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Conformal-Invariant Green Functions Without Ultraviolet Divergences

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It is known that conformal invariance, with anomalous dimensions, determines the 2- and 3-point functions in relativistic quantum field theory up to some constants. It is then natural to use these to construct the skeleton-graph expansion of the general n-point Green function. We demonstrate that this leads to well-defined conformal-invariant expressions, for nonexceptional external momenta, given by ordinary convergent (Riemann) integrals. This also applies to the integrals in the Schwinger-Dyson equation for the 2- and 3-point functions. These results support in particular conjectures recently advanced by Migdal—including self-consistency of conformal invariance. All results are derived for (pseudo-) scalar Yukawa theory.

I. INTRODUCTION

Wilson's suggestion[1,2] that infinite wave-function renormalization may lead to fields of anomalous dimensions opens the way to construct a nontrivial conformal-invariant[3,4] field theory. More precisely one can write down[5] the most general 2- and 3-point functions invariant under infinitesimal conformal transformations which satisfy the correct locality and analyticity properties in coordinate space and positivity constraints that follow from the general principles of relativistic quantum field theory. They do not coincide with the corresponding functions for a free field (if dimensions are anomalous). It is then natural to attempt to use these simple conformal-invariant expressions for the "dressed" propagators and vertex functions in the skeleton-graph expansion of the general n-point function ($n \geq 4$). (These skeleton-graph expansions may be considered as iterative solutions of integral equations for Green functions in Lagrangian field theory.[6]) The purpose of the present paper is to prove that such a skeleton theory is free from ultraviolet divergences. All Green functions are given by ordinary convergent integrals, for nonexceptional external momenta.[7] We consider scalar or pseudoscalar (ps) Yukawa theory as a model of hadronic interactions. Strict chiral invariance (without spontaneous breaking) may also be imposed if desired. Quantum electrodynamics is essentially different, however,[8] and is not covered here.

We emphasize that our result should not be viewed as an exercise in constructive quantum field theory (QFT) only. The conformal-invariant zero-mass theory is expected to reproduce correctly the real short-distance behavior of products of fields and currents in the presence of strong interactions.[1] Indeed, recently Symanzik showed[9-11] that the small-x behavior of the c-number coefficients in Wilson's operator-product expansions are determined by properties of certain zero-mass theories. Some of these coefficients are measurable by electron-positron annihilation[12] or deep-inelastic lepton-hadron scattering.[13] Before going to concrete applications, we must of course include currents in our considerations. This problem will be reserved for a later publication.[14]

For consistency one must also analyze the (remaining) integral equations which are normally used[6,15,16] for determination of the 2- and 3-point functions. A relevant discussion was recently given by Migdal[17] for the 3-point function, and by Parisi and Peliti for the 2-point function.[18] This work will be reviewed in Sec. IV in the light of our results. As a consequence of these integral equations, one obtains three (algebraic) equations for the three parameters g, d, and d'. Assuming that this system of equations is nondegenerate, the theory will not contain any dimensionless-free parameter.

This is in agreement with what one expects on the basis of the Callan-Symanzik analysis of short-distance behavior in renormalizable Lagrangian QFT. In particular, Symanzik has shown[9] for the example of φ^4 theory that its Gell-Mann–Low-limit theory[19]—which is expected to be conformal-invariant on the basis of Schroer's result[20]—contains no dimensionless-free parameter if it exists at all. Accordingly one may look upon the present approach as an attempt to construct directly the Gell-Mann–Low large-momentum asymptote of a massive theory (Sec. IV C).

The material of the present paper is organized as follows. First, we give the most general conformal-invariant expressions for the 2- and 3-point functions (Sec. II). The derivation of these expressions based on the manifestly conformal-covariant six-dimensional formalism[21] (Appendix A) is given in Appendix B. In Sec. III we demonstrate the absence of ultraviolet divergences for dimensions of mass $1<d<3$, $\frac{3}{2}<d'<\frac{5}{2}$ of the "meson" and "nucleon" field, respectively, and also absence of infrared divergences of the "catastrophic kind."[22,23] To save labor the analysis is first performed in the presence of an infrared cutoff (a mass-type parameter) (Sec. III A). Subsequently, it is shown that the infrared cutoff can be removed for nonexceptional external momenta (Sec. III B), and that the limit thus obtained is conformal-invariant (Sec. III C). The proof of conformal invariance is based on rewriting the generalized Feynman rules for the skeleton graphs in a manifestly conformal-invariant form. Section IV is concerned with the integral equations for the 2- and 3-point functions. It is also shown that γ_5 invariance, which implies vanishing of the coupling constant of one of the two possible conformal-invariant vertex functions, can be postulated without violating the bootstrap condition. Appendix C contains the proof of the fundamental covariance lemma which is used in Sec. III C.

