of ϕ can always be set so as to forbid trilinear Q-lepton-ϕ couplings. This question arises if and only if $\mathcal{G}$ contains Q and lepton representations which are abstractly identical. (Among the numerous other ways to forbid such couplings one may note the possible existence of lepton-number gauge fields.) I want to thank H. Pagels for a discussion on this point.

[10] For the technical meaning of this term and for earlier references see, e.g., T. Hagiwara and B. W. Lee, Phys. Rev. D 7, 459 (1973). The symbol $\mathcal{L}(\mathcal{G})$ is meant to imply that specific choices for lepton, quark, and scalar field representations have been made.

[11] To be precise, one more term should be added to $\mathcal{L}$, namely, $|\phi|^2$ times a linear superposition of all $H^{(i)\dagger}H^{(i)}$, where the $H^{(i)}$ are the Higgs fields of representation type (i), as they appear in $\mathcal{L}(\mathcal{G})$. These terms do not affect the argument.

[12] Cf. I. Bars, M. B. Halpern, and M. Yoshimura, Phys. Rev. D 7, 1233 (1973); Y. Achiman, Nucl. Phys. B (to be published); J. C. Pati and A. Salam, Phys. Rev. D 8, 1240 (1973).

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High-Energy Expansions of Scattering Amplitudes*

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The partial-wave series is converted without approximation to a Fourier-Bessel expansion based on a new (infinite series) expansion for the Legendre function. The direct connection between the Fourier-Bessel phase shift and the partial-wave interpolating phase shift is established as an infinite series in powers of K^{-2} (K = wave number). The series contains the Glauber eikonal approximation as a leading term and reproduces the results of an eikonal expansion about the Glauber propagator. Corrections to the eikonal approximation are developed and rules are given for an unambiguous interpretation of the eikonal expansion. The relativistic eikonal expansion is discussed for forward and backward scattering without small-angle approximations.

I. INTRODUCTION

The problem of obtaining high-energy limits of scattering amplitudes is one of general interest in physics. In this paper, attention is focused on the high-energy expansion of the Fourier-Bessel representation of scattering amplitudes, as it has been obvious for a long time that high-energy scattering through small angles is very conveniently treated by means of eikonal (or straight-line path) approximations. One of the simplest and most successful theories of this type was developed by Glauber,[1] who noted the advantages of a straight-line path parallel to the average momentum. By introducing such a path, Glauber obtained a Fourier-Bessel representation of the scattering amplitude which embodies approximate unitarity. The Fourier-Bessel representation is advantageous because its existence can be justified for all angles on general grounds of analyticity in the momentum transfer as shown by Blankenbecler and Goldberger.[2]

However, all derivations of eikonal approximations require a small scattering angle in some sense. As a result many variants of the approximation[3] have arisen in attempts to extend the angular range of validity. In principle, the number of possible variants of the eikonal approximation is unlimited. This is because, for nonforward scattering, the set of rays which represents the eikonal approximation to the scattering wave function can be imagined to propagate through the interaction in innumerable ways, each of which generates a new variant of the approximation. Since the question of angular range of validity for any particular variant of the approximation has remained open, there has been no compelling reason to believe any of them was good for large-angle scattering.[4]

Several studies of the high-energy limit of scattering have been made by means of converting the partial-wave sum to an integral over impact parameters. For example, Glauber showed that his average-momentum-direction eikonal approximation could be obtained in such a manner if the Legendre polynomials were replaced by a Bessel function $J_0(qb)$, where $q = 2K\sin\frac{1}{2}\theta$ and $b = (l+\frac{1}{2})/K$. Similar methods have been used to examine forward or small-angle scattering in relativistic models.[5] Even though assumptions about ray paths are eliminated in this approach, generally a small-angle approximation remains, as there are terms ignored which are only small when the scattering angle is small.

Cottingham and Peierls and also Adachi and Kotani[6] have shown that an exact Fourier-Bessel representation could be obtained which is valid for all physical angles $0 \leq \theta < \pi$ provided the scattering amplitude is identically zero for angles beyond the physical range. This assumption is of course too restrictive, as it destroys analyticity in the momentum transfer.

Another type of approach to short-wavelength scattering has been to systematically expand the scattering T matrix using the eikonal-expansion approach. In this vein, corrections to Glauber's approximation have recently been obtained by developing without approximation a tedious eikonal expansion about the average-momentum-direction propagator of the Glauber theory.[7] For small-angle scattering, unambiguous results were obtained. For large-angle scattering some questions remain open due to non-Fourier-Bessel terms in the eikonal expansion. However, a conjecture was advanced that all the non-Fourier-Bessel terms cancel in the average-momentum-direction eikonal expansion which proved numerically successful at large angles.

The present paper develops a complete high-energy expansion of the Fourier-Bessel representation of the scattering amplitude. The expansion is exact in the sense that no small-angle approximation is employed. For this reason many of the questions about angular range of validity of the Glauber-type eikonal approximation, which is the leading term in the expansion, can be unambiguously discussed and corrections can be developed. The expansion is developed in Sec. II by direct conversion of the partial-wave sum to a Fourier-Bessel integral based on an expansion of the Legendre function developed in Appendix A.

Section III discusses high-energy approximations to the interpolating function for the partial-wave phase shifts for the case of potential scattering. The Glauber approximation is derived without use of a small-angle assumption, and some obvious corrections to it are discussed. Section III also develops the leading terms in the Fourier-Bessel expansion and shows that they agree with the results of an eikonal expansion about the eikonal propagator of the Glauber theory. Based on this agreement, some rules are given for interpreting the content of an average-momentum-direction eikonal expansion. Section IV considers the higher-order corrections to the Fourier-Bessel amplitude $S_F(b)$. By using known results in the classical limit, a large class of contributions to $S_F(b)$ are summed to given connecting formulas between the Fourier-Bessel and partial-wave phase-shift functions.

Section V considers the relativistic (spinless) partial-wave sum. Some results of Blankenbecler and Goldberger are reviewed, and the relativistic Fourier-Bessel expansion is developed. The equivalent of the Glauber theory for small-angle scattering and for large-angle scattering by an exchange interaction is discussed. Finally Sec. VI provides some concluding remarks.

II. FOURIER-BESSEL EXPANSION OF SCATTERING AMPLITUDE

We start with the partial-wave sum[8] for the scattering amplitude at fixed wave number K ($\hbar = c = 1$):

$$f(\theta) = \sum_{j=0}^{\infty} A_j P_j(1-2y^2), \quad y = \sin\tfrac{1}{2}\theta, \tag{2.1a}$$

$$A_j = (-i/K)(j+\tfrac{1}{2})[\exp(2i\delta_j) - 1]. \tag{2.1b}$$

The index j in this sum can be continued from the discrete integer values to the complex angular-momentum plane as shown by Regge,[9] who considered the behavior of the radial Schrödinger equation for arbitrary values of a variable λ. The physical angular momenta are realized when λ assumes positive half-integer values:

$$\lambda = j + \tfrac{1}{2}; \quad \lambda^2 - \tfrac{1}{4} = j(j+1). \tag{2.2}$$

Assuming a potential in the Schrödinger equation which satisfies the bounds

$$\int_0^\infty dr\, r |V(r)| < \infty, \quad |r^2 V(r)| < M = \text{const} \tag{2.3}$$

and is no more singular than r^{-2+2c} ($c > 0$) as $r \to 0$, Regge found that the A_j in (2.1) are smoothly interpolated by a function which we write as

$$A(\lambda) = -i(\lambda/K)\{\exp[2i\delta(\lambda/K)] - 1\}. \tag{2.4}$$

Some established properties of $A(\lambda)$ are that it is analytic for real values of λ in the range $R(\lambda) = -C$ to $+\infty$ (if $C < 1$) or $R(\lambda) > -1$ to $+\infty$ (if $C \geq 1$). The Froissart-Gribov continuation of the relativistic partial-wave amplitudes similarly leads to an $A(\lambda)$ analytic for $R(\lambda) > -\frac{1}{2}$. For large, real values of λ, the phase shift tends to zero as

$$\delta(\lambda/K) = O(1/\lambda). \tag{2.5}$$

Also, for $R(\lambda) > 0$, Regge has obtained the bound

$$\left|\frac{d}{d\lambda}\delta(\lambda/K)\right| < \tfrac{1}{2}\pi. \tag{2.6}$$

The properties above have been reviewed because they permit the partial-wave sum to be converted to an integral over *real* values of λ using the Euler summation formula,[10] i.e.,

$$f(\theta)=\int_0^\infty d\lambda\; A(\lambda)P_{\lambda-1/2}(1-2y^2)-R_1(y)\,. \quad (2.7)$$

The standard interpolation of the Legendre polynomials P_j by the Legendre function $P_{\lambda-1/2}$ is used.[11]

A drawback to Eq. (2.7) is the complicated remainder $R_1(y)$ which arises from derivatives of the argument of the integral

$$F(\lambda;y)\equiv A(\lambda)P_{\lambda-1/2}(1-2y^2)\,, \quad (2.8)$$

with respect to λ at the lower limit of integration, i.e., $\lambda=0$:

$$R_1(y)=\sum_{n=1}^{\infty}\frac{1}{(2n)!}B_{2n}(\tfrac{1}{2})F^{(2n-1)}(0;y)\,. \quad (2.9)$$

Similar terms for $\lambda=\infty$ arise in the Euler formula, but they vanish under the conditions stated above. In (2.9), $B_{2n}(\frac{1}{2})$ is the Bernoulli polynomial of order $2n$ and argument $\frac{1}{2}$.[12] Carrying out the differentiations, which we assume to exist at $\lambda=0$, and separating the remainder into derivatives of $A(\lambda)$ times coefficients depending on $y^2=\sin^2(\frac{1}{2}\theta)$ produces

$$\begin{aligned}R_1(y)&=\sum_{n=1}^{\infty}\frac{1}{(2n)!}B_{2n}(\tfrac{1}{2})\sum_{k=0}^{n-1}\binom{2n-1}{2k}A^{(2n-2k-1)}(0)\left[\left(\frac{d}{d\lambda}\right)^{2k}P_{\lambda-1/2}(1-2y^2)\right]_{\lambda=0}\\&=\sum_{k=1}^{\infty}A^{(2k-1)}(0)\sum_{n=0}^{\infty}\frac{1}{(2n+2k)!}B_{2n+2k}(\tfrac{1}{2})\binom{2n+2k-1}{2n}\left[\left(\frac{d}{d\lambda}\right)^{2n}P_{\lambda-1/2}(1-2y^2)\right]_{\lambda=0}\,. \quad (2.10)\end{aligned}$$

In the classical limit $\hbar\to 0$, K becomes large but $\lambda/K=b$ is the impact parameter which remains finite. Then if the remainder $R_1(y)$ is ignored and the Legendre function is replaced by the asymptotic form[13]

$$P_{Kb-1/2}(\cos\theta)\to\left[\frac{2}{\pi Kb\sin\theta}\right]^{1/2}\cos(Kb\theta-\tfrac{1}{4}\pi)+O(K^{-3/2})\,, \quad (2.11)$$

Eq. (2.7) leads to the well-known classical scattering formulas.

