Weak-Boson Triangle Anomalies*

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The causal methods of source theory are used to calculate the triangle anomalies of weak bosons ($W^{\pm}$ and Z) in a unified gauge theory of weak and electromagnetic interactions (not involving heavy leptons). The pseudoelectric form factors for these particles are also presented.

It is well known that unified gauge theories[1-3] without heavy leptons[4,5] possess axial-vector anomalies[6,7] (the so-called triangle anomalies) which adversely affect the renormalizability of such theories.[6] To our knowledge there has been no explicit evaluation of the anomalies for the weak bosons ($W^{\pm}$ and Z) which occur in Weinberg's[2] and Schwinger's[1] models. In this note we will calculate these anomalies using the causal methods of source theory.[8]

In order to determine the triangle anomaly one compares a certain pseudoscalar coupling with the divergence of the axial-vector one. This can only be done if the variable with respect to which one differentiates is fully generalized and is not subject to the kinematic restrictions of being a free particle. Thus we consider the causal arrangement indicated in Fig. 1 rather than that of Fig. 2.

In the models considered, the primitive interaction between leptons and weak bosons and photons is, in part,

$$\begin{aligned}\mathcal{L}' &= e\psi_-\gamma^0\gamma^\mu\psi_+A_\mu + \lambda_1\psi_-\gamma^0\gamma^\mu\tfrac{1}{2}(1-i\gamma_5)\psi_+Z_\mu \\ &\quad+\lambda_2\psi_-\gamma^0\gamma^\mu\tfrac{1}{2}(1+i\gamma_5)\psi_+Z_\mu \\ &\quad+\lambda_3\psi_0\gamma^0\gamma^\mu\tfrac{1}{2}(1+i\gamma_5)\psi_+W_\mu^- \\ &\quad+\lambda_3\psi_-\gamma^0\gamma^\mu\tfrac{1}{2}(1+i\gamma_5)\psi_0W_\mu^+, \end{aligned}\tag{1}$$

where the $\pm$, 0 labels refer to the charge of the incoming particles and $\psi_\pm$ refer to either electron or muon. Here, in Weinberg's model,[2] the coupling constants are

$$\begin{aligned}\lambda_1 &= g'^2(g^2+g'^2)^{-1/2}, \\ \lambda_2 &= \tfrac{1}{2}(g'^2-g^2)(g^2+g'^2)^{-1/2}, \\ \lambda_3 &= \frac{1}{\sqrt{2}}g,\end{aligned}\tag{2}$$

while in Schwinger's model,[1] $g=g'=\sqrt{2}\,e$.

Using the causal methods of source theory[8] we generate the vacuum amplitude corresponding to Fig. 1. There a virtual W (Z) produces a neutrino and a charged lepton (a lepton pair). These particles then propagate and scatter by exchanging a virtual lepton to produce a real W (Z) and a photon. The vacuum amplitude for the process of Fig. 1(a) is[9]

$$\begin{aligned}\langle 0_+|0_-\rangle &= \frac{ie\gamma}{8\pi^2}\int\frac{(dp)}{(2\pi)^4}\frac{(dk)}{(2\pi)^4}\frac{(dQ)}{(2\pi)^4}(2\pi)^4\delta(Q-p-k) \\ &\quad\times W_-^\mu(-p)W_+^\nu(Q)A^\lambda(-k)\tilde{I}_{\mu\nu\lambda},\end{aligned}\tag{3}$$

where $\gamma=\lambda_3^{\,2}$, and (m = lepton mass)

$$\begin{aligned}\tilde{I}_{\mu\nu\lambda} &= 2\pi^2\mathrm{Tr}\int d\omega_q d\omega_{q'}(2\pi)^4\delta(Q-q-q') \\ &\quad\times(\gamma_\nu\gamma q\gamma_\mu\gamma_5+\gamma_\nu\gamma_5\gamma q\gamma_\mu) \\ &\quad\times\frac{1}{m+\gamma(q'-k)}\gamma_\lambda(m-\gamma q').\end{aligned}\tag{4}$$