II. CONFORMAL-INVARIANT PROPAGATORS AND VERTEX FUNCTIONS: THE POSITIVITY CONSTRAINT

We consider Poincaré-covariant quantum fields $\psi(x)$ transforming under dilatations $x\to\rho x$ according to the law

$$U(\rho)\psi(x)U^{-1}(\rho)=\rho^{d_\psi}\psi(\rho x),\quad \rho>0 \tag{2.1}$$

and under infinitesimal special conformal transformations as

$$U(\epsilon)\psi(x)U^{-1}(\epsilon)=\psi(x)-i\epsilon^\mu[\psi(x),K_\mu]\quad(\epsilon^\mu\to 0),$$

where

$$[\psi(x),K_\mu]=i(2d_\psi x_\mu+2x_\mu x_\nu\partial^\nu-x^2\partial_\mu-2ix^\nu s_{\mu\nu})\psi(x)\,. \tag{2.2}$$

Here $s_{\mu\nu}=\frac{1}{4}i[\gamma_\mu,\gamma_\nu]$ for a Dirac field, and $s_{\mu\nu}=0$ for a spinless field.

Unlike the global conformal transformations, these infinitesimal rules are well defined for a field in Minkowski space, and do not violate the causal order of events.[24] The nonexistence of global unitary transformations corresponding to the infinitesimal law (2.2) (except[25] for the trivial case of a free zero-mass field) indicates that the operators K_μ, though formally Hermitian, are not self-adjoint (cf. Ref. 26).

Henceforth the dimension of mass d_ψ of the (pseudo-) scalar "meson" field will be denoted by d, and the dimension of the spin-$\frac{1}{2}$ "nucleon" field by d'. [Note that dimension of length $l_\psi=-d_\psi$ (rather than d_ψ) is used in some papers,[3] and others (see Ferrara *et al.*, Ref. 12) use l_ψ for the dimension of mass.] We shall also assume that our theory is invariant under space reflection as well as under the γ_5 transformation ($\gamma_5{}^2=-1$)

$$\psi(x)\to\gamma_5\psi(x),\quad \bar\psi(x)\to\bar\psi(x)\gamma_5,\quad \phi(x)\to-\phi(x)\,. \tag{2.3}$$

In this section we write down and discuss the general expression for the invariant 2- and 3-point functions in the case of interacting spinor and (pseudo-) scalar fields. They are derived in Appendix B by the manifestly covariant technique of Appendix A. They can all be written as (products of) generalized Feynman propagators in the sense of Speer,[27] with specified exponents δ. We shall use the following notation for these later:

$$\begin{aligned}&\Delta_\delta^c(x)=i(4\pi)^{-2}\Gamma(\delta)(-\tfrac{1}{4}x^2+i0)^{-\delta}\,,\\&S_\delta^c(x)=i\not\partial\Delta_{\delta-1/2}^c(x),\quad \not\partial=\gamma\cdot\partial\,.\end{aligned} \tag{2.4a}$$

Similarly, as a prototype for Wightman functions we define

$$\Delta_\delta^+(x)=i(4\pi)^{-2}\Gamma(\delta)(-\tfrac{1}{4}x^2+i0x^0)^{-\delta}\,. \tag{2.4b}$$

A. The 2-Point Function in Coordinate and Momentum Space

The general form of the conformal-invariant 2-point functions is derived in Appendix B. It is found that the dressed propagators for a (pseudo-) scalar field ϕ of dimension d and for a spinor field ψ of dimension d' are given by

$$\begin{aligned}&\langle 0|T(\phi(x)\phi^*(0))|0\rangle=-i\Delta_d^c(x)\,,\\&\langle 0|T(\psi(x)\bar\psi(0))|0\rangle=-iS_{d'}^c(x)\,,\end{aligned} \tag{2.4c}$$

with the right-hand side defined by Eq. (2.4a).

