(p)|\overline{\theta}^{ii}-(3-d)u|\alpha(p)\rangle|_{\vec{p}=0} =0. \tag{9}$$

Let us expand $|\alpha(p)\rangle$ around the SU(3) limit, namely,

$$|\alpha(p)\rangle=|\alpha_0(p)\rangle+|\alpha_1(p)\rangle+\cdots, \tag{10}$$

where $|\alpha_0(p)\rangle$ stands for the SU(3)-symmetric state, $|\alpha_1(p)\rangle$ for the first-order SU(3) correction to the symmetric state, and so on.

Inserting Eq. (10) into Eq. (9) and equating to zero each order in SU(3) breaking, we find

$$\langle \alpha_0(p)|\,\overline{\theta}^{ii} - (3-d)\epsilon u_0|\,\alpha_0(p)\rangle|_{\vec{p}=0} = 0 \tag{11}$$

and

$$2\mathrm{Re}\langle\alpha_0(p)|\overline{\theta}^{ii} - (3-d)\epsilon u_0|\,\alpha_1(p)\rangle|_{\vec{p}=0} = (3-d)\epsilon c\langle\,\alpha_0(p)|\,u_8|\,\alpha_0(p)\rangle|_{\vec{p}=0}\,. \tag{12}$$

From Eqs. (4) and (5) it follows that

$$\overline{\theta}^{00} = \overline{\theta}^{ii}\,. \tag{13}$$

For single-particle states at rest, normalized to unity, Eq. (1) yields to

$$\langle\,\alpha(p)|\,\theta^{00}|\alpha(p)\rangle|_{\vec{p}=0} = m_\alpha\,. \tag{14}$$

Ordinary perturbation theory leads to

$$2\mathrm{Re}\langle\alpha_0(p)|\,\overline{\theta}^{00} + \epsilon u_0|\,\alpha_1(p)\rangle|_{\vec{p}=0} = 0\,. \tag{15}$$

By inserting Eqs. (13) and (15) into Eq. (12), we obtain

$$2\mathrm{Re}\langle\alpha_0(p)|u_0|\,\alpha_1(p)\rangle|_{\vec{p}=0} = \frac{3-d}{d-4}\,c\langle\alpha_0(p)|u_8|\alpha_0(p)\rangle|_{\vec{p}=0}\,. \tag{16}$$

Standard techniques in perturbation theory enable us to evaluate the left-hand side of Eq. (16) in terms of matrix elements of u_0 and u_8 between SU(3) symmetrical states. The result is

$$\sum_\beta \frac{\langle\alpha_0|u_0|\,\beta\rangle\langle\,\beta\,|\,u_8|\,\alpha_0\rangle}{E_\beta - E_\alpha} = \frac{3-d}{d-4}\langle\alpha_0|u_8|\alpha_0\rangle\,. \tag{17}$$

The sum includes all those states which do not belong to the same irreducible representation to which $|\alpha_0\rangle$ belongs. In particular, the sum also involves an integral over the continuum.

Due to the success of the Gell-Mann–Okubo mass formula, we have good reason to believe that matrix elements of u_8 between physical states are well approximated by matrix elements between SU(3) states. Therefore, we can replace in Eq. (17) SU(3) states by physical states. The right-hand side of Eq. (17) can then be determined, for various states, from the mass spectrum.

Let us now consider matrix elements of $\theta^\mu{}_\mu$ between states of equal momentum. From Eq. (4) we have

$$\begin{aligned} m_\alpha &= \langle\,\alpha(p)|\,\theta^\mu{}_\mu\,|\alpha(p)\rangle \\ &= (4-d)\epsilon\,\langle\,\alpha(p)|\,u_0 + cu_8|\,\alpha(p)\rangle\,. \end{aligned} \tag{18}$$

Using Eq. (6) and the fact that masses are well approximated by keeping only terms up to first order in SU(3) breaking, Eq. (18) can be approximated by

$$m_\alpha \simeq (4-d)\langle\alpha_0|\,u_0|\,\alpha_0\rangle + c\langle\,\alpha_0|\,u_8|\,\alpha_0\rangle\,. \tag{19}$$

Equations (1), (13), and (18) imply

$$\langle\,\alpha_0|\,\overline{\theta}^{00}|\,\alpha_0\rangle|_{\vec{p}=0} = (3-d)\epsilon\langle\,\alpha_0(p)|\,u_0|\alpha_0(p)\rangle|_{\vec{p}=0}\,, \tag{20}$$

which is the same as Eq. (11).

From Eq. (20) it follows that if $d=3$, then $\overline{\theta}^{00}$ does not contribute to the masses, which are then given by matrix elements of $\epsilon(u_0 + cu_8)$. This, however, does not necessarily imply that in the SU(3)×SU(3) limit all masses vanish.[2]

We want to point out that had we used covariantly normalized states, we would have obtained a similar sum rule to that given by Eq. (17), except for the factor $3-d$, which would then be replaced by a factor $2-d$. Hence, $d=2$ implies that first-order SU(3) corrections to matrix elements of u_0 between single-particle states vanish. In other words, the assumption that matrix elements like $\langle\alpha(p)|\,u_0|\,\alpha(p)\rangle$ can be approximated by their SU(3) values for covariantly normalized states[3] is equivalent to the assumption that $d=2$.[4] In order to prove this result let us consider

$$\langle\alpha|\theta^{00}|\,\alpha\rangle_u = M_\alpha$$

and

$$\langle\alpha|\theta^{00}|\,\alpha\rangle_c = 2M_\alpha{}^2\,,$$

where the subscripts u and c stand for unit and covariantly normalized states, respectively. In particular we notice that $\langle\,\alpha_0|\,u_0|\,\alpha_0\rangle_u$ is constant over the octet while $\langle\,\alpha_0|\,u_0|\,\alpha_0\rangle_c$ is not, so that a different grouping of terms is required in order to write the mass formula in the general form:

(mass) = (constant piece over the octet)

+ (varying piece over the octet).

