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Bound-State Problem in a Model Nonpolynomial-Lagrangian Theory

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We have shown that the bound-state problem in a nonpolynomial-Lagrangian theory can be solved as in the usual polynomial theory assuming that a Wick rotation is admissible. For definiteness we have assumed the interaction of two scalar fields interacting via the exchange of a superfield to be of the form $U(x) = \exp[g\,\phi(x)]$, where g denotes the minor coupling constant and $\phi(x)$ is a massless neutral scalar field. The major coupling constant is introduced through the ladder diagrams in the Bethe-Salpeter formalism. We find that the Wick-rotated Bethe-Salpeter equation reduces to a standard Fredholm equation with a modified kernel corresponding to the exchange of the superfield $U(x)$. To study the physical content in the theory we have investigated the equation in the instantaneous approximation. The resulting nonrelativistic equation is projected onto the surface of a four-dimensional sphere by using Fock's transformation variables. The bound-state eigenvalue problem is solved approximately in the weak-binding limit, using Hecke's theorem, leading to a Balmer-type formula. Finally, the fully relativistic equation at $E = 0$ is considered by transforming it onto the surface of a five-dimensional Euclidean sphere. The approximate-symmetry property of the equation is studied, and the eigenvalue problem is solved in terms of the coupling constants of the theory.

I. INTRODUCTION

Of late there has been some interest in the use of nonpolynomial-Lagrangian models to study the nature of interaction between elementary particles. While a typical nonpolynomial interaction, viz., $H_{\text{int}} = G\exp[g\phi(x)]$, where $\phi(x)$ is a neutral scalar field and G and g are the major and minor coupling constants, respectively, was discussed at length by Okubo[1] as early as 1954, there are two factors that have mainly caused the regeneration of interest in the class of such Lagrangians. These are (a) the demonstration by Efimov[2] that the constraints to be put on the interaction Lagrangian in order to have a divergence-free theory do not exclude such type of interactions, and (b) their occurrence within the chiral group in the framework of current algebra. The particularly attractive feature of the nonpolynomial theories, as has been emphasized by Salam,[3] is the presence of built-in damping effects which might allow the calculation of renormalization constants, and other self-interaction effects, which were perhaps impossible to obtain in polynomial theories. As a consequence, some of the salient fundamental features of this class of Lagrangians have been under extensive investigation. The criteria of their renormalizability have been dealt with[3]; some attention has also been given to the construction of the S matrix and investigation of the unitarity property.[4] Some of the recent investigations have been concerned with two-body scattering processes via the exchange of an infinite number of bubbles.[5,6]

In this paper we study the existence of bound-state solutions for two scalar particles interacting via the exchange of a nonpolynomial field $U(x) = \exp[g\phi(x)]$ within the framework of the Bethe-Salpeter equation. We note that while it is a model Lagrangian, nevertheless it contains all the basic features of a nonpolynomial theory. For instance, the rules for calculating the S-matrix elements analogous to those in ordinary perturbation theory and questions of, e.g., the uniqueness and finiteness of results together with the conditions of unitarity and causality have already been discussed with respect to this interaction. It is worthwhile then to discuss the bound-state problem in the relativistic nonpolynomial theory of the exponential type. As in the usual polynomial theory, we discuss the problem in the ladder-diagram approach. In Sec. II we give a simple derivation of the bound-state Bethe-Salpeter equation for the nonpolynomial interaction following the standard method. In Sec. III we briefly take up the question of the radially symmetric solutions of this equation for zero total energy of the system. Assuming the admissibility of a Wick rotation we succeed in reducing the equation to a form from which it is apparent that the standard Fredholm technique can be applied for its further study. To study the physical content of the theory we investigate, in Sec. IV, the Bethe-Salpeter equation with our model Lagrangian in the instantaneous approximation. The resulting nonrelativistic equation is projected onto the surface of a four-dimensional sphere by using Fock's transformation variables. The bound-state eigenvalue problem is solved approximately in the weak-binding limit leading to a modified form of the

Balmer formula. Finally, in Sec. V, the fully relativistic equation at $E=0$ is considered by transforming it onto the surface of a five-dimensional Euclidean sphere and the approximate-symmetry property of the equation studied. The eigenvalue problem is solved resulting in a transcendental equation involving the parameters of the theory, namely, the major and the minor coupling constants.