Similarly, the Wightman functions,

$$\langle 0|\phi(x)\phi^*(0)|0\rangle = -i\Delta_d^+(x),$$
$$\langle 0|\psi(x)\bar\psi(0)|0\rangle = -iS_{d'}^+(x) \equiv \not\partial\Delta_{d'-1/2}^+(x). \tag{2.4d}$$

The distributions Δ_d^c and Δ_d^+ are (different) boundary values of the same analytic function $iF(z)$ holomorphic in the extended tube[28]

$$\mathcal{T} = \{z \in C^4 | z^2 \neq a \geq 0\}.$$

This domian includes the Euclidean region in which x_0 is pure imaginary ($x_0 = ix_4$, $x \neq 0$). The Fourier transform of Δ^c is most easily evaluated by performing the Wick rotation to Euclidean x and p:

$$\begin{aligned}\tilde\Delta_d^c(ip_4, \underline{p}) &= \int d^4x\, e^{ipx}\Delta_d^c(x)\\ &= \int\int dx_4 d^3x e^{-i\vec{p}\cdot\vec{x}} F(ix_4, \underline{x})\\ &= \Gamma(2-d)|\vec{p}|^{2(d-2)},\end{aligned} \tag{2.5}$$

whence

$$\tilde\Delta_d^c(p) = \Gamma(2-d)(-p^2 - i0)^{d-2}. \tag{2.5$'$}$$

Here

$$\vec{p}\cdot\vec{x} = \underline{p}\cdot\underline{x} + p_4x_4 \quad (p_0 = ip_4), \quad |\vec{p}|^2 = \underline{p}^2 + p_4{}^2.$$

In evaluating the right-hand side of the first equation (2.5), we made use of the known Fourier transform of the distribution r^λ in Euclidean space (cf. Ref. 29, Chap. 4). For the Fourier transform of the Wightman function we obtain

$$\begin{aligned}\tilde\Delta_d^+(p) &= i\int d^4x\, e^{ipx} F(x_0 - i0, \underline{x})\\ &= \frac{2\pi i}{\Gamma(d-1)}\theta(p^0)(p^2)_+^{d-2},\end{aligned} \tag{2.6}$$

where $\tau_+^\lambda = \theta(\tau)\tau^\lambda$ (cf. Ref. 29). To derive (2.6) one uses the identities

$$\begin{aligned}\Delta_d^+(x) &= iF(x_0 - i0, \underline{x})\\ &= \theta(x_0)\Delta_d^c(x) - \theta(-x_0)\bar\Delta_d^c(-x),\end{aligned} \tag{2.7}$$

$$(Q + i0)^\lambda - (Q - i0)^\lambda = 2i\sin\pi\lambda\, Q_-^\lambda \quad [Q_-^\lambda = \theta(-Q)|Q|^\lambda],$$

$$\Gamma(\lambda)\Gamma(1-\lambda)\sin\pi\lambda = \pi.$$

Using (2.5′) and (2.6) we also find the 2-point functions for a Dirac field in momentum space,

$$\tilde S_{d'}^c(p) = \Gamma(\tfrac{5}{2} - d')\not p(-p^2 - i0)^{d'-5/2}, \tag{2.8}$$

$$\tilde S_{d'}^+(p) = \frac{2\pi i}{\Gamma(d' - \frac{3}{2})}\not p\theta(p^0)(p^2)^{d'-5/2}. \tag{2.9}$$

The right-hand side of Eqs. (2.5′) and (2.9) looks at first sight like modified zero-mass propagators. However, the Källén-Lehmann representation[30] for the 2-point functions[31]

$$\tilde\Delta_d^c(p) = \frac{1}{\Gamma(d-1)}\int_0^\infty d\tau \frac{\tau^{d-2}}{\tau - p^2 - i0}, \tag{2.10}$$

$$\tilde\Delta_d^+(p) = \frac{2\pi i}{\Gamma(d-1)}\theta(p^0)\int_0^\infty d\tau\, \tau^{d-2}\delta(\tau - p^2), \tag{2.11}$$

$$\tilde S_{d'}^c(p) = \frac{\not p}{\Gamma(d - \frac{3}{2})}\int_0^\infty d\tau \frac{\tau^{d'-5/2}}{\tau - p^2 - i0} \tag{2.12}$$

exhibit a continuous mass spectrum from 0 to ∞. These representations also give us the range of dimensions d and d' for which the positivity condition for the 2-point function is fulfilled. The requirement is that the spectral function in the Källén-Lehmann representation be positive. The distribution τ_+^λ is positive on the space of fast decreasing test functions $f(\tau)$ if the integral

$$\int_0^\infty d\tau\, \tau^\lambda f(\tau)$$

is convergent at the origin. This demands $\lambda > -1$, or

$$d > 1, \quad d' > \tfrac{3}{2}. \tag{2.13}$$

The canonical values $d = 1$, $d' = \frac{3}{2}$ are also compatible with the positivity requirement since[29]

$$\lim_{\lambda\to -1}\frac{1}{\Gamma(\lambda+1)}\tau_+^\lambda = \delta(\tau). \tag{2.14}$$