Thus, using

$$\langle\,\alpha|\,u_0|\,\alpha\rangle_c = 2M_\alpha\langle\,\alpha|\,u_0|\,\alpha\rangle_u$$

and the definitions

$$\begin{aligned} u_0^0 &= \langle\,\alpha_0|u_0|\,\alpha_0\rangle_u\,, \\ u_0^1 &= 2\mathrm{Re}\,\langle\alpha_0|\,u_0|\,\alpha_1\rangle_u\,, \\ u_8^0 &= \langle\,\alpha_0|u_8|\,\alpha_0\rangle_u\,, \end{aligned}$$

we have

$$\langle\alpha|u_0|\alpha\rangle_c = 2(\overline{\theta}_0^{00} + u_0^0 + cu_8^0)(u_0^0 + u_0^1) + \cdots\,,$$

where the first term on the right-hand side is just

$$\begin{aligned} M_\alpha &= \langle\alpha|\theta^{00}|\,\alpha\rangle_u \\ &= \langle\alpha_0|\overline{\theta}^{00} + u_0|\,\alpha_0\rangle + c\langle\,\alpha_0|\,u_8|\,\alpha_0\rangle + \cdots\,. \end{aligned}$$

Using Eqs. (11) and (16), we obtain, keeping terms up to first order in SU(3),

$$\langle \alpha | u_0 | \alpha \rangle_c = 2u_0^0(4-d)\left(u_0^0 + \frac{d-2}{4-d} c u_8^0\right),$$

which means that the first-order SU(3) correction to matrix elements of u_0 between covariantly normalized states, namely, $(u_0^1)_c$, is given by

$$(u_0^1)_c = \frac{d-2}{4-d} c (u_8^0)_c .$$

Hence our claim that the assumption that matrix elements of u_0 can be approximated by their SU(3) values is equivalent to the assumption that $d=2$, for covariantly normalized states.[5]

Finally, we want to point out that the sum rule given by Eq. (17) might be an indication that $d \approx 3$, the reason being that if $|\alpha_0\rangle$ is an octet, then $|\beta\rangle$ in the sum can only belong to an octet. On the other hand, we know, due to the accuracy of the Gell-Mann–Okubo mass formula for the $\frac{1}{2}^+$ baryon octet (~1%), that there is very little, if any, mixing of two or more $\frac{1}{2}^+$ baryon octets.[6] This means that

$$\sum_{\beta \neq \alpha} \frac{|\langle \alpha_0 | u_8 | \beta \rangle|^2}{E_\beta - E_\alpha}$$

is extremely small compared with $\langle \alpha_0 | u_8 | \alpha_0 \rangle$. Hence, unless $\langle \alpha_0 | u_0 | \beta \rangle \gg \langle \alpha_0 | u_8 | \beta \rangle$, $d \approx 3$.

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[1]See, for example, M. Gell-Mann, in *Proceedings of the Third Hawaii Topical Conference on Particle Physics*, edited by S. F. Tuan (Western Periodicals, North Hollywood, Calif., 1970); J. Ellis in *Broken Scale Invariance and the Light Cone*, 1971 Coral Gables Conference on Fundamental Interactions at High Energy, edited by M. Dal Cin, G. J. Iverson, and A. Perlmutter (Gordon and Breach, New York, 1971), Vol. 2.

[2]P. J. O'Donnell, Phys. Rev. D 3, 1021 (1971); V. S. Mathur, Phys. Rev. Lett. 27, 452 (1971); 27, 700(E) (1971).

[3]This assumption has been widely used. See, for example, M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968).

[4]This result has been derived by Mathur (see Ref. 2), by looking at matrix elements relevant to K_{l3} decay. It is, however, completely general as follows from the present discussion.

[5]This result is explicitly displayed in the effective-Lagrangian calculations of Ellis [J. Ellis, Phys. Lett. 33B, 591 (1970); J. Ellis, P. Weisz, and B. Zumino, *ibid.* 34B, 91 (1971)]. See also Ref. 4.

[6]L. Gomberoff and V. Tolmachev, Phys. Rev. 187, 3185 (1969).

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Two-Component Forms of the Dirac Equation*

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In this paper some aspects of the two-component alternative to the Dirac equation, recently proposed by Biedenharn, Han, and Van Dam (BHvD), are discussed. We consider the possibility that the BHvD equation can be interpreted as another two-component form of the Dirac equation, being brought about by a unitary transformation which resembles the Foldy-Wouthuysen transformation. This point of view also enables us to derive more two-component alternatives to the Dirac equation.

I. INTRODUCTION

Recently, Biedenharn, Han, and Van Dam (BHvD) proposed an alternative to Dirac's factorization of the Klein-Gordon equation which yields two-component, $m \neq 0$ equations describing a particle with spin $\frac{1}{2}$.[1-8] The essential point in their procedure was to present a factorization of the Klein-Gordon equation in terms of two-dimensional matrices in contrast to the usual Dirac factorization which uses four-dimensional matrices $\vec{\alpha}$, β. The important difference with Dirac's procedure is that Dirac demanded the matrices $\vec{\alpha}, \beta$ to be independent of space-time, whereas BHvD use matrices which are explicitly space-time–dependent. Both in the usual Dirac equation and in the BHvD equation the momenta appear linearly only. When electromagnetic interactions are present, BHvD factorize the Kramers equation. The Kramers equation is a two-component second-order wave equation which can be obtained from the iterated Dirac equation. The two-component