II. THE BETHE-SALPETER EQUATION FOR A NONPOLYNOMIAL INTERACTION

We consider the interaction Hamiltonian

$$H_I = G'\psi^2(x)U(x), \tag{2.1}$$

where $\psi(x)$ is a neutral scalar field of mass m and $U(x)$ stands for the nonpolynomial field

$$U(x) = \exp[g\phi(x)]; \tag{2.2}$$

$\phi(x)$ is taken to be a massless neutral scalar field, G' is the renormalized[1] major coupling constant,

$$G' = G\exp[g^2 D_F(0)],$$

and g determines the minor interaction strength. In the standard way, we adopt the following definition for the bound-state wave function for the two-body system of ψ fields:

$$\chi(x,y) = \langle 0|\,T(\psi(x)\psi(y))\,|p\rangle, \tag{2.3}$$

where $|0\rangle$ and $|p\rangle$ are the state vectors for the vacuum and the two-body system, respectively. If we regard the two-body interaction processes to be taking place owing to the exchange of the $U(x)$ field alone, then

$$\chi(x,y) = -G'^2\int d^4x_1 d^4y_1 \Delta_F(x-x_1)\Delta_F(y-y_1) \times F(x_1-y_1)\chi(x_1,y_1), \tag{2.4}$$

where Δ_F is the usual Feynman propagator function for the ψ field and

$$F(x_1-y_1) = \langle 0|\,T(U(x_1)U(y_1))\,|0\rangle. \tag{2.5}$$

Separating the center-of-mass motion from $\chi(x,y)$, we can obtain the relevant equation for the relative motion through the substitution

$$\chi(x,y) = \exp[\tfrac{1}{2}iE(x+y)]\tau(x-y)$$

in Eq. (2.4). Introducing the Fourier transforms

$$\tau(x-y) = \int d^4p\, e^{ip(x-y)}\tau(p), \tag{2.6a}$$

and

$$F(x-y) = \frac{-i}{(2\pi)^4}\int d^4p\, e^{ip(x-y)}F(p), \tag{2.6b}$$

we can rewrite Eq. (2.4) in momentum space as

$$\tau(p) = \frac{G'^2}{(2\pi)^4 i}\frac{1}{(\frac{1}{2}E+p)^2+m^2}\frac{1}{(\frac{1}{2}E-p)^2+m^2} \times\int d^4p' F(p-p')\tau(p'). \tag{2.7}$$

We now proceed to evaluate $F(p)$. We know that the factor that corresponds to the propagation of the superfield in coordinate space between space-time points 0 and x is $F(x)-1$, where

$$F(x) = \exp[g^2 D_F(x)]. \tag{2.8}$$

By analytic continuation we set, following Okubo,[1]

$$g^2 = -i\lambda \quad (\lambda > 0),$$

whence

$$F(x) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{1}{u+D_F(x)}e^{i\lambda u}du. \tag{2.9}$$

Then $F(p)$, the Fourier transform of $F(x)$, is seen to be given by

$$F(p) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} f(p)e^{i\lambda u}du, \tag{2.10}$$

where $f(p)$ = Fourier transform of $1/[u+D_F(x)]$. Writing $f(x) = 1/[u+D_F(x)]$, we easily find that $f(p)$ satisfies the following integral equation:

$$f(p) = \frac{(2\pi)^4}{u}\delta^4(p) + \frac{i}{u(2\pi)^4}\int\frac{d^4p'}{(p-p')^2}f(p'), \tag{2.11}$$

where we have used the fact that

$$D_F(x) = \frac{-i}{(2\pi)^4}\int\frac{1}{p^2-i\epsilon}e^{ipx}d^4x.$$

Setting

$$f(p) = \frac{(2\pi)^4}{u}\delta^4(p) + g(p), \tag{2.12}$$

we obtain

$$g(p) = \frac{i}{u^2}\frac{1}{p^2} + \frac{i}{(2\pi)^4}\frac{1}{u}\int\frac{1}{(p-p')^2}g(p')d^4p'. \tag{2.13}$$

For solving Eq. (2.13) we indicate below a straightforward method.[6] By effecting Wick rotation we go over from the Lorentz metric to the Euclidean metric in Eq. (2.13). The application of the operator $\Box_p^2$ converts this integral equation into a differential equation[7]:

$$\left(\Box_p^2 - \frac{1}{4\pi^2 u}\right)g(p) = \frac{i}{u^2}\Box_p^2\left(\frac{1}{p^2}\right). \tag{2.14}$$