However, they enforce a free-field theory.[32,33]

Knowledge of the Fourier transforms allows us to write down the explicit expressions for the inverse propagators defined by

$$\int[\Delta_d^c(x - x')]^{-1}\Delta_d^c(x' - y)d^4x' = \delta(x - y). \tag{2.15}$$

We obtain

$$\begin{aligned}[\Delta_d^c(x)]^{-1} &= \int\frac{d^4p}{(2\pi)^4}\frac{(-p^2 - i0)^{2-d}}{\Gamma(2-d)}e^{-ipx}\\ &= \frac{i\Gamma(4-d)(-\frac{1}{4}x^2 + i0)^{d-4}}{(4\pi)^2\Gamma(d-2)\Gamma(2-d)}\\ &= \frac{1}{\Gamma(d-2)\Gamma(2-d)}\Delta_{4-d}^c(x)\\ &= \frac{1}{\pi}(2-d)\sin\pi d\,\Delta_{4-d}^c(x),\end{aligned} \tag{2.16}$$

$$\begin{aligned}[S_{d'}^c(x)]^{-1} &= -\int\frac{d^4p}{(2\pi)^4}\frac{\not p(-p^2 - i0)^{3/2-d'}}{\Gamma(\frac{5}{2} - d')}e^{-ipx}\\ &= \frac{-1}{\Gamma(\frac{5}{2} - d')\Gamma(d' - \frac{3}{2})}S_{4-d'}^c(x)\\ &= -\frac{\cos\pi d'}{\pi}S_{4-d'}^c(x).\end{aligned} \tag{2.17}$$

B. The 3-Point Function

Let us consider now as a model of hadronic interactions the conformal- and γ_5-invariant interaction of pseudoscalar mesons (or gluons) with spin-$\frac{1}{2}$ nucleons (or quarks). The general form of the invariant 3-point function for this case is derived in Appendix B.

We shall later need in particular the vertex function $\Gamma(x;y,\tilde{y})$ which is obtained from the time-ordered Green function by full-propagator amputation, viz.,

$$\Gamma(x;y,\tilde{y})=-i\int d^4x'd^4y'd^4\tilde{y}'[\Delta_d^c(x-x')]^{-1}[S_{d'}^c(y-y')]^{-1}\langle 0|T(\phi(x)\psi(y)\tilde{\psi}(\tilde{y}))|0\rangle[S_{d'}^c(\tilde{y}-\tilde{y}')]^{-1}. \tag{2.18a}$$

It was determined in Appendix B as

$$\Gamma(x;\,y,\tilde{y})=g\Gamma(4-\Sigma\delta_i)^{-1}S_{\delta_1}^c(y-x)\times\gamma_5 S_{\delta_2}^c(x-\tilde{y})\Delta_{\delta_3}^c(y-\tilde{y})\ , \tag{2.18b}$$

with

$$\delta_1=\delta_2=2-\tfrac{1}{2}d\ ,\quad \delta_3=2+\tfrac{1}{2}d-d'. \tag{2.18c}$$

The notation was explained in Eq. (2.4a), and we have chosen a normalization factor according to convenience.

The inclusion of internal symmetries is obvious. For instance, in the chiral SU(2)×SU(2)-invariant σ model[34] the 2-point functions will be diagonal with respect to the internal-symmetry indices, while the vertex function has to be replaced by the SU(2)×SU(2) 4-vector Γ_λ whose components are 8×8 matrices

$$\Gamma_j(x;y,\tilde{y})=g\Gamma(4-\Sigma\delta_i)^{-1}S_{\delta_1}^c(y-x)\underline{\tau}_j\gamma_5\times S_{\delta_2}^c(x-\tilde{y})\Delta_{\delta_3}^c(y-\tilde{y})\quad\text{for } j=1,2,3$$

and (2.18d)

$$\Gamma_4(x;y,\tilde{y})=g\Gamma(4-\Sigma\delta_i)^{-1}S_{\delta_1}^c(y-x)\underline{1}\times S_{\delta_2}^c(x-\tilde{y})\Delta_{\delta_3}^c(y-\tilde{y}).$$

δ_i are given by Eq. (2.18c); $\underline{\tau}_j$ are the 2×2 isospin matrices, and $\underline{1}$ is the 2×2 unit matrix.