However, the remainder $R_1(y)$ can be eliminated without approximation when Eq. (2.7) is converted to a Fourier-Bessel form with finite K ($\hbar\neq 0$). This conversion is based on an expansion of the Legendre function which is developed in Appendix A. The expansion takes the following form involving the Bessel function $J_0(x)$:

$$P_{\lambda-1/2}(\pm z)=\sum_{k=0}^{\infty}\frac{1}{(2k)!}\left(\frac{d}{d\lambda}\right)^{2k}b_k\left(\frac{d}{d\lambda}\tfrac{1}{2}\lambda\right)\times J_0(2\lambda[(1\mp z)/2]^{1/2})\,. \quad (2.12)$$

Thus the object is to convert Eq. (2.7) to Fourier-Bessel form using this result which is valid for $|1\pm z|<2$. In Eq. (2.12) the polynomials $b_k(x)$ are of degree k in x and are defined in terms of generalized Bernoulli polynomials as follows:

$$b_k(x)=B_{2k}^{(2x)}(x)\,.$$

These polynomials are discussed in Ref. 14. Table I lists the first five of the b_k polynomials. Note that the highest powers of x take a simple form.

For orders higher than 5 the polynomials are obtained from those in Table I by means of a recursion formula [see Eq. (A5) of Appendix A]. Actually the first three terms in the expansion are very old, as they are equivalent to results derived in another form by MacDonald[15] in 1914. When we write $z=1-2y^2$, the leading term is $J_0(2\lambda y)$ and it has been often used to show that a Fourier-Bessel integral can be used as a small-angle approximation to the partial-wave series. To remove the small-angle assumption it is necessary to keep the higher-order terms in the expansion.

When (2.12) is substituted into the integral in (2.7), the derivatives of the Bessel function can be integrated by parts to cast then onto $A(\lambda)$. The Fourier-Bessel expansion which results takes the form

TABLE I. Generalized Bernoulli polynomials $b_k(x)\equiv B_{2k}^{(2x)}(x)$.

$b_0(x)=1$
$b_1(x)=-\frac{1}{6}x$
$b_2(x)=x(5x+1)/60$
$b_3(x)=-x(35x^2+21x+4)/504$
$b_4(x)=x(175x^3+210x^2+101x+18)/2160$
$b_5(x)=-x(385x^3+770x^2+671x^2+286x+48)/3168$
$\frac{1}{(2k)!}b_k(x)=\frac{1}{k!}(-\frac{1}{12}x)^k-\frac{1}{5}\frac{1}{(k-2)!}(-\frac{1}{12}x)^{k-1}+\dots$

$$f(\theta)=\int_0^\infty d\lambda\, J_0(2\lambda y)\sum_{k=0}^\infty \frac{1}{(2k)!}\, b_k\left(-\tfrac{1}{2}\lambda\frac{d}{d\lambda}\right)\left(\frac{d}{d\lambda}\right)^{2k} A(\lambda) + R_2(y)-R_1(y)\,. \tag{2.13}$$

A second remainder $R_2(y)$ appears in this formula because there are nonvanishing terms from the integrations by parts which are given by

$$R_2(y)=\sum_{k=1}^\infty \frac{1}{(2k)!}\sum_{l=0}^{2k-1}(-)^{l+1}A^{(l)}(0) \times\left[\left(\frac{d}{d\lambda}\right)^{2k-l-1} b_k\left(\frac{d}{d\lambda}\,\tfrac{1}{2}\lambda\right)J_0(2\lambda y)\right]_{\lambda=0}\,. \tag{2.14}$$

Again corresponding terms from the $\lambda=\infty$ limit have vanished.

The remarkable feature of the expansion is that the new remainder $R_2(y)$ in Eq. (2.13) cancels the previous one, i.e.,

$$R(y)=R_2(y)-R_1(y) =0\,. \tag{2.15}$$

This property is demonstrated by direct calculation. Using the expansion [see Eq. (A2) of Appendix A] of the Bessel function $J_0(2\lambda y)$, we find from (2.14) that

$$R_2(y)=\sum_{k=1}^\infty A^{(2k-1)}(0)\sum_{q=0}^\infty \frac{(2q)!}{(q!)^2}(-y^2)^q \times\left[\frac{1}{(2k+2q)!}\, b_{q+k}(q+\tfrac{1}{2})\right]. \tag{2.16}$$

Similarly, using (2.12) to evaluate (2.10), we find that

$$R_1(y)=\sum_{k=1}^\infty A^{(2k-1)}(0)\sum_{q=0}^\infty \frac{(2q)!}{(q!)^2}(-y^2)^q\left[\sum_{n=0}^{q}\frac{1}{(2n+2k)!}B_{2n+2k}(\tfrac{1}{2})\binom{2n+2k-1}{2n}\frac{1}{(2q-2n)!}\, b_{q-n}(q+\tfrac{1}{2})\right]. \tag{2.17}$$

So both remainders involve the same derivatives of $A(\lambda)$ at $\lambda=0$ and also the same powers of y^2. The difference lies in the factors contained in curly brackets, which involve the generalized Bernoulli polynomials in a nontrivial way. In Appendix B, we show that these factors are also identical, which then suffices to prove (2.15).

With the usual identification $b=\lambda/K$ of the impact parameter, it becomes clear that (2.13), when it exists, is the expansion of an exact Fourier-Bessel representation of the scattering amplitude. The partial-wave sum has been converted, without approximation, to the integral

$$f(\theta)=(K/i)\int_0^\infty db\, b\, J_0(qb)[S_F(b)-1]\,, \qquad q=2K\sin\tfrac{1}{2}\theta \tag{2.18}$$

where the amplitude $S_F(b)$ is defined by the following expansion in powers of K^{-2}:

$$S_F(b)=b^{-1}\sum_{k=0}^\infty \frac{1}{(2k)!}\, b_k\left(-\tfrac{1}{2}b\frac{d}{db}\right)\left(\frac{1}{K}\frac{d}{db}\right)^{2k} b \times\exp[2i\delta(b)]. \tag{2.19}$$

If the exponential phase factor is commuted leftward, the expansion factors into separate functions of the impact parameter as follows:

$$S_F(b)=\exp[2i\delta(b)]\,W[\delta]\,, \tag{2.20}$$

where using $\delta'\equiv(d/db)\delta(b)$, we write

$$W[\delta]=b^{-1}\sum_{k=0}^\infty \frac{1}{(2k)!}\, b_k\left(-ib\delta'-\tfrac{1}{2}b\frac{d}{db}\right) \times\left(2i\frac{\delta'}{K}+\frac{1}{K}\frac{d}{db}\right)^{2k} b \tag{2.21}$$

as an expansion in powers of K^{-2}, with unity as its leading-order term. Equation (2.21) defines $W[\delta]$ as the function of b one obtains by substituting the $b_k(x)$ polynomials from Table I [now with powers of x replaced by powers of the operator $-ib\delta' -\frac{1}{2}b(d/db)$] and then performing the indicated differentiations of $\delta(b)$. Equation (2.20) is enlightening, in that it clearly displays the distinction between an exact Fourier-Bessel amplitude and the interpolating function of the partial-wave phase shifts.

For a fixed value of b, the expansion obviously will only be well defined provided K is sufficiently large compared with the derivatives of the phase-shift function. Qualitative conditions which must hold if the series is to converge, for fixed b, are that a positive length a exists which bounds the derivatives of the phase function

$$|\delta^{(n)}(b)|<a^{-n}|\delta(b)|\,,$$

and a finite positive length R exists which bounds the range of the interaction in the sense

$$|b^n\delta^{(n)}(b)|<(R/a)^n|\delta(b)|\,. \tag{2.22}$$

Then if $Ka>1$, $R/(Ka^2)<1$, and $|\delta|<1$, the series converges. Generally the series converges faster for larger values of $b \geq R$ due to Eq. (2.6). When the series does not converge, it is asymptotic.

The role of the factor $W[\delta]$ is clarified somewhat by considering the exact unitarity constraint[7] on $S_F(b)$. For potentials no more singular than r^{-1} as $r \to 0$, the elastic unitarity constraint in the form $(4\pi/K)\,\mathrm{Im}f(0) = \sigma$ [$f(0)$ = forward scattering amplitude, σ = total scattering cross section] is satisfied, provided $|S_F(b)| \neq 1$, i.e.,

$$2\pi \int_0^\infty db\, b[|S_F(b)|^2 - 1] = \sigma' = \frac{2\pi}{K^2}\int_{4K^2}^\infty d(q^2)\,|f(q^2)|^2. \quad (2.23)$$

Here σ' is a small $[O(K^{-2})]$ positive number which represents the total scattering cross section to unphysical momentum transfers, $q^2 > 4K^2$. One way of expressing the above unitarity constraint is to write (for later convenience)

$$S_F(b) = [1 - 2\omega(b)]^{1/2} e^{i\chi(b)}. \quad (2.24)$$

The Fourier-Bessel phase function $\chi(b)$ contains $2\delta(b)$ plus contributions from $W[\delta]$; however, the factor $[1 - 2\omega(b)]^{1/2}$ arises solely from $W[\delta]$.[16]

The angular range of validity of Eq. (2.18) is, in principle, now only limited by the singularities of $S_F(b)$ since the Bessel function is an analytic function. In practice, the angular range of validity is limited by the accuracy to which we can construct $\chi(b)$ and $\omega(b)$ in Eq. (2.24).

III. CONNECTION WITH THE GLAUBER APPROXIMATION AND ITS EXTENSIONS

In the high-energy limit ($K \to \infty$, $v = K/E \to$ const) of scattering, the phase function $\delta(b)$ can be approximated by its Born approximation value:

$$2\delta(b) \simeq 2\delta_0(b) = -v^{-1}\int_{-\infty}^\infty dz\; V([z^2 + b^2]^{1/2}). \quad (3.1)$$

Then the leading term of (2.20) produces the Glauber approximation:

$$S_F(b) \simeq S_F^0(b) = \exp[2i\delta_0(b)]. \quad (3.2)$$

Although the cancellation of the remainders in Eq. (2.13) was not known, essentially this method of deriving the high-energy approximation was given in Glauber's 1958 lectures.[1] The purpose at that time was to justify the use of $2\sin\frac{1}{2}\theta$ in the Bessel-function argument as in (2.18) as opposed to $\sin\theta$ or θ. It is apparent that the approximation involves no intrinsic small-angle assumption even though the original derivation of (3.2) did assume θ was small. Glauber's approximation contains the Born approximation when expanded to leading order in $\delta_0(b)$, embodies approximate unitarity, and has the property of reproducing the exact Coulomb scattering amplitude except for a constant phase.

It is worth emphasizing the fact that the high-energy approximation of Glauber does not depend upon the existence of a potential. This is because $\delta_0(b)$ is defined to be the leading term in the expansion of the phase function in powers of the coupling constant. By performing the inverse Fourier-Bessel transformation of the leading-order term from a perturbation-theory calculation [which we denote by $f_B(q^2)$] the phase $\delta_0(b)$ can be defined from

$$2\delta_0(b) \equiv K^{-1}\int_0^\infty dq\, q\, J_0(qb) f_B(q^2). \quad (3.3)$$

Then Eq. (3.2) can be used with this extended definition of $\delta_0(b)$.