Taking $g \equiv g(p^2)$, and putting $s=p^2$, Eq. (2.14) becomes

$$\left(4s\frac{d^2}{ds^2}+8\frac{d}{ds}-\frac{1}{4\pi^2 u}\right)g(s)=0\,. \tag{2.15}$$

The introduction of a new function

$$v=wg(w)\,,$$

where

$$w=\left(\frac{s}{4\pi^2 u}\right)^{1/2},$$

reduces Eq. (2.15) to an equation for the modified Bessel functions:

$$\frac{d^2v}{dw^2}+\frac{1}{w}\frac{dv}{dw}-\left(\frac{1}{w^2}+1\right)v=0\,. \tag{2.16}$$

The solution of this equation which satisfies the boundary condition, viz.,

$$g(p^2)\rightarrow\frac{i}{u^2}\frac{1}{p^2}\quad\text{as}\quad\frac{1}{4\pi^2 u}\rightarrow 0\,,$$

is given by

$$g(p^2)=\frac{i}{4\pi^2u^3}\,\frac{1}{(p^2/4\pi^2u)^{1/2}}\,K_1\!\left(\left(\frac{p^2}{4\pi^2u}\right)^{1/2}\right).$$

Using a suitable representation[8] for the K function, we obtain

$$g(p^2)=\frac{2i}{\pi^2u^3}\int_0^\infty\frac{\beta^2J_1(\beta)d\beta}{(\beta^2+p^2/4\pi^2u)^3}\,, \tag{2.17}$$

whence the substitution of Eq. (2.17) together with Eq. (2.12) into Eq. (2.10) yields

$$F(p)=(2\pi)^4\delta^4(p)-\frac{2\lambda}{p^2}\int_0^\infty d\beta\frac{J_2(\beta)}{\beta}\exp\!\left(\frac{-ip^2\lambda}{4\pi^2\beta^2}\right). \tag{2.18}$$

Recalling that the factor that corresponds to the superpropagator in momentum space is the Fourier transform of $F(x)-1$, we can drop the $\delta^4(p)$ term from Eq. (2.18). Upon substituting it in Eq. (2.7) we obtain

$$\tau(p)=-\frac{2\lambda f}{i\pi^2}\,\frac{1}{(\frac{1}{2}E+p)^2+m^2}\,\frac{1}{(\frac{1}{2}E-p)^2+m^2}\times\int_0^\infty d\beta\frac{J_2(\beta)}{\beta}\int\frac{d^4p'}{(p-p')^2}\times\exp[-i\lambda'(p-p')^2]\tau(p')\,, \tag{2.19}$$

where

$$f=(G'/4\pi)^2$$

and

$$\lambda'=\lambda/4\pi^2\beta^2\,.$$

We note that our result for $F(p)$ is the same as obtained previously by Okubo.[1]

III. RADIALLY SYMMETRIC SOLUTIONS OF THE BETHE-SALPETER EQUATION WITH NONPOLYNOMIAL FIELD

The momentum-space Bethe-Salpeter amplitude for the bound state of two scalar particles interacting via the exchange of the nonpolynomial field $U(x)=\exp[g\phi(x)]$ in the ladder approximation (for zero total energy of the system) is given by

$$(p^2+m^2)^2\tau(p)=-\frac{2\lambda f}{i\pi^2}\int_0^\infty d\beta\frac{J_2(\beta)}{\beta}\times\int\frac{d^4p'}{(p-p')^2}\exp[-i\lambda'(p-p')^2]\tau(p')\,, \tag{3.1}$$

where the symbols have the meaning ascribed to them in Sec. II.

To make Eq. (3.1) tractable, we first employ Wick rotation. This is followed by a transition to spherical polar coordinates in four dimensions. Then, for the radially symmetric solution, i.e., for $\tau\equiv\tau(p^2)$, we obtain

$$(s+m^2)^2\tau(s)=-\frac{4\lambda f}{\pi}\int_0^\infty d\beta\frac{J_2(\beta)}{\beta}\times\int_0^\infty ds'\,s'\exp[-i\lambda'(s+s')]\,\tau(s')I_\chi\,, \tag{3.2}$$

where

$$I_\chi=\int_0^\pi d\chi\,\frac{\sin^2\chi\exp[i2\lambda'(ss')^{1/2}\cos\chi]}{[s-2(ss')^{1/2}\cos\chi+s']}\,,$$

and

$$s=p^2\,,$$

$$s'=p'^2\,.$$

The χ integration can be performed by expanding the denominator in the integrand in a series of Gegenbauer polynomials. The resulting series may be expressed in terms of Lommel functions of two variables[8]:

$$I_\chi=\tfrac{1}{2}\pi\left(\frac{U_0}{s}+i\frac{U_1}{s}+\frac{U_2}{s'}+i\frac{U_3}{s'}\right),$$

where

$$U_i\equiv U_i(2\lambda's',\,2\lambda'(ss')^{1/2})\,.$$

Consequently, Eq. (3.2) becomes

$$\tau(s)=(-2\lambda f)\int_0^\infty ds'K(s,s')\tau(s')\,, \tag{3.3}$$

where

$$K(s,s')=\int_0^\infty d\beta\frac{J_2(\beta)}{\beta}\,s'\,\frac{\exp[-i\lambda'(s+s')]}{(s+m^2)^2}\times\left[\frac{U_0}{s}+i\,\frac{U_1}{s}+\frac{U_2}{s'}+i\,\frac{U_3}{s'}\right]. \tag{3.4}$$

The eigenvalue condition for the existence of solutions of Eq. (3.2) is given by the vanishing of the Fredholm determinant of the problem. It is thus seen that the standard Fredholm technique can be applied to the nonpolynomial field of the form $\exp[g\phi(x)]$ as in the usual polynomial theory.

IV. INSTANTANEOUS-INTERACTION APPROXIMATION AND THE BALMER FORMULA

We now proceed to investigate Eq. (2.19) in the approximation where the propagation time for the exchanged particle is neglected, following closely the method given by Basu and Biswas.[9] In this approximation, Eq. (2.19) reduces to

$$[\vec{p}^2-(p_0+\tfrac{1}{2}E)^2+m^2][\vec{p}^2-(p_0-\tfrac{1}{2}E)^2+m^2]\tau(p)=-\frac{2\lambda f}{i\pi^2}\int_0^\infty d\beta\frac{J_2(\beta)}{\beta}\int\frac{d^4p'}{(\vec{p}-\vec{p}')^2}\exp[-i\lambda'(\vec{p}-\vec{p}')^2]\tau(p'), \tag{4.1}$$

where we have used the rest frame,

$$E=(0,0,0,iE).$$

We now introduce a function $S(\vec{p})$, given by

$$[\vec{p}^2-(p_0+\tfrac{1}{2}E)^2+m^2][\vec{p}^2-(p_0-\tfrac{1}{2}E)^2+m^2]\tau(p)=S(\vec{p}), \tag{4.2}$$

and carry out the p_0 integration. Defining

$$\phi(\vec{p})=\frac{S(\vec{p})}{(\vec{p}^2+m^2)^{1/2}(\vec{p}^2+m^2-\tfrac{1}{4}E^2)}, \tag{4.3}$$

one can easily obtain

$$(\vec{p}^2+m^2)^{1/2}(\vec{p}^2+m^2-\tfrac{1}{4}E^2)\phi(\vec{p})=\frac{-\lambda f}{\pi}\int_0^\infty d\beta\frac{J_2(\beta)}{\beta}\int\frac{\phi(\vec{p})d^3p'}{(\vec{p}-\vec{p}')^2}\exp[-i\lambda'(\vec{p}-\vec{p}')^2]. \tag{4.4}$$

It is thus seen that in the instantaneous-interaction approximation, the additional degree of freedom of the relative time variable appearing in the fully covariant equation is removed. Further, since Eq. (4.4) is O(3)-symmetric, one can write

$$\phi(\vec{p})=g_l(p)Y_l^m(\theta,\phi)\quad(|\vec{p}|=p). \tag{4.5}$$

Substituting Eq. (4.5) in Eq. (4.4) and using the properties of spherical harmonics,[10] one can separate out the angular variables to obtain

$$X(\rho^2)(1+\rho^2)g_l(\rho)=\frac{-\lambda f}{\pi c m^2}\int d\beta\frac{J_2(\beta)}{\beta}\int\int\int\rho'^2g_l(\rho')P_l(\cos\theta')\frac{\sin\theta'd\theta'd\phi'd\rho'}{\rho^2+\rho'^2-2\rho\rho'\cos\theta'}\times\exp[-i\lambda''(\rho^2+\rho'^2-2\rho\rho'\cos\theta')] \tag{4.6}$$

where

$$c=\left(1-\frac{E^2}{4m^2}\right),\quad p=mc\rho,$$
$$X(\rho^2)=(1+c^2\rho^2)^{1/2},\ \text{and}\ \lambda''=\lambda'm^2c^2. \tag{4.7}$$