These expressions can be graphically represented by an ("infraparticle") triangular diagram (see Fig. 2 below). A simple application of the results of Weinberg[35] and Speer[27] proves that the expressions in the right-hand side of Eqs. (2.18b) and (2.18d) are well-defined distributions (cf. Sec. III below).

Note finally that the coupling constant g is a dimensionless number if we adopt the "noncanonical" normalization convention described by Wilson.[1]

III. CONSTRUCTION OF THE n-POINT AMPLITUDE

In Lagrangian field theory one derives skeleton-graph expansions for the n-point Green function (see, e.g., Ref. 6). In the simplest case—which applies whenever this leads to finite results—they allow the general time-ordered n-point Green function ($n \geqslant 4$) to be computed in terms of the "dressed" propagators and the "dressed" 3-point vertex function. For instance, the elastic meson-scattering amplitude would then be given by an infinite series of terms, the first of which are presented in Fig. 1. In a skeleton graph G_s each line stands for a "dressed" propagator, and each vertex for a "dressed" vertex function. There are no proper subgraphs which are self-energy insertions or vertex corrections. It will, however, be necessary to consider graphs G_s where the *whole* graph has three external legs. Contributions of such graphs occur in the integral equation for the 3-point function (see Sec. IV).

We shall postulate that these skeleton-graph expansions hold for the conformal-invariant theory to be constructed here.[36] The purpose of the present chapter is to show that in this way well-defined conformal-invariant expressions are obtained for the n-point Green function ($n \geqslant 3$, see above) which are free from ultraviolet divergences. In other words, the contribution from any skeleton graph is given by an ordinary convergent integral, if we use the result of Sec. II for dressed propagators and vertex functions, with anomalous dimensions obeying the inequalities (< means less than and not equal to)

$$\tfrac{3}{2}<d'<\tfrac{5}{2},\quad 1<d<3\quad (d\neq 2). \tag{3.1}$$

A. Absence of Ultraviolet Divergences

In order to save labor we shall first investigate the problem of ultraviolet divergences in the presence of an infrared cutoff. This permits known results[27] to be used. Subsequently we shall show that the infrared cutoff may be removed for nonexceptional external momenta, and that the

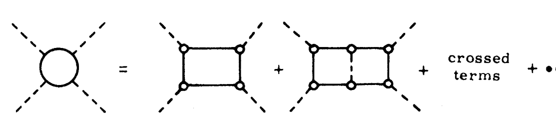

FIG. 1. Skeleton graph expansion for the mesonic 4-point function.

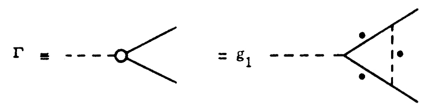

FIG. 2. Triangular representation of the dressed vertex. The coupling constant $g_1 = g\Gamma(d' + \frac{1}{2}d - 2)^{-1}$.

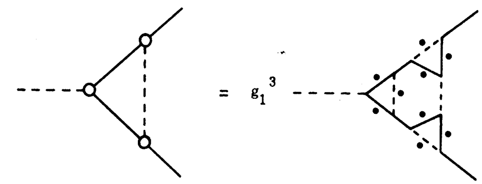

FIG. 3. The simplest skeleton graph occurring in the vertex bootstrap (Fig. 10) and associated Migdal graph.

result obtained in this way is conformal-invariant.

Let us define the infrared-cutoff generalized Feynman propagators by

$$\tilde{\Delta}^c_{h,\epsilon}(p;m) = \Gamma(\lambda_h) Z^h(p)(-p^2 + m^2 - i\epsilon)^{-\lambda_h}, \qquad m^2 > 0, \quad \epsilon > 0, \tag{3.2}$$

where $Z^h(p)$ is, in general, a homogeneous matrix-valued polynomial of p. In the end we shall let $\epsilon \to 0$, $m \to 0$. The limit $\epsilon \to 0$ will, as usual, present no problem as long as $m > 0$. The removal of the infrared cutoff $m \to 0$ will, however, need careful investigation (Sec. III B).

The Fourier transform of (3.2) is, for $\epsilon = +0$,

$$\Delta^c_{h,\epsilon}(x;m) = i2^{-\lambda_h} Z^h(i\partial) m^{2-\lambda_h}(-x^2 + i0)^{-(2-\lambda_h)/2} K_{2-\lambda_h}[m(-x^2+i0)^{1/2}] \xrightarrow[m\to 0]{} \text{const}\, Z^h(i\partial)(-x^2+i0)^{-\delta_h}$$
$$\text{with } \delta_h = 2 - \lambda_h\,. \tag{3.3}$$

We see that in the limit $m \to 0$ the $\Delta^c_{h,\epsilon}(x, m)$ go over into the expressions denoted by $\Delta^c_{\delta_h}(x)$ and $S^c_{\delta_h}(x)$, respectively, in (2.4a). Therefore both dressed propagators, defined by Eq. (2.4c) as well as the dressed vertex, Eq. (2.18) can be expressed in terms of such generalized Feynman propagators (3.3) in the limit $\epsilon \to 0$, $m \to 0$.