Here we have considered only the γ_5 part of the trace, since only this contributes to the anomaly. In order to perform space-time extrapolation one must enquire whether there are any physical requirements which render necessary contact terms. There does not seem here to be any normalization requirement but there is a requirement of consistency among different causal arrangements. In particular we will see later that the corresponding process [Fig. 2(a)], in which the photon is timelike and both W's are real, yields a pseudoelectric form factor which vanishes (by gauge invariance) for a real photon. This in turn imposes the requirement that the generalized amplitude $I_{\mu\nu\lambda}$ vanish when all particles are on shell. We thus have, after space-time generalization,

$$\begin{aligned}I_{\mu\nu\lambda} &= \frac{1}{i}\int_{m^2}^{\infty}\frac{dM^2}{Q^2+M^2-i\epsilon}\frac{1}{(M^2-m_W^{\,2})^2} \\ &\quad\times\left(\frac{Q^2+m_W^{\,2}}{m_W^{\,2}-M^2}B_1|\mu\nu\lambda k|+B_2Q_\nu|\mu\lambda Qk|\right).\end{aligned}\tag{5}$$

Here B_1 and B_2 are given below, and $|\mu\nu\lambda k| = \epsilon_{\mu\nu\lambda\alpha}k^\alpha$, etc. The factor $(Q^2+m_W^{\,2})/(m_W^{\,2}-M^2)$ (which equals 1 under the causal situation) is necessary in order that the amplitude vanish when all particles are real, when $Q^2=-m_W^{\,2}$.

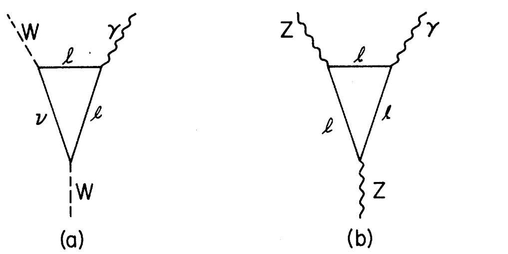

FIG. 1. Lepton triangle with incoming virtual bosons.

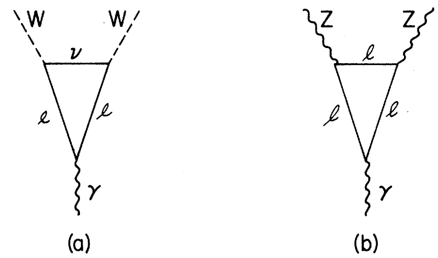

FIG. 2. Lepton triangle with incoming virtual photon.

From this point we can obtain the anomaly essentially by following the procedure given in Appendix A of Ref. 10. The first step is to consider

$$iQ^\nu I_{\mu\nu\lambda} = \int_{m^2}^{\infty} \frac{dM^2}{Q^2+M^2-i\epsilon}\,\frac{1}{(M^2-m_W{}^2)^2} \times\left(\frac{Q^2+m_W{}^2}{M^2-m_W{}^2}B_1\,|\mu\lambda Qk| + B_2Q^2\,|\mu\lambda Qk|\right). \tag{6}$$

The anomaly is the difference between the actual divergence and the naive divergence which is obtained, in this case, by the replacement $\gamma^\nu \to -m$ in Eq. (4). This is just $Q^\nu \tilde{I}_{\mu\nu\lambda}$, which upon space-time extrapolation leads to the naive divergence

$$I_{\mu\lambda} = -m\int_{m^2}^{\infty} \frac{dM^2}{Q^2+M^2-i\epsilon}\,\frac{1}{(M^2-m_W{}^2)^2} I(M^2)\,|\mu\lambda Qk|, \tag{7}$$

where

$$-mI(M^2) = -B_1(M^2) - M^2B_2(M^2). \tag{8}$$

Then the anomaly, c, is defined by

$$iQ^\nu I_{\mu\nu\lambda} = I_{\mu\lambda} + c\,|\mu\lambda Qk|. \tag{9}$$

For this case, the anomaly is

$$c = \int_{m^2}^{\infty} \frac{dM^2}{(M^2-m_W{}^2)^2}\left[\frac{B_1(M^2)}{M^2-m_W{}^2} + B_2(M^2)\right]. \tag{10}$$