In order to solve Eq. (4.6) in the weak-binding limit, we follow the procedure given by Fock.[11] Thus, by means of the transformations

$$\rho=\tan\tfrac{1}{2}\psi,\quad\rho'=\tan\tfrac{1}{2}\psi', \tag{4.8}$$

Eq. (4.6) is projected onto the surface of a four-dimensional sphere leading to

$$[1+\tfrac{1}{2}c^2\tan^2(\tfrac{1}{2}\psi)]\sec^2(\tfrac{1}{2}\psi)g_l(\tan\tfrac{1}{2}\psi)=\frac{-\lambda f}{4\pi m^2c}\cos^2(\tfrac{1}{2}\psi)\int d\beta\frac{J_2(\beta)}{\beta}\int\sec^4(\tfrac{1}{2}\psi')g_l(\tan\tfrac{1}{2}\psi')\frac{P_l(\cos\theta')d\Omega_4'}{(1-\cos\Theta)}\times\exp\{-i\lambda''\tfrac{1}{2}\sec^2(\tfrac{1}{2}\psi)\sec^2(\tfrac{1}{2}\psi')(1-\cos\Theta)\}, \tag{4.9}$$

where Θ is the angle between two unit four-dimensional vectors of polar angles $(\psi, 0, 0)$ and (ψ', θ', ϕ'),

$$\cos\Theta = \cos\psi\cos\psi' + \sin\psi\sin\psi'\cos\theta',$$

and $d\Omega_4'$ is the four-dimensional solid angle:

$$d\Omega_4' = \sin^2\psi'\sin\theta'\, d\psi' d\theta' d\phi'.$$

If we further define

$$H(\psi) = \sec^4(\tfrac{1}{2}\psi)\, g_l(\tan(\tfrac{1}{2}\psi)),$$

Eq. (4.9) takes the form

$$(\alpha + \beta'\cos\psi)H(\psi) = \frac{-\lambda f}{4\pi m^2 c}(1+\cos\psi)\int d\beta \frac{J_2(\beta)}{\beta}\int \frac{H(\psi')P_l(\cos\theta')d\Omega_4'}{(1-\cos\Theta)} \times \exp\{-\tfrac{1}{2}i\lambda''(1-\cos\Theta)[1+\tan^2(\tfrac{1}{2}\psi)+\tan^2(\tfrac{1}{2}\psi')+\tan^2(\tfrac{1}{2}\psi)\tan^2(\tfrac{1}{2}\psi')]\}, \tag{4.10}$$

where

$$\alpha = (1+\tfrac{1}{2}c^2), \quad \beta' = (1-\tfrac{1}{2}c^2). \tag{4.11}$$

In the region of extremely small binding energies, we may replace $\alpha = \beta' \simeq 1$ (we have neglected terms of the order c^2). In this region we see that Eq. (4.10) possesses O(4) symmetry if we neglect

$$f(\psi, \psi') = \tan^2(\tfrac{1}{2}\psi) + \tan^2(\tfrac{1}{2}\psi') + \tan^2(\tfrac{1}{2}\psi)\tan^2(\tfrac{1}{2}\psi'). \tag{4.12}$$

With this approximation, Eq. (4.10) reduces to

$$H(\psi) = -\frac{\lambda f}{4\pi m^2 c}\int d\beta \frac{J_2(\beta)}{\beta} \times \int \frac{H(\psi')P_l(\cos\theta')d\Omega_4'}{(1-\cos\Theta)} \times \exp[-\tfrac{1}{2}i\lambda''(1-\cos\Theta)]. \tag{4.13}$$

The solution of Eq. (4.13) can be taken as

$$H(\psi) = P_{N-1,\,l}^{(2)}(\cos\psi), \quad N = 1, 2, 3, \ldots, \tag{4.14}$$

where the functions $P_{N,\,l}^{(2)}(\cos\psi)$ are related to the Gegenbauer polynomials in a simple manner:

$$P_{N,\,l}^{(2)}(\cos\psi) = \frac{1}{N+1}\sin^l\psi\, C_{N-l}^{l+1}(\cos\psi).$$