Still it is worth pursuing the simple potential scattering case further to explore the corrections to Glauber's high-energy approximation. One aspect of improving upon (3.2) is to use a more sophisticated phase function $\delta(b)$ even if the approximation $W[\delta] = 1$ is not improved. For example, it has been suggested[13] that the WKB phase function be used for $\delta(b)$. If there is a single turning point (r_0) in the radial motion, the WKB phase function is

$$\delta_{\mathrm{WKB}}(b) = K\int_{r_0}^\infty dr\,[1 - b^2/r^2 - 2\epsilon U(r)]^{1/2} - K\int_b^\infty dr\,[1 - b^2/r^2], \quad (3.4)$$

where we define a dimensionless expansion parameter ϵ by the equations

$$\epsilon = V_0/(Kv), \qquad V(r) = V_0 U(r), \quad (3.5)$$

and r_0 is the zero of the first integrand. Here the Langer form which involves b as defined previously is assumed. In this form the WKB phase contains the Glauber phase function $\delta_0(b)$ as its leading term in an expansion in powers of the potential,

$$\delta_{\mathrm{WKB}}(b) = \sum_{n=0}^\infty \delta_n(b). \quad (3.6)$$

The general term in this expansion may be defined in several ways but always involves $n+1$ powers of the potential and, regarding V_0/v fixed, a coefficient of order K^{-n}. For a Yukawa potential $V(r)=V_0 e^{-\mu r}/r$ the expansion (3.6) involves the modified Bessel function $K_n(x)$ of integer order,[7]

$$\delta_n(b)=\frac{K}{\mu}(-\mu\epsilon)^{n+1}\sum_{k=0}^{N}\frac{(n+1)^{n-1}(-)^k}{(n-2k)!(2k)!!}\times\frac{K_k((n+1)\mu b)}{[(n+1)\mu b]^k}\,, \qquad (3.7)$$

where $N=\frac{1}{2}n$ or $\frac{1}{2}(n-1)$. The expansion is useful because the $\delta_n(b)$ are analytic functions of the impact parameter for the special cases of Yukawa and Gaussian potentials. Thus an impact-parameter amplitude containing WKB corrections to the phase function is

$$S_F(b)\simeq S_{\rm WKB}(b)=\exp[2i\delta_{\rm WKB}(b)]\,, \qquad (3.8)$$

with $W[\delta]$ still approximated by unity. There is a disadvantage to this modification in that it no longer reproduces the Coulomb scattering amplitude.

While the WKB phase function is an obvious extension of the Glauber approximation to $\delta(b)$, it is of course not exact. Corrections to the WKB phase shift may be similarly incorporated. The leading correction to the Langer form (3.4) has been developed by Rosen and Yennie.[17] Their result is equivalent to the following expression:

$$\delta_{\rm RY}(b)=-\frac{1}{24K}\int_{r_0}^{\infty}dr\,r^{-1}[1-b^2/r^2-2\epsilon U(r)]^{-1/2}\times\frac{d}{dr}r\frac{d}{dr}F(r)\,, \qquad (3.9a)$$

where

$$F(r)=\ln[2-4\epsilon U(r)-2\epsilon rU'(r)]\,. \qquad (3.9b)$$

To check the validity of the WKB approximation at high energy it is sufficient to consider only the leading terms in $\delta_{\rm YR}(b)$ obtained by expanding in powers of ϵ. We find that these can be expressed as follows:

$$\delta_{\rm RY}(b)=\frac{-1}{24K^2}\left(\frac{d}{db}\right)^3 b[\delta_0(b)+\delta_1(b)]+\tfrac{1}{2}\phi_3(b)+O(K^{-4})\,, \qquad (3.10)$$

where for a Yukawa potential[7]

$$\varphi_3(b)=-\frac{1}{6}\frac{K}{\mu}(\mu\epsilon)^2\left(\frac{\mu}{K}\right)^2[K_0(2\mu b)+2\mu bK_1(2\mu b)] \qquad (3.11)$$

is of order K^{-3} (regarding V_0/v fixed as is our convention), which accounts for the subscript. The functions $\delta_0(b)$ and $\delta_1(b)$ are as given above in Eq. (3.7). A modified WKB approximation to the impact-parameter amplitude is then

$$S_{\rm MWKB}(b)=\exp[2i\delta_{\rm WKB}(b)+2i\delta_{\rm RY}(b)]\,. \qquad (3.12)$$

Clearly the procedure can be continued to higher order so long as the WKB expansion converges. Thus for high-energy potential scattering, it is possible to interpolate the phase shifts in an obvious and well-defined manner, and in this way several suggestions for improving upon the Glauber approximation have been made. However, the situation is not quite so simple due to the higher-order terms from $W[\delta]$, which are competitive with the above-defined WKB phase expansion.

A. Connection with Eikonal Expansion

At some point, it is clear that one can no longer ignore the correction factor $W[\delta]$ which involves derivatives $\delta'(b)$ of the phase function with respect to b. Consider, for example, a term like $\delta_2(b)$, obtained from (3.6), which is of order $K\epsilon^3$. This is the same order as the leading term $Kb[\delta_0'/K]^3=O(K\epsilon^3)$ which comes from the derivatives involved in $W[\delta]$, i.e.,

$$W[\delta]=1-\tfrac{1}{3}iKb[\delta'/K]^3-\tfrac{1}{2}b\delta'\nabla^2\delta/K^2+\tfrac{1}{2}i[b\delta]'''/K^2+O(K^{-4})\,, \qquad (3.13)$$

where $\nabla^2\delta=\delta''(b)+b^{-1}\delta(b)$ is used. Neither type of correction can be said to dominate the other in a systematic grouping of terms of the same order. Thus starting from order $K\epsilon^3$, the expansion of $S_F(b)$ generally involves combinations of terms from expanding $\exp[2i\delta(b)]$ and also $W[\delta]$ by use of Eq. (3.6). In this manner systematic corrections to the Glauber formula (3.2) can be obtained. When like powers of K and ϵ are grouped together, one obtains correction terms to (3.2) identical to a subset of the corrections previously found from an eikonal expansion of the T matrix[7]:

$$S_F(b)=\exp[i\chi_0(b)]\{1+i\chi_1(b)+i\chi_2(b)-\omega_2(b)+[i\chi_1(b)]^2/2!+[i\chi_1(b)]^3/3!+i\chi_1(b)[i\chi_2(b)-\omega_2(b)]+i[\chi_3(b)+\varphi_3(b)]-\omega_3(b)\}+O(K^{-4})\,. \qquad (3.14)$$

Omitting the argument b upon which all these functions depend, the eikonal corrections are equivalent to those given in Ref. 7:

$$\begin{aligned}
\chi_0 &= 2\delta_0 && = O(K\epsilon),\\
\chi_1 &= 2\delta_1 && = O(K\epsilon^2),\\
\chi_2 &= 2\delta_2 - \tfrac{1}{3}b[\delta_0']^3/K^2 && = O(K\epsilon^3),\\
\chi_3 &= 2\delta_3 - b\delta_1'[\delta_0']^2/K^2 && = O(K\epsilon^4),\\
\omega_2 &= \tfrac{1}{2}b\delta_0'\nabla^2\delta_0/K^2 && = O(\epsilon^2),\\
\omega_3 &= \tfrac{1}{2}b[\delta_0'\nabla^2\delta_1 + \delta_1'\nabla^2\delta_0]/K^2 && = O(\epsilon^3).
\end{aligned} \tag{3.15}$$

In the present notation we write $\chi_n(b)$ in place of $\tau_n(b)$.[7] Note that the high-order derivatives from the term involving $[b\delta]'''$ in (3.13) have canceled with the leading portion of $2i\delta_{\rm RY}(b)$ from expanding Eq. (3.12).[18] Just the portion $\varphi_3(b)$ of the Rosen-Yennie phase remains to order ϵ^2/K. Indeed the systematic grouping of powers of K and ϵ has an intriguing consequence: Each of the correction terms to the Glauber phase χ_0 in Eqs. (3.15) as well as φ_3 from Eq. (3.11) vanishes for the special case of a Coulomb potential. This is shown in Ref. 7, where the eikonal expansion containing Eq. (3.14) is developed. Thus even though the δ_n do not vanish (except for δ_1) if $U(r) = r^{-1}$, the combinations in Eqs. (3.15) do vanish. It is intriguing because the Glauber approximation already gives an exact result for the r^{-1} potential. As emphasized by Moore's analysis,[19] this seemingly accidental property of the Glauber approximation should persist to all orders in perturbation theory and thus must be a property of the exact Fourier-Bessel expansion. However, to preserve the Coulomb symmetry of the Fourier-Bessel amplitude, the expansion must be carried out consistently in terms of powers of K and ϵ as above. It then follows that the simpler extensions of the Glauber theory afforded by Eqs. (3.8) or (3.12) are not consistent beyond $\delta(b) \simeq \delta_0(b) + \delta_1(b)$ because they ignore all but the leading term $W[\delta] = 1$ of the Fourier-Bessel expansion.

The term-by-term correspondence between the eikonal expansion and the Fourier-Bessel expansion (2.18) taken together with the WKB phase-shift expansion (3.6) is not complete. In addition to the terms shared in common, the eikonal amplitude $S_E(b;q^2)$ has been shown[7] to contain the following additional terms, through order K^{-3}, on the right side of Eq. (3.14):

$$-\frac{\nabla^2+q^2}{8K^2}\{1 - i\chi_0(b) - i\chi_1(b)[2 + i\chi_0(b)]\}\, e^{i\chi_0(b)}, \tag{3.16}$$

where ∇^2 differentiates all factors. These terms however produce no contribution to the scattering amplitude as $\nabla^2 + q^2$ is the null operator of the Fourier-Bessel transformation (2.18), i.e.,

$$\int_0^\infty db\, b\, J_0(qb)(\nabla^2 + q^2) f(b) = 0; \quad \text{all } q. \tag{3.17}$$

In other words, there are some terms in the leading-order eikonal expansion amplitude $S_E(b;q^2)$ which do not contribute to the scattering amplitude, to order K^{-3} at least, and the remaining subset of terms which do contribute to the scattering amplitude are precisely those obtained from the Fourier-Bessel expansion (2.18) as described above.

B. A Conjecture Regarding the Eikonal Expansion

It is clear that the corrections to the Glauber theory of high-energy scattering involve two ingredients. First, there are the corrections to $\delta_0(b)$, which derive from expansion of the phase functions as in Eqs. (3.8) and (3.12). In addition to these corrections, extra terms arise from $W[\delta]$, which represents the transformation of the l-representation interpolating phase function $S(b) = \exp[2i\delta(b)]$ into a Fourier-Bessel amplitude $S_F(b)$. The resulting Fourier-Bessel amplitude, in turn, has been found through order K^{-3} to be connected in a term-by-term way with the amplitude obtained from an average-momentum-direction eikonal expansion. It is this connection which helps to clarify a previous conjecture regarding the eikonal expansion.