Let us use a graphical notation. A line stands for a generalized Feynman propagator as defined in Eq. (3.2) above, with $Z^h(p) = 1$ for a dashed line, and $Z^h(p) = \not{p}$ for a solid line. Lines will be labeled by h, and the value of λ_h will be indicated by dots as follows:

$$\begin{aligned}
&\text{(a)} \;\text{———} \quad \lambda_h = -d' + \tfrac{5}{2}, \\
&\quad\;\; \text{- - - - - - -} \quad \lambda_h = -d + 2, \\
&\text{(b)} \;\text{———} \quad \lambda_h = \tfrac{1}{2}d + \tfrac{1}{2}, \\
&\quad\;\; \text{- - - - - - -} \quad \lambda_h = d' - \tfrac{1}{2}d\,.
\end{aligned} \tag{3.4}$$

The conformal-invariant dressed propagators found in Sec. II can be represented by the undotted lines in this notation. The expression for the vertex function is shown in Fig. 2.

This observation allows us to write the contribution of any given skeleton graph G_s to some n-point function (see Fig. 1 or Fig. 11 below) in the form of a generalized Feynman integral in the sense of Speer.[27] For instance, the simplest graph occurring on the right-hand side of Migdal's "bootstrap" equation (Fig. 10 below) is presented in Fig. 3.

Thus to every skeleton graph G_s we may assign a (generalized) Feynman graph G by substituting Fig. 2 for the dressed vertex function. This graph G will henceforth be referred to as the Migdal graph associated with skeleton graph G_s. Migdal graphs can be characterized by the following requirements:

(i) They can be obtained from a primitive (skeleton ~) graph by substitution of Fig. 2.

(ii) Therefore, there are no self-energy subgraphs.

(iii) And with the exception of the whole uncut graphs on the right-hand side of Migdal's bootstrap equation (Fig. 10) there are no vertex subgraphs except those of Fig. 2.

The set of all graphs satisfying (i), (ii), and (iii) will be denoted by $S\Gamma$. The contribution of a given skeleton graph G_s to some amputated time-ordered n-point function $F_G(p)$ ($p = p_1 \cdots p_n$, $n \geq 3$) in momentum space may be written in the general form:

$$F_G(p) = \lim_{m\to 0} F_G(p;m), \tag{3.5}$$

$$F_G(p,m)(2\pi)^4\delta\Big(\sum_a p_a\Big) = \lim_{\epsilon\to +0} \int\cdots\int \Big(\prod_i d^4x_i\Big)\Big(\exp i\sum_{\text{ext}} p_i\cdot x_i\Big) \prod_{h\in\mathcal{L}(G)} \Delta^c_{h,\epsilon}(x_{i_h} - x_{f_h}; m)\,. \tag{3.6}$$

Here $\mathcal{L}(G)$ is the set of all (internal) lines h of the Migdal graph $G \in S\Gamma$. Vertices in graph G are labeled by i; $\sum_{\text{ext}}$ means a sum over external vertices (i.e., with external lines attached); i_h and f_h are initial and final vertex of line h in G. To every vertex i in G there is assigned a variable of integration $x_i \in R_4$. Traces for internal fermion loops (loops of solid lines) are understood. The parameters λ_h in definition (3.2) of generalized Feynman propagators Δ_h^c are chosen according to prescription (3.4); they depend on the dimensions d and d' of "meson" and "nucleon" field.

The convergence properties and singularities *qua* analytic functions of λ_h of generalized Feynman amplitudes (3.6) have been investigated by Speer.[27] The result is that singularities are concentrated on hypersurfaces

$$\sum_{h\in \mathfrak{L}(H)} (\lambda_h - 1) = [\tfrac{1}{2}\mu_c] - k, \quad k = 0, 1, 2, \ldots, \tag{3.7}$$

and the generalized Feynman amplitude exists as an ordinary convergent integral (after extraction of energy-momentum-conserving δ function) if

$$\sum_{h\in \mathfrak{L}(H)} (\lambda_h - 1) > [\tfrac{1}{2}\mu_c] \tag{3.8}$$

for all one-particle irreducible subgraphs H of G (exclusive of subgraphs with one vertex and no line). $\mathfrak{L}(H)$ is the set of lines of the subgraph H of G; μ_c is the "canonical superficial degree of divergence" of the graph H, viz.,

$$\mu_c(H) = 2L(H) - 4\{n(H) - 1\} + \sum_{h\in \mathfrak{L}(H)} r_h\,. \tag{3.9}$$