Substituting the solution (4.14) in Eq. (4.13), applying Hecke's theorem, and noting that

$$\int \frac{P_l(\cos\theta')P_{N-1,\,l}^{(2)}(\cos\psi')d\Omega_4'}{(1-\cos\Theta)} \times \exp[-\tfrac{1}{2}i\lambda''(1-\cos\Theta)] = \frac{2\pi^{3/2}}{\Gamma(3/2)}\int_{-1}^{+1} dx \frac{P_{N-1,\,0}^{(2)}(x)}{1-x}(1-x^2)^{1/2} \times \exp[-\tfrac{1}{2}i\lambda''(1-x)]P_{N-1,\,l}^{(2)}(\cos\psi), \tag{4.15}$$

we easily obtain

$$\frac{-4\pi m^2 c}{\lambda f} = \frac{4\pi}{N}\int_0^\infty d\beta \frac{J_2(\beta)}{\beta} \times \int_{-1}^{+1} \frac{dx(1-x^2)^{1/2}}{(1-x)} \times \exp[-\tfrac{1}{2}i\lambda''(1-x)]C_{N-1}^1(x). \tag{4.16}$$

In Eq. (4.16) we first perform the β integration by a method given by Okubo.[1] Letting

$$I(x) = \int_0^\infty d\beta \frac{J_2(\beta)}{\beta}\exp[-\tfrac{1}{2}i\lambda''(1-x)] = \frac{\phi(u)}{u}, \tag{4.17}$$

where

$$u = \frac{-\lambda(1-x)m^2c^2}{8\pi^2}$$

and

$$\phi(u) = u\int_0^\infty d\beta \frac{J_2(\beta)}{\beta}\exp\left(\frac{iu}{\beta^2}\right),$$

one can show that $\phi(u)$ satisfies the following third-order differential equation:

$$\left(\frac{d^3\phi}{du^3} - \frac{\phi}{4iu^2}\right) = 0. \tag{4.18}$$

Using the solution[1] of this equation which is analytic in the range $-1 \leqslant x \leqslant +1$ we have

$$I(x) = -\tfrac{1}{4}i\sum_{n=0}^\infty \frac{1}{(n+2)!\,(n+1)!\,n!}\left(\frac{i\lambda(1-x)m^2c^2}{32\pi^2}\right)^{n+1}, \tag{4.19}$$

the other two linearly independent solutions being nonanalytic at $x=0$. Substituting Eq. (4.19) in Eq. (4.16), we have

$$\frac{-m^2c}{\lambda f}=\frac{-i}{4N}\sum_{n=0}^{\infty}\frac{1}{(n+2)!\,(n+1)!\,n!}\left(\frac{-g^2m^2c^2}{32\pi^2}\right)^{n+1}L_n\,, \tag{4.20}$$

where[8]

$$\begin{aligned}L_n&=\int_{-1}^{+1}dx\frac{(1-x^2)^{1/2}}{(1-x)}C_{N-1}^1(x)(1-x)^{n+1}\\&=\frac{2^n\pi}{(N-1)!}\sum_{k=0}^{N-1}\frac{\Gamma(-N+1+k)\Gamma(N+1+k)\Gamma(n+k+\frac{3}{2})}{\Gamma(-N+1)\Gamma(k+\frac{3}{2})\Gamma(n+k+3)}\frac{1}{K!}\,.\end{aligned} \tag{4.21}$$

From Eq. (4.21) we get the corresponding Balmer formula in our model nonpolynomial-Lagrangian theory as

$$\begin{aligned}c&=\left(1-\frac{E^2}{4m^2}\right)^{1/2}\\&=\frac{-fg^2\pi}{4m^2N!}\sum_{n=0}^{\infty}\sum_{K=0}^{N-1}\frac{2^n}{(n+2)!(n+1)!n!}\left(\frac{-g^2m^2c^2}{32\pi^2}\right)^{n+1}\frac{\Gamma(-N+1+k)\Gamma(N+1+k)\Gamma(n+k+\frac{3}{2})}{\Gamma(-N+1)\Gamma(k+\frac{3}{2})\Gamma(n+k+3)}\frac{1}{k!}\,,\end{aligned} \tag{4.22}$$

where the infinite series in Eq. (4.22) in terms of the square of the minor coupling constant is absolutely convergent for all values of g^2.