The average-momentum-direction eikonal expansion is unambiguous for forward scattering ($q = 0$). However, for nonzero q, the expansion yields a Fourier-Bessel representation of the T matrix only in the following sense:

$$(-M/2\pi)\langle \vec{k}_f|T|\vec{k}_i\rangle = (K/i) \times \int_0^\infty db\, b\, J_0(qb)[S_E(b;q^2) - 1], \tag{3.18}$$

i.e., the eikonal amplitude generally depends on q^2 in a nontrivial way. For this reason an obvious connection with the standard Fourier-Bessel representation is lacking. A formal expression for the eikonal impact parameter amplitude $S_E(b;q^2)$ explicitly displays the extent of the q^2 dependence through a parameter

$$\beta = 1/\cos\tfrac{1}{2}\theta = [1 - q^2/(4K^2)]^{-1/2}. \tag{3.19}$$

The following expression together with (3.18) generates the average-momentum-direction eikonal expansion of Ref. 7,

$$S_E(b;q^2)=\beta^{-1}\exp[i\beta\chi_0(b)]\times\mathfrak{z}\left\{\exp\left[-i\beta/(2K)\int_{-\infty}^{\infty}dZ\,N(\vec{r},\vec{p})\right]\right\}, \quad (3.20)$$

when (1) it is expanded in powers of $N/2K$ and (2) the parameter β is expanded about unity. The operator

$$N(\vec{r},\vec{p})=\exp[i\beta\chi_+(\vec{r})](\vec{p}+\vec{q})\cdot\vec{p}\exp[-i\beta\chi_+(\vec{r})]$$

incorporates corrections to the linearized propagator of the eikonal approximation. It involves the momentum operator $\vec{p}=-i\vec{\nabla}$ and the position $\vec{r}=(\vec{b},Z)$ through

$$\chi_+(\vec{r})=-v^{-1}\int_{-\infty}^{Z}dz\,V([b^2+z^2]^{1/2}). \quad (3.21)$$

The operator $N(\vec{r},\vec{p})$ does not commute with itself at different points $\vec{r}$. This necessitates Z ordering of the integrals obtained by expansion of the exponential function. The Z ordering is represented by $\mathfrak{z}$ in (3.20). It enforces the ordering constraint $Z_1\geq Z_2\geq Z_3\ldots$ on the integration arguments starting with Z_1 leftmost, etc.

At first sight the eikonal amplitude $S_E(b;q^2)$ exhibits no obvious connection with the Fourier-Bessel expansion. It is only by explicitly carrying out the expansion to leading orders as in Ref. 7 that the very definite connection discussed above emerges. As expressed in Eq. (3.20), the apparent $K\to\infty$ limit is not Glauber's approximation at all but rather a related approximation attributable to Abarbanel and Itzykson[20] (AI), i.e., by approximating $N(\vec{r},\vec{p})/K\to 0$, one obtains

$$S_E(b;q^2)\to S_{\rm AI}(b;q^2)=\beta^{-1}\exp[i\beta\chi_0(b)]. \quad (3.22)$$

This only agrees with the high-energy limit of the Fourier-Bessel expansion when $q^2=0$ ($\beta=1$). In order to resolve the conflict, it becomes necessary to understand the q^2 dependence and why $N/K\to 0$ is not a valid high-energy limit.

The way out of this difficulty in interpreting the eikonal expansion has been discussed in Ref. 7. All q^2 dependence in the eikonal expansion (3.20) was conjectured to disappear in the following way. By expanding Eq. (3.20) in powers of N/K and expanding β^{-1} according to

$$\beta^{-1}=1-q^2/(8K^2)-q^4/(128K^4)-\cdots,$$

it is possible to perform a systematic regrouping of like powers of K^{-1} (regarding now q^2 fixed as well as V_0/v). Then the conjecture is that the regrouped terms in the expansion of eikonal amplitude $S_E(b;q^2)$ are related to the Fourier-Bessel amplitude $S_F(b)$ in the following way:

$$S_E(b;q^2)=S_F(b)+\sum_{n=1}^{\infty}[(\nabla^2+q^2)/K^2]^n t_n(b). \quad (3.23)$$

Here all of the q^2 dependence in the eikonal impact parameter amplitude is of the type explicitly found in the leading orders in Eq. (3.16), i.e., it is of the type that does not contribute to the scattering amplitude for any scattering angle due to Eq. (3.17). There is no doubt that all powers of q^2/K^2 will be present on the right side of (3.23) as they come directly from the expansion of β. Thus the conjecture is really that the $(\nabla^2/K^2)^n$ terms also arise in just the right way when the expansion of Eq. (3.20), through order $2n$ in $N(\vec{r},\vec{p})/2K$, is carried out.

The reason for emphasizing this conjecture is because the Fourier-Bessel expansion of this paper can be seen to provide just the conjectured form. We find an exact term-by-term agreement for $S_F(b)$ to order K^{-3} between the eikonal expansion and the Fourier-Bessel expansion modulo the null-type terms in Eq. (3.23). More importantly, the Fourier-Bessel expansion proves that the only dependence on q^2 arises from the Bessel function $J_0(qb)$. The q^2 dependence in $S_E(b;q^2)$, therefore, must all cancel in the end.

The correspondence with the exact Fourier-Bessel expansion provides, we believe, the essential rule to a sensible interpretation of the average-momentum-direction eikonal expansion. To obtain the physics from the eikonal expansion, one should expand systematically in powers of K^{-1} (q^2, v fixed) and retain only those terms consistently determined. For example, powers such as $(q^2/K^2)^n$ in Eq. (3.23) are known to exist in $S_E(b;q^2)$. However, if one does not know the competing terms from $[N(\vec{r},\vec{p})/K]^{2n}$, the exact K^{-2n} term has not been consistently determined and therefore all K^{-2n} terms should be dropped. The point is that $[N(\vec{r},\vec{p})/K]^2$ can never be regarded as negligible compared with q^2/K^2. Clearly this rule relies on the Fourier-Bessel symmetry of the eikonal expansion conjectured in (3.23). It explains why the AI approximation (3.22) fails to be a good approximation whenever the deviation of β from unity is significant. Indeed the above discussion points out a general pitfall which should be expected when expanding in an operator involving the kinetic energy as does $N(\vec{r},\vec{p})$.

The reason for emphasizing the above rule is because it is necessary to properly interpret the content of a simple eikonal expansion (i.e., for potential scattering) before systematic corrections to the Glauber multiple-diffraction theory (for scattering by compound targets) can be developed using the eikonal expansion method.

IV. HIGHER-ORDER CORRECTIONS TO $S_F(b)$: EXPONENTIATION

The Fourier-Bessel expansion of Eqs. (2.20) and (2.21) gives a rather complicated recipe for constructing the amplitude $S_F(b)$. However, some relatively simple results can be obtained. The first is obtained by using the leading term in the generalized Bernoulli polynomials given in Table I. Then summing the leading portion of each term in the series (2.21) leads to

$$W[\delta] \simeq \sum_{k=0}^{\infty} \frac{1}{k!}(\tfrac{1}{12} i b\delta')^k (2i\delta'/K)^{2k} = \exp[-\tfrac{1}{3} iKb(\delta'/K)^3]\,. \tag{4.1}$$

Thus the Fourier-Bessel phase $\chi(b)$ defined in (2.24) is approximately

$$\chi(b) = 2\delta(b) - \tfrac{1}{3}Kb[\delta'(b)/K]^3\,, \tag{4.2}$$

where an error term of leading order $K\epsilon^5$ arises from the K^{-4} portion of $W[\delta]$, which is not incorporated in (4.1). Still Eq. (4.2) completely defines the first four terms in the expansion of $\chi(b)$ to be

$$\chi(b) = \chi_0(b) + \chi_1(b) + \chi_2(b) + \chi_3(b) + O(K\epsilon^5)\,, \tag{4.3}$$

i.e., the eikonal corrections defined in Eqs. (3.15) have exponentiated. This is just what was assumed (but without definite proof) in Ref. 7, where analytic forms for these phases are given in the cases of Yukawa and Gaussian potentials.

A good deal more insight into the structure of $S_F(b)$ is obtained by considering the classical limit ($\hbar \to 0$, $K \to \infty$). The well-known classical expression[13] for the scattering amplitude takes the form, for fixed scattering angle θ,

$$f(\theta) = (1/i)\left[\frac{KB}{2\sin\theta\,|\delta''(B)|}\right]^{1/2} \times \exp\{2i[\delta(B) - B\delta'(B)] + i\alpha[\delta(B)]\}\,, \tag{4.4}$$

where

$$\alpha[\delta(B)] = (\tfrac{1}{4}\pi)\,\mathrm{sgn}[\delta'(B)] + (\tfrac{1}{4}\pi)\,\mathrm{sgn}[\delta''(B)] \tag{4.5}$$

and B is the point of stationary phase in the integral obtained from Eqs. (2.7) and (2.11). B is given by the condition

$$\pm\theta = 2\delta'(B)/K, \tag{4.6}$$

and we have assumed a single stationary phase point for each value of θ. Equation (4.6) can alternatively be thought of as defining $B(\theta)$.

The above result is to be compared with the one obtained when $S_F(b) = |S_F(b)|e^{i\chi(b)}$ is substituted into (2.18) and the integral evaluated by the stationary-phase method. The result is

$$f(\theta) = (K/i)\left[\frac{b|S_F(b)|^2}{q|\chi''(b)|}\right]^{1/2} \times \exp\{i[\chi(b) - b\chi'(b)] + i\alpha[\chi(b)]\}\,, \tag{4.7}$$

where the same functional defines the phase constant α. However, the value $b(\theta)$ at the Fourier-Bessel stationary phase point is given by

$$\pm q = \pm 2K\sin\tfrac{1}{2}\theta = \chi'(b)\,. \tag{4.8}$$

In general $b(\theta) \neq B(\theta)$ as is apparent from (4.6) and (4.8).