Here $L(H)$ and $n(H)$ are the number of lines and vertices of the Feynman graph H, respectively, and r_h is the degree of Z^h in (3.2), i.e., 0 or 1 for solid and dashed lines, respectively. Finally $[x]$ is, as usual, the largest integer $\leq x$.

Since we only deal with Yukawa-type vertices, it can be shown in the usual way[6] that μ_c may also be written in the form

$$\mu_c = 4 - B - \tfrac{3}{2}F, \tag{3.10}$$

B being the number of external boson lines and F the number of external fermion lines of H. We shall show that conditions (3.1) guarantee that inequality (3.8) is fulfilled. This will then prove the convergence of the generalized Feynman integral (3.6). Existence of the limit $\epsilon \to 0$ has been shown by Speer.[27] It must be emphasized that the result (3.7), (3.8), is valid *only* when $m > 0$. The limit $m \to 0$ will be studied in Sec. III B.

First we observe the relation

$$\sum_{h\in S(i)} (\lambda_h - 1) = 0, \tag{3.11}$$

where summation is over the three lines h incident at any vertex i of the Migdal graph G. This is verified by explicit computation from (3.4). This "conservation of dimension" is a consequence of conformal symmetry.[37]

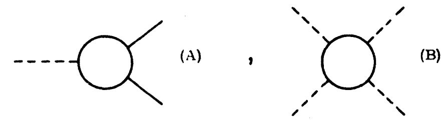

FIG. 4. Subgraphs with $\mu_c \geq 0$ and undotted external lines.

As a result of this, inequality (3.8) may be written in the equivalent form

$$-\tfrac{1}{2}\sum_{\text{ext}} (\lambda_h - 1) > [\tfrac{1}{2}\mu_c]\,, \tag{3.12}$$

where summation is over the external lines attached to subgraph H.

Thus we have managed to reduce the problem to one of enumeration of all possible configurations of external lines for graphs H which can occur as one-particle irreducible subgraphs of a Migdal graph G. They will be termed "admissible" graphs.

The two diagrams presented in Fig. 4 are the only admissible graphs with undotted external lines for which $\mu_c \geq 0$. In both cases actually $\mu_c = 0$. It follows from (3.1) and (3.4) that $\lambda_h - 1 < 0$ for undotted external lines, so that the left-hand side of (3.12) is positive. This implies the validity of (3.12) for all admissible graphs H with undotted external lines only.

Going to the general case, we observe that a subgraph H with some dotted external lines can only be obtained by cutting (among others) an internal line of a dressed vertex Γ in the skeleton graph G_s. Hence dotted external lines of H only appear in pairs. Let the set E of external lines of an admissible graph H consist of k dotted fermion lines, n dotted boson lines, κ undotted fermion lines, and b undotted boson lines, $n+k$ and $\kappa + k$ being even numbers. A necessary condition that H can occur as a one-particle irreducible subgraph of a Migdal graph G is given by the following inequality on the number of external lines:

$$\min(n, k) + \tfrac{1}{2}(k + 3\kappa) + b \geq 4. \tag{3.13}$$

To verify this we note that pairs of dotted external lines of H must be imbedded in G in one of the following ways:

Two dotted external boson lines of H can be connected in G, i.e., they form the two ends of one and the same line in G. In an analogous manner, two dotted external fermion lines of H may be connected in G. Then there is the possibility that a dotted fermion line is linked with a dotted boson line to a vertex with incident undotted fermion line not in H. Finally, a pair of dotted fermion lines may be linked at a vertex of G with an incident undotted boson line not in H. In this listing we made use of the one-particle irreducibility of H. This excludes subgraphs as shown, e.g., in Fig. 5.

In this way every subgraph H can be imbedded in a minimal larger subgraph $H' \subseteq G$ with only undotted external lines. Inequality (3.13) then follows by noting that for H' the inequality $\frac{3}{2}F + B \geq 4$ will hold, because a skeleton graph has no proper 2- or 3-point subgraphs. (Three-boson vertex parts are absent because of γ_5 invariance. Indeed every such graph would involve a fermion loop with an odd number of lines, and hence the trace of an odd number of γ matrices which is zero.)