V. APPROXIMATE O(5) SYMMETRY FOR $E=0$

Setting the masses of the ψ fields equal to unity, the Wick rotated Bethe-Salpeter equation with the nonpolynomial interaction under consideration may be written as

$$(1+p^2)^2\tau(p)=\frac{f}{\pi^2}\int F((p-p')^2)\tau(p')d^4p'\,, \tag{5.1}$$

where we have confined our attention to the case $E=0$; $F(p)$, the superpropagator for the field $U(x)$, is given by

$$F(p)=-\frac{2\lambda}{p^2}\int_0^\infty d\beta\frac{J_2(\beta)}{\beta}\exp(-i\lambda' p^2)\,. \tag{5.2}$$

In order to investigate the symmetry structure of the above equation, we transform it onto the surface of a five-dimensional sphere by following the stereographic-projection method of Fock,[11] Lévy,[12] and Cutkosky.[13] For this purpose we go over to a polar coordinate system in the five-dimensional Euclidean space with polar angles χ, ψ, θ, and ϕ, χ being given by

$$|p|=\tan\tfrac{1}{2}\chi\,,$$

and ψ being the angle between four-vector p and the (imaginary) relative time axis. Defining

$$H(\chi,\psi,\theta,\phi)=\sec^6(\tfrac{1}{2}\chi)\tau(p)\,, \tag{5.3}$$

we may rewrite Eq. (5.1) as

$$\begin{aligned}H(\chi,\psi,\theta,\phi)&=\frac{f}{8\pi^2}\int d\Omega_5'\frac{H(\chi',\psi',\theta',\phi')}{1-\cos\alpha}\\&\quad\times(-2\lambda)\int_0^\infty d\beta\frac{J_2(\beta)}{\beta}\exp\{-\tfrac{1}{2}i\lambda'(1-\cos\alpha)[1+\tan^2(\tfrac{1}{2}\chi)+\tan^2(\tfrac{1}{2}\chi')+\tan^2(\tfrac{1}{2}\chi)\tan^2(\tfrac{1}{2}\chi')]\}\,,\end{aligned} \tag{5.4}$$

where $d\Omega_5'$ is an element of the solid angle in five dimensions, and α is the angle between two five-dimensional unit vectors defined by the polar angles $(\chi,0,0,0)$ and $(\chi',\psi',\theta',\phi')$. From the above equation we immediately see that if we neglect

$$f(\chi,\chi')=\tan^2(\tfrac{1}{2}\chi)+\tan^2(\tfrac{1}{2}\chi')+\tan^2(\tfrac{1}{2}\chi)\tan^2(\tfrac{1}{2}\chi')\,, \tag{5.5}$$

the equation under study is invariant under all rotations in five dimensions.

Neglecting the symmetry-breaking part $f(\chi,\chi')$ in Eq. (5.4), we can obtain its solution in terms of five-dimensional spherical harmonics as well as the corresponding eigenvalues $(f)_N$. Thus, regarding $C^{3/2}_{N-1}(\cos\psi)$ as a solution we have for $\chi=0$

$$\frac{1}{(f)_N}C^{3/2}_{N-1}(1)=\frac{1}{8\pi^2}\int d\Omega_4'\frac{C^{3/2}_{N-1}(\cos\psi')\sin^3\chi' d\chi'}{1-\cos\chi'}(-2\lambda)\int_0^\infty d\beta\frac{J_2(\beta)}{\beta}\exp[-\tfrac{1}{2}i\lambda'(1-\cos\chi')]. \tag{5.6}$$

Multiplying both sides of Eq. (5.6) with r^{N-1}, summing over N, and using the definition of the generating function of the Gegenbauer polynomials, we obtain

$$\sum_{N=1}^\infty\frac{1}{(f)_N}r^{N-1}C^{3/2}_{N-1}(1)=R(r), \tag{5.7}$$

where

$$R(r)=\tfrac{1}{4}\int_0^\pi\frac{\sin^3\chi' d\chi'}{(1-\cos\chi')(1-2r\cos\chi'+r^2)^{3/2}}(-2\lambda)\int_0^\infty d\beta\frac{J_2(\beta)}{\beta}\exp[-\tfrac{1}{2}i\lambda'(1-\cos\chi')]. \tag{5.8}$$

We now make a Taylor-series expansion of $R(r)$ about $r=0$, and use

$$C^{3/2}_{N-1}(1)=\tfrac{1}{2}N(N+1).$$

Comparing the coefficients of r^{N-1}, we obtain from Eqs. (5.7) and (5.8)

$$(f)_N=\tfrac{1}{2}(N+1)!\frac{1}{R^{(N-1)}(0)}, \tag{5.9}$$

where

$$R^{(N-1)}(0)=\tfrac{1}{4}(N-1)!\int_0^\infty(-2\lambda)d\beta\frac{J_2(\beta)}{\beta}\int_{-1}^{+1}(1+x)C^{3/2}_{N-1}(x)\exp[-\tfrac{1}{2}i\lambda'(1-x)]dx. \tag{5.10}$$