The direct evaluation of Eq. (4) yields

$$B_1 = (M^2-m^2)(m^2+m_W{}^2) - (M^2+m_W{}^2)m^2\ln\frac{M^2}{m^2}, \tag{11}$$

$$B_2 = -\frac{M^2-m^2}{M^2}\left(m_W{}^2 + 2m^2 - \frac{m^2m_W^2}{M^2}\right) + 2m^2\ln\frac{M^2}{m^2}.$$

The anomaly cited in the literature[6] is independent of the lepton mass; to see if this holds here, we first consider the normal threshold case, where $m^2 > m_W{}^2$. We find then

$$c = -\tfrac{1}{2} + 1 = \tfrac{1}{2}, \tag{12}$$

which is indeed mass-independent. The other case, $m^2 < m_W{}^2$, leads to an unstable anomalous threshold. One could proceed there by performing mass extrapolation[11] upon the constant result, Eq. (12). (This is probably equivalent to the contour deformation of Fronsdal and Norton,[12] so that the integral runs from m^2 to $-\infty$. Explicitly, such a prescription also leads to the anomaly $c=\frac{1}{2}$.) Finally, there is a prescription which is based upon a physical interpretation of the singularity present at $M^2 = m_W{}^2$. We know why this singularity is present: The W source is not localized for $M^2 = m_W{}^2$, so we must exclude this value from the mass spectrum when $m^2 < m_W{}^2$. Following the example of the ρ meson[13] we take

$$c = \left[\frac{d}{dz}\,\mathrm{P}\int_{m^2}^{\infty}\frac{dM^2}{M^2-z}B_2(M^2) + \frac{1}{2}\frac{d^2}{dz^2}\,\mathrm{P}\int_{m^2}^{\infty}\frac{dM^2}{M^2-z}B_1(M^2)\right]_{z=m_W{}^2} = 1 - \tfrac{1}{2} = \tfrac{1}{2}. \tag{13}$$

The result being mass-independent, the muon and electron contribute equally, giving a total anomaly of $2c = 1$.

The anomaly corresponding to the Z can be obtained in a very similar manner. The vacuum amplitude corresponding to Fig. 1(b) differs from Eq. (3) in that $W \to Z$ and now $\gamma = 2(\lambda_2{}^2 - \lambda_1{}^2)$; in place of Eq. (4),

$$\tilde{I}_{\mu\nu\lambda} = 2\pi^2\mathrm{Tr}\int d\omega_q d\omega_{q'}(2\pi)^4\delta(Q-q-q') \times[\gamma_\nu(m+\gamma q)\gamma_\mu\gamma_5 + \gamma_\nu\gamma_5(m+\gamma q)\gamma_\mu] \times\frac{1}{m+\gamma(q'-k)}\gamma_\lambda(m-\gamma q'). \tag{14}$$

The extrapolated amplitude is of the form of Eq. (5) and its divergence has the form of Eq. (6). Here the naive divergence is again the generalized form of $Q^\nu\tilde{I}_{\mu\nu\lambda}$ [which arises only from the second term of Eq. (14) and is obtained by replacing $\gamma_\nu \to -2m$ there]. As before, the anomaly is

$$c' = \int_{4m^2}^{\infty}\frac{dM^2}{(M^2-m_Z{}^2)^2}\left[\frac{B_1'(M^2)}{M^2-m_Z{}^2} + B_2'(M^2)\right], \tag{15}$$

where

$$B_1'(M^2) = -M^2 m_Z{}^2 v + m^2(M^2 + m_Z{}^2)\ln\frac{1+v}{1-v}\,,$$
$$B_2'(M^2) = m_Z{}^2 v - 2m^2 \ln\frac{1+v}{1-v}\,, \tag{16}$$

and

$$v^2 = 1 - \frac{4m^2}{M^2}\,. \tag{17}$$

Again it is straightforward to perform the integrations; we find, whether $4m^2$ is greater than or less than $m_Z{}^2$,

$$c' = \tfrac{1}{2} - 1 = -\tfrac{1}{2} \tag{18}$$

giving a total (electron + muon) anomaly of $2c' = -1$. These results were also obtained by noncausal calculations.