Obviously the content of (4.4) and (4.7) when $\hbar \to 0$ is the same only if there is a definite relation between $\delta(B)$ and $\chi(b)$. The constant phase factors α and absolute values do not affect this relationship since θ is the same sign as q and also $d\theta/dB$ is the same sign as dq/db. If one demands that both values of $f(\theta)$ be the same in the classical limit, the following conditions must be satisfied at fixed θ:

$$\chi'(b) = \pm 2K\sin(\tfrac{1}{2}\theta) = 2K\sin[\delta'(B)/K]\,, \tag{4.9}$$

$$\chi(b) - b\chi'(b) = 2[\delta(B) - B\delta'(B)]\,, \tag{4.10}$$

$$b|S_F(b)|^2/\chi''(b) = B/[2\cos\tfrac{1}{2}\theta\,\delta''(B)]\,. \tag{4.11}$$

Condition (4.9) implies a function $B(b)$ so long as there are unique stationary-phase points b and B for each θ. A simple example of this situation is scattering by a monotonically decreasing potential. In this event we may differentiate both sides of (4.9) and (4.10), with respect to b, and eliminate θ to obtain the relation

$$b\{1 - [\chi'(b)/2K]^2\}^{1/2} = B. \tag{4.12}$$

From (4.10) and (4.12) it follows that

$$\frac{dB}{db} = \frac{b\chi''(b)}{2B\delta''(B)} = \left[1 - \frac{b\chi'(b)\nabla^2\chi(b)}{4K^2}\right]\left\{1 - \left[\frac{\chi'(b)}{2K}\right]^2\right\}^{-1/2}, \tag{4.13}$$

and finally one obtains an interesting result by using (4.12) and (4.13) in (4.11):

$$|S_F(b)|^2 = 1 - b\chi'(b)\nabla^2\chi(b)/(4K^2)\,. \tag{4.14}$$

The magnitude of the Fourier-Bessel amplitude is simply obtained from the phase $\chi(b)$. Thus if we define

$$\omega(b) = b\chi'(b)\nabla^2\chi(b)/(8K^2)\,, \tag{4.15}$$

a Fourier-Bessel amplitude which reproduces the classical approximation to the partial-wave amplitude is given by Eq. (2.24). The phase $\chi(b)$ is given a complex mapping:

$$\chi(b=B/\cos[\delta'(B)/K])=2\delta(B)+2KB \times\{\tan[\delta'(B)/K]-\delta'(B)/K\}, \tag{4.16}$$

which follows from conditions (4.9), (4.10), and (4.12). For example, the WKB approximation to $\delta(B)$ could be used to determine $\chi(b)$ by this equation. By construction, Eq. (4.16) sums all the contributions to $S_F(b)$ which do not vanish in the $K\to\infty$ (finite but arbitrarily large ϵ) limit. If the expansion parameter $\epsilon=V_0/(Kv)$ is small, $\chi(b)$ can be alternatively calculated from Eq. (4.3).

It is not difficult to deduce an expansion for $\chi(b)$ which contains corrections to Eq. (4.3) by expanding the right side of (4.16) in a Taylor series about $B=b$ and using (4.12). However, it is important to observe that the connection between $\chi(b)$ and $\delta(b)$ does not depend on the existence of a potential but rather upon the knowledge of $\delta(B)$. This is the reason that the connection obtained above using the notion of classical scattering by a monotonic potential has much more general content. Although the classical limit was used to find the mapping between $\chi(b)$ and $\delta(B)$, this procedure is really only a simple trick for summing the terms in $W[\delta]$ which survive the $K\to\infty$ limit for fixed ϵ.

For this reason, (4.3) or (4.16) together with (4.15) can be used for any potential. For nonclassical scattering the integral (2.18) must be carried out numerically and nonclassical corrections such as $\varphi_3(b)$ from the Rosen-Yennie phase shift correction may be needed. Still, the above formulas go well beyond the Glauber approximation and should prove quite useful in improving the angular range of validity of the eikonal approximation.

V. RELATIVISTIC FOURIER-BESSEL EXPANSION

It is almost self-evident that a similar connection between the partial-wave expansion and Fourier-Bessel representation will hold in the relativistic case. However, whether or not such an expansion is used to relate the two, a Fourier-Bessel representation exists and is as interesting for relativistic scattering as is the partial-wave expansion. This was first emphasized by Blankenbecler and Goldberger [2] (BG), who established most of the formal properties of the exact Fourier-Bessel representation by using analyticity in the momentum transfer in place of the potential scattering notions stated at the beginning of this paper. This section reviews some general results of BG as a preliminary to showing the explicit conversion of the partial-wave sum to Fourier-Bessel integrals.

The Fourier-Bessel scattering amplitude equivalent to the Mandelstam representation is

$$f(s,t)=(K/i)\int_0^\infty db\, b\, J_0(b(-t)^{1/2})[S_F^t(b;s)-1] + (K/i)\int_0^\infty db\, b\, J_0(b(-u)^{1/2}) \times[S_F^u(b;s)-1], \tag{5.1}$$

where the Mandelstam variables s,t,u are defined in terms of the mass m (assumed to be the same for each particle), c.m. momentum K, and $z=\cos\theta$ by

$$\begin{aligned} s+t+u&=4m^2,\\ t&=-2K^2(1-z),\\ u&=-2K^2(1+z),\\ 4K^2&=s-4m^2, \end{aligned} \tag{5.2}$$

and the normalization is such that

$$\frac{d\sigma}{d\Omega}=|f(s,t)|^2 =\left|\frac{1}{2s^{1/2}}\mathfrak{M}(s,t)\right|^2.$$

The Fourier-Bessel amplitudes in (5.1) can be defined in terms of the spectral functions $A_t(s,t)$ and $A_u(s,u)$ (which include Born terms) in the Mandelstam representation

$$\mathfrak{M}(s,t)=\pi^{-1}\int_{t_0}^\infty dt'\,(t'-t)^{-1}A_t(s,t') +\pi^{-1}\int_{u_0}^\infty du'\,(u'-u)^{-1}A_u(s,u'). \tag{5.3}$$

Blankenbecler and Goldberger define Fourier-Bessel amplitudes equivalent to the following ones (using $\bar{s}=[s(s-4m^2)]^{1/2}$):

$$S_F^t(b,s)-1=\frac{i}{\pi\bar{s}}\int_{t_0}^\infty dt\, A_t(s,t)K_0(bt^{1/2}), \tag{5.4}$$

$$S_F^u(b,s)-1=\frac{i}{\pi\bar{s}}\int_{u_0}^\infty du\, A_u(s,u)K_0(bu^{1/2}). \tag{5.5}$$

By use of the Fourier-Bessel transform of the modified Bessel function $K_0(Z)$ which appears above,

$$\int_0^\infty db\, b\, J_0(b(-t)^{1/2})K_0(bt^{1/2})=(t'-t)^{-1}, \tag{5.6}$$

one can easily verify that (5.4) and (5.5) are equivalent to (5.3). The Mandelstam representation is known to be valid for scattering by Yukawa potentials; however, the spectral function $A_u(s,u)$ vanishes if there is no exchange potential.[21]

The important restriction on the Fourier-Bessel amplitudes of (5.4) and (5.5) arises from the asymptotic unitarity condition. Ignoring small s^{-1} contributions [equivalent to ignoring $\omega(b)$ and σ' in Eqs. (2.23) and (2.24)] BG showed that if the signatured amplitudes

$$S_F^{\pm}(b,s) = S_F^t(b,s) \pm S_F^u(b,s) \mp 1 \tag{5.7}$$

are unimodular, then elastic unitarity can be satisfied in both the forward and backward directions. Dispersion relations can be written for the amplitudes $S_F^{\pm}-1$. However, it is more interesting for our purpose to express unitarity by introducing two Fourier-Bessel phases [22] in analogy to (2.24) as follows:

$$S_F^{\pm}(b,s) = \exp[i\chi^{\pm}(b,s)]\,. \tag{5.8}$$

The + and − denote signature, and the phases $\chi^{\pm}(b,s)$ are real for elastic scattering but become complex when absorption is present. In terms of these signatured amplitudes the scattering amplitude (5.1) becomes

$$f(s,t) = (K/i)\int_0^\infty db\, b\, J_0(b(-t)^{1/2})[\tfrac{1}{2}S^+(b,s) + \tfrac{1}{2}S^-(b,s) - 1] + (K/i)\int_0^\infty db\, b\, J_0(b(-u)^{1/2})[\tfrac{1}{2}S^+(b,s) - \tfrac{1}{2}S^-(b,s)]\,. \tag{5.9}$$

The Fourier-Bessel phases $\chi^{\pm}$ can now be related to the signatured partial-wave amplitudes.

The Froissart-Gribov projection [23] defines the correct interpolating functions for the partial-wave amplitudes by using the integral [24]

$$\tfrac{1}{2}\int_{-1}^{1} dz\, P_J(z)(z'-z)^{-1} = Q_J(z')\,. \tag{5.10}$$

By separating the two independent parity components, the amplitudes

$$A^{\pm}(\lambda,s) = -i(\lambda/K)\left\{\frac{i}{\pi\tilde{s}}\int_{t_0}^\infty dt\, A_t(s,t)Q_{\lambda-1/2}(1+t/(2K^2)) \pm \frac{i}{\pi\tilde{s}}\int_{u_0}^\infty du\, A_u(s,u)Q_{\lambda-1/2}(1+u/(2K^2))\right\} \tag{5.11}$$

are found, which interpolate the physical amplitudes $\lambda = J+\frac{1}{2}$. In these formulas $Q_{\lambda-1/2}(z)$ is the Legendre function of the second kind and is analytic for $\mathrm{Re}\lambda > -\frac{1}{2}$. The partial-wave expansion can be written as a sum over the physical values as

$$f(s,t) = \tfrac{1}{2}\sum_{J\,\mathrm{even}} A^+(J+\tfrac{1}{2},s)[P_J(z)+P_J(-z)] + \tfrac{1}{2}\sum_{J\,\mathrm{odd}} A^-(J+\tfrac{1}{2},s)[P_J(z)-P_J(-z)]\,, \tag{5.12}$$

or alternatively as

$$f(s,t) = \tfrac{1}{2}\sum_J\{[A^+(J+\tfrac{1}{2},s)+A^-(J+\tfrac{1}{2},s)]P_J(z) + [A^+(J+\tfrac{1}{2},s)-A^-(J+\tfrac{1}{2},s)]P_J(-z)\}\,. \tag{5.13}$$

It is customary to express the unitarity constraint by writing the partial-wave amplitudes in exponential form involving phase shifts rather than as in Eq. (5.11):

$$A^{\pm}(\lambda,s) = -i(\lambda/K)\{\exp[2i\delta^{\pm}(\lambda/K,s)]-1\}\,. \tag{5.14}$$

Together with (5.11), (5.14) defines the interpolating functions for the signatured phase shifts.

The set of relations is now complete and the partial-wave sums of Eq. (5.13) can be converted to Fourier-Bessel integrals in the manner of Sec. II. The only difference arises from the $P_J(-z)$ sum which [see Eqs. (2.12) and (5.2)] produces the second integral of (5.9) involving $J_0(b(-u)^{1/2})$. The high-energy expansion can be written down by inspection:

$$S_F^{\pm}(b,s) = \exp[2i\delta^{\pm}(b,s)]W[\delta^{\pm}(b,s)]\,, \tag{5.15}$$

where the factor $W[\delta]$ is just as defined by Eq. (2.21). The Fourier-Bessel phases $\chi^{\pm}(b,s)$ are related to the partial-wave interpolating phases $\delta^{\pm}(b,s)$ in the same way as was found in Sec. IV. The present treatment shows that the θ dependence resides in the Bessel functions as in (5.1) rather than $J_0(Kb\sin\theta)$ as found by Noble.[25] As before, if $\delta^{\pm}(b,s)$ can be expanded as in Eq. (3.6) in powers of the coupling constant, then by virtue of (5.15) an expansion just like Eqs. (4.3) and (3.15) for $\chi^{\pm}(b,s)$ will hold.