The β integration in the above equation may be performed as in Sec. IV, followed by the x integration, to yield

$$R^{(N-1)}(0)=\frac{(N-1)!}{256\pi^2}g^4\sum_{n=0}^\infty\frac{1}{(n+2)!(n+1)!n!}\left(\frac{-g^2}{32\pi^2}\right)^n Q_n, \tag{5.11}$$

where

$$\begin{aligned}Q_n&=\int_{-1}^{+1}dx(1-x)^{n+1}(1+x)C^{3/2}_{N-1}(x)\\&=\frac{2^{n+2}}{(N-1)!}\sum_{k=0}^{N-1}\frac{\Gamma(-N+k+1)\Gamma(N+k+2)\Gamma(n+k+2)}{\Gamma(-N+1)\Gamma(k+2)\Gamma(n+k+4)}\frac{1}{\Gamma(k+1)}.\end{aligned} \tag{5.12}$$

Equation (5.9) together with Eqs. (5.11) and (5.12) gives the different discrete eigenvalues $(f)_N$—expressed as an infinite alternating series in terms of the square of the minor coupling constant. The infinite series (5.11) is absolutely convergent for all values of g^2.

Thus, as an approximate solution we obtain the Gegenbauer polynomials as solutions of our problem with eigenvalues given by Eq. (5.9). We note that to obtain the exact eigenvalues, one should assume the solution as a linear combination of Gegenbauer polynomials with unknown coefficients, which would lead to a set of difference equations for the coefficients. One can proceed thereafter as in the ordinary Bethe-Salpeter equation.[14]

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Method of Characteristics and Causality of Field Propagation*

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It is pointed out that the method of characteristics gives only a necessary condition (not a sufficient one) for causality of field propagation. It is shown that the usual theory of vector fields with anomalous magnetic moment coupling to an external magnetic field is noncausal, contrary to what has been believed, as one of its modes of propagation turns out to be tachyonic, with the velocity of light as minimum velocity.

It is well known that the components of higher-spin ($s \geq 1$) fields satisfying manifestly covariant relativistic wave equations are subject to constraints (usually implicit in the field equations) which ensure that the number of field components is just what is demanded by the spin degrees of freedom. When the field is in interaction with other (external) fields, these constraints depend in general on the latter, and this fact leads to serious problems of self-consistency. In particular, for arbitrary external fields, causality may be violated. The systematic investigation of such problems by Johnson and Sudarshan[1] in the context of spin-$\frac{3}{2}$ fields is well known. Recently Velo and Zwanziger[2] have suggested that the question whether field propagation remains causal in the presence of specific interactions can be resolved by inspection of the characteristic surfaces associated with the system of partial differential equations[3] constituted by the field equations. They have investigated a number of examples from this approach—in particular, the case of spin-1 particles with anomalous magnetic moments or electric quadrupole moments. In the presence of quadrupole interaction with external electric fields it turns out that field propagation is noncausal, as evidenced by the existence of spacelike characteristic surfaces. In the case of the anomalous magnetic dipole interaction the characteristic surfaces coincide with the null cone (irrespective of the space-time dependence of the magnetic field), and it is concluded therefore that causality is not violated.

There seems to be good reason, however, to doubt the validity of this last conclusion. Firstly, there is the work of Lee and Yang[4] showing that when the electromagnetic interaction of a spin-1 particle includes an anomalous magnetic moment term, the interaction Hamiltonian to be used for generating the S matrix (through covariant Feynman rules) has a part which is noncovariant and non-Hermitian. A more direct demonstration of difficulties with the theory comes from the recent finding by Tsai and Yildiz[5] that the energy spectrum of such a particle (in sufficiently strong magnetic fields) includes imaginary values. These results are clearly at variance with the conclusion of Velo and Zwanziger, and a reexamination of the latter is therefore essential. This is our objective in this note.

In the work of Velo and Zwanziger[2] and others,[6] the ratio $n_0/|\vec{n}|$ associated with the normal $n \equiv (n_0, \vec{n})$ to a characteristic vector is identified as the maximum velocity of propagation of some mode of the field. Our basic observation is that this identification is not always correct; $n_0/|\vec{n}|$ may well be the *minimum* velocity in certain cases. This point will be clear from the simple example of the Klein-Gordon equation. The equation for the normal to the characteristic surface is $n^2=0$, so that $n_0/|\vec{n}|=1$. This is indeed the maximum velocity in the case of ordinary particles, but if one takes $m^2<0$ (tachyons) it becomes the mini-