In the above discussion, to obtain the space-time generalized form, Eq. (5), we used the fact that the pseudoelectric contribution from Fig. 2 vanished when all particles were real. This can be easily verified by explicit calculation. The vacuum amplitude for the coupling of a photon to two real W's is ($\gamma = \lambda_3{}^2$)

$$\langle 0_+|0_-\rangle = \frac{e\gamma}{16\pi}\int \frac{(dp)}{(2\pi)^4}\frac{(dp')}{(2\pi)^4} W_+^\mu(-p)\tilde{I}'_{\mu\nu\lambda} W_-^\nu(-p')A^\lambda(Q), \tag{19}$$

where, corresponding to Fig. 2(a), the pseudoelectric part is

$$\tilde{I}'_{\mu\nu\lambda} = 8\pi\int d\omega_q d\omega_{q'}(2\pi)^4\delta(Q-q-q') \times \mathrm{Tr}\left[(m+\gamma q)\gamma_\mu \frac{1}{\gamma(p-q)}\gamma_\nu i\gamma_5(m-\gamma q')\gamma_\lambda\right]. \tag{20}$$

After simplification, we obtain (in a Lorentz gauge)

$$\tilde{I}'_{\mu\nu\lambda} = i\, F(M^2)\,|\mu\nu\lambda(p-p')|\,, \tag{21}$$

where

$$F(M^2) = -\tfrac{1}{2}v\frac{1}{\zeta^4}\Big[5\zeta^2 - 3v^2 - \zeta^2(\zeta^2+v^2) + \tfrac{1}{2}(v^2-\zeta^2)^2(\zeta^2+3)\frac{1}{\zeta v}\ln\frac{v+\zeta}{v-\zeta}\Big], \tag{22}$$

and where v is defined in Eq. (17), and

$$\zeta^2 = 1 - \frac{4m_W{}^2}{M^2}\,. \tag{23}$$

Equation (20) is gauge-invariant, and we must take care to see that space-time generalization does not destroy this property.[14] In a Lorentz gauge, this can be done by supplying the factor $-Q^2/M^2$. Then the generalized pseudoelectric part is

$$I'_{\mu\nu\lambda} = f(Q^2)\,|\mu\nu\lambda(p-p')|\,, \tag{24}$$

with

$$f(Q^2) = -\frac{Q^2}{2\pi}\int \frac{dM^2}{M^2}\frac{F(M^2)}{Q^2+M^2-i\epsilon}\,. \tag{25}$$

Because of the factor of Q^2 (due to gauge invariance), the pseudoelectric form factor indeed vanishes for a real photon, $f(0)=0$.

Again for the Z [Fig. 2(b)] the result is expressed in terms of Eq. (19) [with $W\to Z$ and $\gamma = 2(\lambda_2{}^2 - \lambda_1{}^2)$] with the same spectral form as Eq. (24), where now in place of $F(M^2)$

$$F'(M^2) = -\frac{1}{4}\frac{v}{\xi^4}\left\{-3 + 4\xi^2 - \xi^4 + [3 - \xi^2(1+4v^2) + \xi^4(5-4v^2) + \xi^6]\frac{1}{4\xi v}\ln\frac{1+\xi^2+2v\xi}{1+\xi^2-2v\xi}\right\}, \tag{26}$$

where

$$\xi^2 = 1 - \frac{4m_Z{}^2}{M^2}\,. \tag{27}$$

Incidentally, note that this is the only form factor for the Z, since it is a neutral particle having no chargelike property. Therefore, this result is complete (there is no contribution to the Z form factor from boson intermediate states).