Clearly the Born approximation can be used to define the leading portions of the phases $\chi^{\pm}(b,s)$ just as was done in Eq. (3.3). Yukawa terms in the Born approximation like

$$\mathfrak{M}_B(s,t) = \frac{g^2}{\mu^2-t} + \frac{g'^2}{\mu'^2-u} \tag{5.16}$$

correspond to a term $\pi g^2\delta(t-\mu^2)$ in the spectral function $A_t(s,t)$ plus a term $\pi g'^2\delta(u-\mu'^2)$ in the spectral function $A_u(s,u)$ of Eq. (5.3). These one-"particle" exchanges produce Fourier-Bessel phases which are obtained from (5.4) and (5.5):

$$S_F^t(b,s)-1\simeq i(g^2/\bar{s})K_0(\mu b) = \tfrac{1}{2}i[\chi_0^+(b,s)+\chi_0^-(b,s)], \tag{5.17}$$

$$S_F^u(b,s)-1\simeq i(g'^2/\bar{s})K_0(\mu' b) = \tfrac{1}{2}i[\chi_0^+(b,s)-\chi_0^-(b,s)]. \tag{5.18}$$

Here we have written leading-order contributions for the phase functions $\chi^\pm(b,s)$ as $\chi_0^\pm(b,s)$. The solution of the above equation is

$$\chi_0^+(b,s)=[g^2K_0(\mu b)+g'^2K_0(\mu' b)]/\bar{s}, \tag{5.19}$$

$$\chi_0^-(b,s)=[g^2K_0(\mu b)-g'^2K_0(\mu' b)]/\bar{s}. \tag{5.20}$$

Some interesting features are: (1) The limit $\mu'\to\infty$ eliminates the u-channel contribution and reproduces the result of Eq. (3.1) for a Yukawa potential. (2) The limit $\mu'=\mu$, $g'=g$ causes $\chi_0^-(b,s)$ to vanish and gives the Pauli symmetry in Eq. (5.9). Equations (5.19) and (5.20) with (5.8) provide a simple and unambiguous method of unitarizing the single-particle exchange contributions in general.

We know that there are contributions to $\chi^\pm(b,s)$ in Eq. (5.8) to order g^2 and g'^2 and the point is that they must be given by the Born amplitude. Exponentiation of the phases $\chi_0^\pm(b,s)$ is forced by unitarity. What is not evident is whether exponentiation of the leading-order contributions to $\chi^\pm(b,s)$ will produce the leading contribution to the scattering amplitude as $s\to\infty$.

A. Relativistic Eikonal Expansion

In some very impressive papers,[26-30] Cheng and Wu and others have perforce extracted the leading terms in the high-energy limit ($s\to\infty$, t finite) from infinite sequences of Feynman diagrams. By summing the sequences of leading terms, they have shown that an eikonal (Fourier-Bessel) representation can emerge. One result is that when no particles are produced (virtually) in the intermediate state, a simple eikonal result equivalent to (5.19) is found. However, as the energy increases, Cheng and Wu find that absorptive amplitudes due to many-particle intermediate states (tower diagrams) become dominant and these contributions indeed grow very fast with increasing energy. By summing the leading-order contributions from multitower diagrams, they find that the absorptive contribution from one-tower diagrams exponentiates. It becomes apparent from such examples that the perturbation expansion is often equivalent to an expansion of the eikonal phase. If the results of Cheng and Wu are not misleading, such a relativistic eikonal expansion is needed in order to extract field-theoretic predictions for high-energy processes. Fried[29] has also considered such an expansion based on an absorption picture arising from bremsstrahlung of soft mesons.

To simply understand what the relativistic eikonal expansion may mean, it helps to inspect the exact formula for two-particle to two-particle scattering:

$$\chi^\pm(b,s)=-i\ln\left[1+(i/\pi\bar{s})\int_{t_0}^\infty dt\,A_t(s,t)K_0(bt^{1/2})\pm(i/\pi\bar{s})\int_{u_0}^\infty du\,A_u(s,u)K_0(bu^{1/2})\right], \tag{5.21}$$

which restates the connection given in Eqs. (5.4) to (5.8). Consider the perturbation expansion of the spectral functions:

$$A_t(s,t)=\sum_{n=0}^\infty A_t^{(n)}(s,t),\qquad A_u(s,u)=\sum_{n=0}^\infty A_u^{(n)}(s,u), \tag{5.22}$$

which we assume to exist and where $n=0$ designates the Born contribution, i.e., the power of the coupling constant is $n+1$. The relativistic eikonal expansion is the corresponding expansion of the Fourier-Bessel phases in powers of the coupling constant obtained from using Eqs. (5.22) in (5.21) and expanding the logarithm:

$$\chi^\pm(b,s)=\sum_{n=0}^\infty \chi_n^\pm(b,s). \tag{5.23}$$

The leading term is the eikonal approximation $\chi^\pm(b,s)\simeq\chi_0^\pm(b,s)$, which is given by (5.19) and (5.20).

The motivation for the expansion of the Fourier-Bessel phase is twofold. First, it is a perturbation expansion which order by order automatically incorporates asymptotic 2-particle→2-particle unitarity. Second, the expansion can be safely assumed to converge when the impact parameter is large enough. The reason for convergence at large b is that the lowest mass which contributes to each order in the perturbation expansion of the spectral functions is $(n+1)\mu$ or $(n+1)\mu'$. This means that the $\chi_n^\pm(b,s)$ will fall off as $\exp[-(n+1)\mu b]$ or $\exp[-(n+1)\mu' b]$, whichever is decreasing more slowly as $b\to\infty$. In this respect, the behavior is analogous to Eq. (3.7) for Yukawa-potential scattering. So for a fixed energy, the ratio of two consecutive terms $\chi_{n+1}^\pm/\chi_n^\pm$ for large b is proportional to $e^{-\mu b}$ or $e^{-\mu' b}$ and may be made as small as we please. However, the short-distance behavior $(b\to 0)$ is not well understood. For the present it will be assumed that one does not make the behavior as $b\to 0$ worse by exponentiating the perturbation expansion.

In the above generalization of the eikonal expansion, there is a systematic way of generating unitary approximations for scalar two-particle scattering. The leading-order correction to the eikonal (at least for large b) is found from (5.21) to be

$$\chi_1^\pm(b,s)=(\pi\tilde{s})^{-1}\int_{t_0}^\infty dt\,A_t^{(1)}(s,t)K_0(bt^{1/2}) \pm(\pi\tilde{s})^{-1}\int_{u_0}^\infty du\,A_u^{(1)}(s,u)K_0(bu^{1/2})-\tfrac{1}{2}i[\chi_0^\pm(b,s)]^2, \tag{5.24}$$

where the pure imaginary term comes from expansion of the logarithm. To see why the imaginary term is needed, it is helpful to consider Yukawa-potential $[V(r)=V_0e^{-\mu r}/r]$ scattering. Then the spectral function $A_t^{(1)}(s,t)$ comes from the well-known second Born amplitude. The spectral function $A_u^{(1)}(s,u)$ can be ignored since there is no exchange potential.

As shown in Ref. 7, the result of carrying out the inverse Fourier-Bessel transformation of the second Born amplitude is to produce the following expansion for the integral term in (5.24):

$$(\pi\tilde{s})^{-1}\int_{t_0}^\infty dt\,A_t^{(1)}(s,t)K_0(bt^{1/2})=\tfrac{1}{2}i\left[-2\left(\frac{K}{\mu}\right)\mu\epsilon K_0(\mu b)\right]^2+2\frac{K}{\mu}(\mu\epsilon)^2K_0(2\mu b) -i(\mu\epsilon)^2[\tfrac{1}{2}\mu bK_0(\mu b)K_1(\mu b)]\,\frac{1}{6}\,\frac{K}{\mu}(\mu\epsilon)^2\left(\frac{\mu}{K}\right)^2[K_0(2\mu b)+2\mu bK_1(2\mu b)]+O(\epsilon^2/K^2). \tag{5.25}$$

The leading imaginary term here is just $\frac{1}{2}i[\chi_0^\pm(b,s)]^2$ and this cancels the imaginary term in (5.24) when we substitute (5.25). The cancellation then explains that (1) the eikonal correction is predominantly real as it should be for potential scattering and (2) the term in Eq. (5.24) which came from expanding the logarithm is necessary to cancel the leading imaginary term in (5.24). What survives when (5.25) is substituted in (5.24) corresponds exactly to the following terms defined in Eq. (3.14):

$$\chi_1(b)+i\omega_2(b)+\varphi_3(b)+O(\epsilon^2/K^2),$$

that is, the definition given in Eq. (5.24) contains all the terms proportional to $\epsilon^2=(V_0/Kv)^2$ which appear in the expansion of the eikonal phase for a Yukawa potential.

The first nonvanishing imaginary term $i\omega_2(b)$ is of order K^{-2} relative to the leading eikonal phase $\chi_0(b)$ and is equivalent to the elastic unitarity correction discussed in Sec. II, i.e., it produces

$$|S_F(b)|^2=e^{-2\omega_2(b)}\simeq 1-2\omega_2(b). \tag{5.26}$$

This is the same as the result of Eq. (2.24) with $\omega(b)$ approximated by its leading term $\omega_2(b)$. The above considerations show that Eq. (5.24) contains the known results in the potential scattering case where it is clear that $\chi_1(b)$ exponentiates. The succeeding correction terms to the eikonal phase, $\chi_2^\pm(b,s)$, etc., can be found from the expansion of (5.21). In the potential scattering limit (infinite mass of one particle) the expansion must be equivalent to the results of Secs. III and IV. In this case all terms which arise from expanding the logarithm actually cancel with terms of the same order arising from the t and u integrals. In a field theory calculation, the absorptive effects arise from $\chi_1^\pm(b,s)$ and much higher-order contributions in the analyses of Cheng and Wu.

This discussion of the relativistic eikonal expansion is not intended to produce predictions for the eikonal phases $\chi^\pm(b,s)$, but rather to clarify the basis for their calculation and to clarify the angular range of validity of the eikonal expansion. Indeed the justification for the relativistic eikonal expansion given above is in the spirit of S-matrix theory (since we have only assumed analyticity and unitarity) rather than field theory. The only new connection developed is the relation (5.15) between eikonal and partial-wave phase shifts. However, just as in the potential scattering case, there is varying opinion about how to extend the small-angle approximation which generally must be made in deriving eikonal approximations. The small-angle approximation is absent in the above discussion of the eikonal expansion.

It is believed that the problem discussed in Sec. II B generally arises when one linearizes the propagator in the momentum. Thus linearized propagator expansions are generally valid only for small-angle scattering unless one finds some rules, like those given in Sec. III B, for avoiding the problem and extracting the physics.

VI. CONCLUDING REMARKS

In this paper a direct connection between partial-wave and Fourier-Bessel descriptions of scattering amplitudes is developed. This connection is most useful at high energies and provides some unambiguous corrections to the well-known Glauber theory. The content of the present expansion is equivalent to that of an eikonal expansion about the Glauber propagator, provided a proper interpretation of the eikonal expansion is made.