Observe, again, that normal threshold behavior holds only when $m^2 > m_W{}^2$, so then the integral runs from $4m^2$ to ∞. For the unstable anomalous threshold regime, $m^2 < m_W{}^2$, one must perform mass extrapolation. This subject, in general, merits detailed investigation.

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Asymptotic Behavior of the Anomalous Vertex Functions†

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Asymptotic behaviors of the axial-vector-current–two-photon vertex and of its divergence equation are studied in the region where virtual-photon squared masses are large compared with the nucleon mass. By assuming a sum rule of asymptotic functions, we derive some anomalous commutation relations. We also study the asymptotic properties of some particular two-photon processes.

I. INTRODUCTION

It has recently been established by several authors[1] that the $\pi^0\gamma\gamma$ vertex does not vanish in the soft-pion limit, as a consequence of the anomaly[2] in partial conservation of axial-vector current (PCAC) in the presence of electromagnetism. Wilson[3] showed that the PCAC anomaly is essentially caused by the short-distance behavior of products of local operators appearing in the anomalous vertices. Crewther[4] found on the basis of Wilson's theory[3] of broken scale invariance that the anomalous constant which appears in the $\pi^0 \to 2\gamma$ decay amplitude can be determined by a product of parameters in high-energy electroproduction and annihilation cross sections. Also, using Wilson's operator-expansion technique,[3] Brandt and Preparata[5] showed that the scaling limit of the anomalous vertices is strongly controlled by the light-cone behavior of products of local operators. In the framework of a gluon-quark model, Gross and Treiman[6] investigated extensively the scaling limit and the Bjorken-Johnson-Low[7] (BJL) limit of an amplitude appearing in an arbitrary two-photon process. Terazawa[8] studied the behavior of the $\pi^0\gamma\gamma$ vertex in terms of off-shell form factors associated with the axial-vector-current–two-photon matrix element in the limit of large virtual-photon mass squared and found some interesting results.

In this paper we study an asymptotic limit of the form factors which appear in the axial-vector-current–two-photon vertex and in its divergence equation. In Sec. II we introduce some asymptotic functions associated with the form factors and consider a sum rule satisfied by the asymptotic functions. We also derive anomalous commutation relations by applying the BJL theorem to the matrix element of the divergence equation for the axial-vector current. These results are compared with those derived from the triangle diagrams in spinor electrodynamics. In Sec. III we consider some particular two-photon processes such as $\pi^0 \to e^+ + e^- + \gamma$ and $e^+ + e^- \to \mu^+ + \mu^- + \pi^0$ in order to study the off-shell behavior of the $\pi^0\gamma\gamma$ form factor.

II. THE ASYMPTOTIC LIMIT, SUM RULE, AND ANOMALOUS EQUAL-TIME COMMUTATORS

Let us define the off-shell form factors $F_i(p, q)$ ($i = 1, \ldots, 4$) by the equation[9]

$$\begin{aligned}\langle 0|j^5_\lambda(0)|p, \epsilon; q, \epsilon'\rangle[(2\pi)^6 2p_0 2q_0]^{1/2} = \frac{e^2}{\pi^2}\epsilon_\mu\epsilon'_\nu p_\alpha q_\beta\big[&(q_\mu\epsilon_{\alpha\lambda\nu\beta} - q_\alpha\epsilon_{\mu\lambda\nu\beta})F_1(p, q) \\ &+ (p_\beta\epsilon_{\mu\alpha\lambda\nu} - p_\nu\epsilon_{\mu\alpha\lambda\beta})F_2(p, q) \\ &+ (p_\mu\epsilon_{\alpha\lambda\nu\beta} - p_\alpha\epsilon_{\mu\lambda\nu\beta})F_3(p, q) \\ &+ (q_\beta\epsilon_{\mu\alpha\lambda\nu} - q_\nu\epsilon_{\mu\alpha\lambda\beta})F_4(p, q)\big],\end{aligned} \tag{1}$$

where p, q and ϵ, ϵ' denote the four-momenta and polarization vectors of the two photons, respectively. Note that gauge invariance is automatical-