The essential question of angular range of validity of the eikonal approximation has been found to be complicated. For classical scattering, expressions were developed in Sec. IV which show how to add the missing curvature effects back into the straight-line path approximation of the Glauber theory. The corrections are complicated but tractable. For nonclassical scattering, we believe that the expressions developed in this paper will extend the angular range of validity of the Glauber theory as was in fact demonstrated in Ref. 7 using equivalent forms of the eikonal corrections. It is essential however that unitarity be maintained throughout a sequence of approximations to obtain improvement. This is accomplished by expanding consistently the Fourier-Bessel phase shift as in Eq. (4.3), where unitarity is guaranteed by Eq. (2.24).

Many variants of the eikonal approximation involve use of the Glauber approximation as in Eq. (3.2) but also use $J_0(Kb\theta)$ or $J_0(Kb\sin\theta)$ in place of $J_0(2Kb\sin\frac{1}{2}\theta)$ in the impact-parameter integration. These other forms are not likely to be valid away from $\theta=0$.

The present work has dealt only with the spin-zero scattering amplitudes. It is a trivial matter to handle also scattering of a spin-$\frac{1}{2}$ particle by one of spin zero since the only difference [25] arises from the presence of $(d/dz)P_J(z)$ in the partial-wave sum. Equation (2.12) shows that such a sum can be converted to an integral involving $\frac{1}{2}\lambda[(1-z)/2]^{-1/2}J_1(2\lambda[(1-z)/2]^{1/2})$ in place of $J_0(2\lambda[(1-z)/2]^{1/2})$.

For higher spins, the usual partial-wave sum is replaced by the Jacob-Wick helicity expansion.[26] As shown in Eq. (A9) of the Appendix, an expansion of the Wigner d functions appearing in the Jacob-Wick expansion can be made which is quite analogous to the expansion developed for the Legendre function. However, an exact Fourier-Bessel expansion is not readily obtained because a suitable impact parameter is lacking. One expects the impact parameter to range from 0 to ∞. However, the usual definition $b=(J+\frac{1}{2})/K$ fails in this regard since J ranges from λ_m to ∞ in the helicity sum. (λ_m = maximum helicity > 0.) For this reason approximate Fourier-Bessel integrals can be obtained, but not exact ones. The correct definition of the impact parameter appears to be $b=[(J+\frac{1}{2})^2-\lambda_m(\lambda_m+1)]^{1/2}/K$, which is effectively $(J_\perp+\frac{1}{2})/K$, where $J_\perp$ is the component of total angular momentum perpendicular to the momentum. However, in this event, Eq. (A9) does not provide the appropriate expansion of the d function.

The discussions of eikonal expansions given in this paper provide evidence that the straight-line path propagation assumed in the eikonal approximation is not arbitrary after all. To obtain the Fourier-Bessel high-energy limit, it is necessary to use the Glauber propagator [7] $g^{-1}=\vec{v}\cdot(\vec{k}-\vec{p})-V(r)+i\eta$, where $\vec{k}=\frac{1}{2}(\vec{k}_i+\vec{k}_f)$ is the average momentum and $\vec{v}=(K/M)\hat{k}$ is the velocity of the incident particles. This propagator is not the one obtained by simply expanding $\vec{p}$ about $\vec{k}$ since that procedure gives $\vec{v}_{AI}=\vec{k}/M$ as discussed in Sec. III B. The reason that the choice of eikonal propagator is not arbitrary is that there is a Fourier-Bessel symmetry of the scattering amplitude which must be respected to obtain unambiguous results for nonforward angles. The appealing choice of expanding about $\vec{k}_i$ (the initial momentum) is valid for $\theta=0$ since it then agrees with the Glauber propagator, but for nonforward scattering the Fourier-Bessel form is not obtained.

APPENDIX A

An expansion of the Legendre function $P_{\lambda-1/2}(\pm z)$ in terms of the Bessel function $J_0(2\lambda[(1\pm z)/2]^{1/2})$ and its derivatives with respect to λ is developed. The expansion is similar in spirit to the expansion developed by Fields for hypergeometric functions; however, it differs in some essential details. Luke (Ref. 14, p. 52) has summarized Fields's results and also gives formulas for the generalized Bernoulli polynomials (Ref. 14, p. 18) which are used in this Appendix.

The Legendre function is uniquely defined in the circle $|1-z|<2$ and for arbitrary λ by the hypergeometric series

$$P_{\lambda-1/2}(z)={}_2F_1(-\lambda+\tfrac{1}{2},\lambda+\tfrac{1}{2};1;(1-z)/2)$$
$$=\sum_{n=0}^{\infty}\frac{\Gamma(n+\lambda+\frac{1}{2})}{\Gamma(-n+\lambda+\frac{1}{2})}\frac{[-\frac{1}{2}(1-z)]^n}{(n!)^2}. \quad \text{(A1)}$$

For large λ, the ratio of Γ functions tends to λ^{2n}, in which case one obtains a well-known limit valid when $|1-z|$ is small:

$$P_{\lambda-1/2}(z) \to \sum_{n=0}^{\infty} \frac{[-\frac{1}{4}\chi^2]^n}{(n!)^2} = J_0(\chi),$$

$$\chi = 2\lambda[(1-z)/2]^{1/2}. \quad \text{(A2)}$$

The corrections to this formula become important when $|1-z|$ is not small. MacDonald (see Ref. 11, p. 147) has developed the leading corrections to (A2). However, the entire expansion which contains (A2) as the leading term can be developed quite simply by expanding the ratio of Γ functions in (A1). Using a formula due to Fields (see Ref. 14, p. 34), we obtain

$$\frac{\Gamma(n+\lambda+\frac{1}{2})}{\Gamma(-n+\lambda+\frac{1}{2})} = \sum_{k=0}^{n} B_{2k}^{(2n+1)}(n+\tfrac{1}{2})\binom{2n}{2k}\lambda^{2n-2k}. \quad \text{(A3)}$$

In the present case, the series for the ratio of Γ functions terminates because n is an integer. It gives an exact result. The generalized Bernoulli polynomials $B_{2k}^{(2\rho)}(\rho)$ appearing in (A3) are polynomials of order k in the parameter ρ. For our purpose it is convenient to emphasize this polynomial character by introducing a new notation:

$$b_k(\rho) = B_{2k}^{(2\rho)}(\rho). \quad \text{(A4)}$$

Table I of the text lists the polynomials $b_k(\rho)$, $k=0(1)5$. For values of k greater than 5, the polynomials can be calculated from the recursion

$$b_k(\rho) = -2\rho \sum_{r=0}^{k-1} \binom{2k-1}{2r+1} \frac{B_{2r+2}}{2r+2} b_{k-r-1}(\rho), \quad \text{(A5)}$$

where B_{2r+2} is a Bernoulli number.

The expansion (A3) is equivalent to the following derivative operator acting on λ^{2n}:

$$\frac{\Gamma(n+\lambda+\frac{1}{2})}{\Gamma(-n+\lambda+\frac{1}{2})} = \left[\sum_{k=0}^{n} \frac{1}{(2k)!}\left(\frac{d}{d\lambda}\right)^{2k} b_k\left(\frac{d}{d\lambda}\tfrac{1}{2}\lambda\right)\right]\lambda^{2n}. \quad \text{(A6)}$$

Powers of $n+\frac{1}{2}$ in the polynomial $b_k(n+\frac{1}{2})$ of (A3) are here replaced by powers of $(d/d\lambda)\frac{1}{2}\lambda$ acting on λ^{2n}. Similarly the binomial coefficient $\binom{2n}{2k}$ is replaced by $1/(2k)!$ times an over-all $2k$th-order derivative of the factor λ^{2n}, which survives the action of $b_k((d/d\lambda)\frac{1}{2}\lambda)$. Now substituting (A6) into (A1) leads directly to the desired expansion:

$$P_{\lambda-1/2}(1-2y^2) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{(2k)!}\left(\frac{d}{d\lambda}\right)^{2k} b_k\left(\frac{d}{d\lambda}\tfrac{1}{2}\lambda\right)\frac{[-\frac{1}{2}\chi^2]^n}{(n!)^2}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(2k)!}\left(\frac{d}{d\lambda}\right)^{2k} b_k\left(\frac{d}{d\lambda}\tfrac{1}{2}\lambda\right)\sum_{n=0}^{\infty}\frac{[-\frac{1}{2}\chi^2]^n}{(n!)^2}. \quad \text{(A7)}$$

The k and n sums have been interchanged in writing down the second line of (A7) and the proper lower limit $n=k$ of the n sum has been replaced by $n=0$, since the derivative operator of order $2k$ annihilates all terms added by this change. The interchange of sums is permissible because the Bessel function $J_0(\chi)$ possesses continuous derivatives to all orders. Thus the expansion for the Legendre function which contains (A2) as its leading term is

$$P_{\lambda-1/2}(z) = \sum_{k=0}^{\infty} \frac{1}{(2k)!}\left(\frac{d}{d\lambda}\right)^{2k} b_k\left(\frac{d}{d\lambda}\tfrac{1}{2}\lambda\right) \times J_0(2\lambda[\tfrac{1}{2}(1-z)]^{1/2}). \quad \text{(A8)}$$

This expansion converges for $|1-z|<2$ since it is just a rearrangement of the original series (A1).

The expansion of (A8) is fairly rich in mathematical content since many related expansions of special functions commonly used in mathematical physics can be obtained by slight generalization of the derivation.

Associated Legendre functions $P_\nu^\mu(z)$ and Jacobi polynomials $P_\nu^{(\alpha,\beta)}(z)$ as well as the Wigner rotation functions $d_{\lambda\mu}^j(z)$ can all be expanded in a series of Bessel functions in the manner used above. For example, an expansion for the Wigner rotation functions which appear in the Jacob-Wick expansion of the scattering amplitude is

$$d_{\mu,\nu}^{\lambda-1/2}(z) = \operatorname{sgn}(\mu,\nu)\Phi_{\mu,\nu}(\lambda-\tfrac{1}{2}) \times \sum_{k=0}^{\infty} \frac{1}{(2k)!}\left(\frac{d}{d\lambda}\right)^{2k} b_k\left(\frac{d}{d\lambda}\tfrac{1}{2}\lambda\right)[\tfrac{1}{2}\lambda^2(1+z)]^{\beta/2} \times J_\Delta(2\lambda[\tfrac{1}{2}(1-z)]^{1/2}), \quad \text{(A9)}$$

where $\Delta = |\mu-\nu|$ is the helicity change,

$$\beta = \min\{|\nu|, |\mu|\},$$

$$\Phi_{\mu,\nu}(\lambda-\tfrac{1}{2}) = \left[\frac{\Gamma(\lambda+\frac{1}{2}+\beta-\Delta)\Gamma(\lambda+\frac{1}{2}-\beta)}{\Gamma(\lambda+\frac{1}{2}-\beta+\Delta)\Gamma(\lambda+\frac{1}{2}+\beta)}\right]^{1/2}, \quad \text{(A10)}$$

and $\operatorname{sgn}(\mu,\nu)$ is a phase factor reflecting the relations

$$d_{\mu,\nu}^j(z) = (-)^{\mu-\nu}d_{\nu,\mu}^j(z) = (-)^{\mu-\nu}d_{-\mu,-\nu}^j(z) = d_{-\nu,-\mu}^j(z),$$

and is given by

$$\operatorname{sgn}(\mu,\nu) = \begin{cases} 1, & \mu-\nu \geq 0 \\ (-)^{\mu-\nu}, & \mu-\nu < 0 \end{cases}. \quad \text{(A11)}$$

The expansion in (A9) is not developed in detail because it does not appear to give a desirable impact-parameter representation for scattering as discussed in Sec. VI. However, the expansions of the above special functions $P_\nu^\mu(z)$ and $P_\nu^{(\alpha,\beta)}(z)$ can be simply obtained from (A9).

APPENDIX B: A RECURSION RELATION FOR GENERALIZED BERNOULLI POLYNOMIALS

In this appendix, a key formula which is used in Sec. II of the text is derived. The formula involves a nontrivial relation among the generalized Bernoulli polynomials $B_{2k}^{(2a)}(a) \equiv b_k(a)$, which is not available in the literature. Thus it is necessary to show that [see Eqs. (2.16) and (2.17)]

$$\frac{1}{(2k+2q)!}\, b_{k+q}(q+\tfrac{1}{2}) = \sum_{n=0}^{q} \binom{2n+2k-1}{2n} b_{q-n}(q+\tfrac{1}{2}) \times \frac{1}{(2q-2n)!}\, \frac{B_{2k+2n}(\frac{1}{2})}{(2k+2n)!}\,, \tag{B1}$$

where $B_{2k+2n}(x)$ is the ordinary Bernoulli polynomial. The ordinary and generalized Bernoulli polynomials are discussed in Ref. 14, where one also finds the two basic formulas

$$aB_k^{(a+1)}(x) = (a-k)B_k^{(a)}(x) + k(x-a)B_{k-1}^{(a)}(x)\,, \tag{B2}$$

which relates polynomials of differing superscript order, and

$$B_k^{(a)}(x) = (-)^k B_k^{(a)}(a-x), \tag{B3}$$

which shows that when $a=2x$, as in the present case, only even-order (in k) polynomials are non-vanishing.

From (B2) one obtains (by using $a \to 2a-1$, $k \to 2k$, $x=a$)

$$(2a-1)B_{2k}^{(2a)}(a) = (2a-1-2k)B_{2k}^{(2a-1)}(a) + 2k(1-a)B_{2k-1}^{(2a-1)}(a)\,.$$

Now using (B3) to replace $B_{2k}^{(2a-1)}(a)$ by $B_{2k}^{(2a-1)}(a-1)$ and again applying (B2) to reduce each superscript from $2a-1$ to $2a-2$ on the right side, one finds a recursion for the polynomials $b_k(a)$ as follows:

$$(2a-2)_2\, b_k(a) = (2a-2k-2)_2\, b_k(a-1) - (2k-1)_2(a-1)^2\, b_{k-1}(a-1)\,, \tag{B4}$$

where

$$(n)_m = n(n+1)\cdots(n+m-1). \tag{B5}$$

To derive (B1), first substitute $a=q+\frac{1}{2}$ and $k=k+q$ in (B4), where q and k are integers, to obtain the recursion

$$(2q-1)_2\, b_{k+q}(q+\tfrac{1}{2}) = (-1-2k)_2\, b_{k+q}(q-\tfrac{1}{2}) - (2k+2q-1)_2(q-\tfrac{1}{2})^2\, b_{k+q-1}(q-\tfrac{1}{2})\,, \tag{B6}$$

and then iterate this formula $q-1$ more times to reduce the argument $q-\frac{1}{2}$ of the b polynomials on the right side down to $+\frac{1}{2}$. For example, two more iterations of (B6) help to illuminate the progression which results:

$$\begin{aligned}(2q-5)_6\, b_{k+q}(q+\tfrac{1}{2}) &= (-5-2k)_6\, b_{k+q}(q-\tfrac{5}{2}) \\ &+ (2k+2q-1)_2(-3-2k)_4[-(q-\tfrac{1}{2})^2-(q-\tfrac{3}{2})^2-(q-\tfrac{5}{2})^2]b_{k+q-1}(q-\tfrac{5}{2}) \\ &+ (2k+2q-3)_4(-1-2k)_2[(q-\tfrac{1}{2})^2(q-\tfrac{3}{2})^2+(q-\tfrac{1}{2})^2(q-\tfrac{5}{2})^2+(q-\tfrac{3}{2})^2(q-\tfrac{5}{2})^2]b_{k+q-2}(q-\tfrac{5}{2}) \\ &+ (2k+2q-5)_6[-(q-\tfrac{1}{2})^2(q-\tfrac{3}{2})^2(q-\tfrac{5}{2})^2]b_{k+q-3}(q-\tfrac{5}{2})\,.\end{aligned}$$

The coefficients in square brackets here arise from derivatives with respect to x of the polynomial

$$[x-(q-\tfrac{1}{2})^2][x-(q-\tfrac{3}{2})^2][x-(q-\tfrac{5}{2})^2],$$

evaluated at the point $x=0$. Thus the progression above can be generalized to the following formula valid for p iterations of (B6):

$$(2q-2p+1)_{2p}\, b_{k+q}(q+\tfrac{1}{2}) = \sum_{n=0}^{p} (-1-2k-2n)_{2n}\,(2k+2q-2p+2n+1)_{2p-2n} \times b_{k+q-p+n}(q-p+\tfrac{1}{2})\left\{\frac{1}{n!}\,\frac{d^n}{dx^n}[x-(q-\tfrac{1}{2})^2][x-(q-\tfrac{3}{2})^2]\cdots[x+(q-p+\tfrac{1}{2})^2]\right\}_{x=0}\,. \tag{B7}$$

The factor in the curly bracket is now evaluated by noting that it involves the ratio of two Γ functions when we set $p=q$:

$$\{\ \}_{x=0} = \left\{\frac{1}{n!}\,\frac{d^n}{dx^n}\,\frac{\Gamma(\sqrt{x}+q+\frac{1}{2})}{\Gamma(\sqrt{x}-q+\frac{1}{2})}\right\}_{x=0}\,,$$

and can be evaluated by use of the expansion (A3) of Appendix A. The result is

$$\{\ \}_{x=0} = b_{q-n}(q+\tfrac{1}{2})\binom{2q}{2q-2n}\,. \tag{B8}$$

Then choosing $p=q$ in (B7), using $b_{k+n}(\frac{1}{2}) = B_{2k+2n}(\frac{1}{2})$, and substituting (B8) for the curly bracket produces

$$(2q)!b_{k+q}(q+\tfrac{1}{2}) = \sum_{n=0}^{q}(-1-2k-2n)_{2n} \times (2k+2n-1)_{2q-2n}B_{2k+2n}(\tfrac{1}{2}) \times \binom{2q}{2q-2n} b_{q-n}(q-\tfrac{1}{2}). \tag{B9}$$

Finally, one obtains (B1) from this formula by replacing the Pochammer symbols by factorials using (B5), which completes the derivation.

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$$\hat{f}(\theta) = \begin{cases} f(\theta), & 0 \le \theta < \pi \\ 0, & \theta > \pi \end{cases}$$

in place of the analytic $f(\theta)$. The replacement of $f(\theta)$ by $\hat{f}(\theta)$ is needed to permit use of a discontinuous integral representation of the Legendre polynomials which vanishes identically for $\theta > \pi$. However, the use of $\hat{f}(\theta)$ in place of $f(\theta)$ *is* an approximation and can be seen to be equivalent to approximating $\sigma' = 0$, $|S_F(b)| = 1$, and $\omega(b) = 0$ in Eqs. (2.23) and (2.24) of the present paper. The error induced by the use of $\hat{f}(\theta)$ is of order K^{-2} and is thus small when K is large. However, it explains the difference between the K^{-2} term in Eq. (2.18) and a K^{-2} correction $S_F(b)$ given by Cottingham and Peierls (Ref. 6) based on $\hat{f}(\theta)$.

The discontinuous cutoff at $\theta = \pi$ in $\hat{f}(\theta)$ also causes high-frequency oscillations

$$S_F(b)_{b \simeq \infty} f(\pi)\cos(2Kb - \tfrac{1}{4}\pi)/(2Kb)$$

in the large-b behavior of the impact-parameter amplitude. This unphysical behavior is avoided only by adopting a new representation of the Legendre function such as is given in the present work.

These differences of order K^{-2} and higher turn out to be essential in establishing a term-by-term correspondence between Eq. (2.18) and an eikonal expansion of $f(\theta)$ as discussed in Sec. III.

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Equal-Time Commutator of Charge Densities in Quantum Electrodynamics

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We show that one-electron and one-photon expectation values of the equal-time commutator of charge densities in quantum electrodynamics, which are usually assumed to be zero, contain δ-function derivatives of order $n \geq 4$ when evaluated in fourth-order perturbation theory.

I. INTRODUCTION

A few years back Drell and Hearn[1] obtained a sum rule from dispersion relations and the low-energy theorem which relates the anomalous magnetic moment of the proton to an integral over photoabsorption cross sections. Many authors[2] have given an equal-time commutator method of derivation of this sum rule which is applicable to any spin-$\frac{1}{2}$ particle. In this derivation it is assumed that the electric charge densities commute at equal times. On the other hand, in the derivation given by Drell and Hearn, it is assumed that the spin-flip Compton amplitude satisfies an unsubtracted dispersion relation. As we shall show, this would be true if the electric current $\vec{j}(x)$ commutes with the potential $\vec{A}(y)$ at equal times. This assumption is equivalent to the vanishing of the commutator of the charge density with itself at equal times. The vacuum expectation value of this commutator can be shown to be zero. It is however, not clear whether its one-particle expectation value is also zero. A simple way to settle this question is to examine how far unsubtracted dispersion relations for Compton scattering as well as photon-photon scattering are satisfied when cross sections and amplitudes appearing in these relations are approximated by their perturbation-theory values. We find that these are not satisfied, showing thereby that $\vec{j}(x)$ does not commute with $\vec{A}(y)$ at equal times. Consequently $\rho(x)$ does not commute with $\rho(y)$ at equal times. We give explicit expressions for the one-electron and one-photon expectation values of

$$[\rho(x),\rho(y)]\delta(x_0-y_0)\,.$$

II. ONE-ELECTRON EXPECTATION VALUE

To obtain the one-electron expectation value of the commutator under discussion, we start with the on-shell forward Compton scattering amplitude

$$f_{rs}(\omega)=t_{rs}(\omega)-i\int d^4x\,e^{ikx}\delta(x_0)\langle p|[j_s(0),\dot{A}_r(x)]|p\rangle+\omega\int d^4x\,e^{ikx}\delta(x_0)\langle p|[j_s(0),A_r(x)]|p\rangle\,, \tag{1}$$

where

$$t_{rs}(\omega)=i\int d^4x\,e^{ikx}\theta(x_0)\langle p|[j_s(0),j_r(x)]|p\rangle\,. \tag{2}$$

It follows from the structure of $t_{rs}(\omega)$ that the dispersive part $d_2(\omega^2)$, given by