To fill in the details, let us write $B = b + \overline{B}$, $F = \kappa + \overline{F}$, where $\overline{B}$ ($\overline{F}$) is the number of boson (fermion) lines which are external lines of H' but not of H. To make up one $\overline{F}$ line we link at a vertex one dotted fermion line and one dotted boson line. Consequently, $\overline{F} \leq \min(n, k)$. Let us next look at the balance of external dotted fermion lines of H. We need one of them to make an $\overline{F}$ line, and two to make a $\overline{B}$ line. Therefore, $\frac{1}{2}\overline{F} + \overline{B} \leq \frac{1}{2}k$. Combining these inequalities we get

$$\begin{aligned} 4 &\leq \tfrac{3}{2}F + B \\ &= \tfrac{3}{2}\kappa + b + \overline{F} + (\tfrac{1}{2}\overline{F} + \overline{B}) \\ &\leq \tfrac{3}{2}\kappa + b + \min(n, k) + \tfrac{1}{2}k , \end{aligned}$$

which proves (3.13). Putting the result in another way, we have shown that if condition (3.13) is violated, then the completed graph H' will have no more than two external lines (provided it is one-particle-irreducible and nonzero). All such graphs are excluded from our set $S\Gamma$. Some examples of imbedding graphs violating (3.13) in self-energy graphs are shown on Fig. 6.

Let then the condition (3.13) be satisfied. Taking into account (3.4) we see that the convergence condition (3.12) is verified if

$$\begin{aligned} \mu_c + \sum_{\text{ext}} (\lambda_h - 1) = (4 - b - m - \tfrac{3}{2}\kappa - \tfrac{1}{2}k) \\ + b(1 - d) + n(d' - \tfrac{5}{2}) \\ + [m - k - \tfrac{1}{2}(k - n)(1 - d)] + \kappa(\tfrac{3}{2} - d') \\ < 0 , \end{aligned} \qquad (3.14)$$

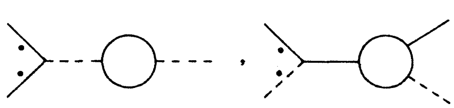

FIG. 5. One-particle reducible graphs.

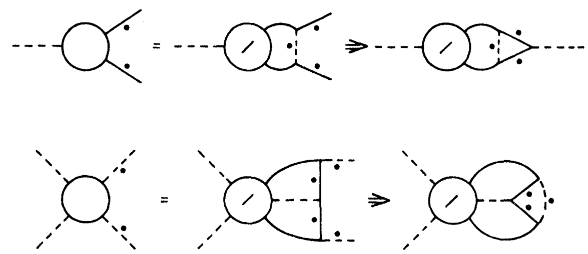

FIG. 6. Imbedding in self-energy graphs.

where

$$\begin{aligned} m &= \min(n, k) \\ &= \tfrac{1}{2}(n + k - |n - k|) . \end{aligned}$$

But inequality (3.14) is indeed fulfilled if field dimensions lie in the range (3.1), since each of the terms in parentheses in (3.14) is then negative semi-definite. In particular,

$$\begin{aligned} m - k - \tfrac{1}{2}(k - n)(1 - d) &= \tfrac{1}{2}[(n - k)(2 - d) - |n - k|] \\ &\leq \tfrac{1}{2}|n - k|(|2 - d| - 1) \\ &\leq 0 \end{aligned}$$

because of (3.1). The possibility that each term on the right-hand side of (3.14) is zero is readily disposed of by noting that this would require $b = n = \kappa = 0$ and $k = n$.

We have thus proven that the contribution of any skeleton graph to an arbitrary n-point Green function ($n \geq 4$), and also to the right-hand side of Migdal bootstrap equation (Fig. 10) below ($n = 3$) is given by an ordinary convergent generalized Feynman integral, in the presence of an infrared cutoff $m > 0$.

This result is not obvious. First, canonical perturbation theory does involve divergent skeleton diagrams with four external meson lines (an infinite $\lambda\phi^4$ counterterm is needed in that case). Also, the right-hand side in Fig. 10 below would not be finite in canonical perturbation theory, since it involves graphs with $\mu_c = 0$. Second, it should be noted that the dimensions d, d' enter into expressions (3.4) for λ_h sometimes with positive and sometimes with negative sign. [This is one reason why we had to restrict our discussion to dimensions in the range (3.1).] Therefore, there exist subgraphs H for which $\sum(\lambda_h - 1)$ is independent of dimensions d, d'. If subgraphs of this variety existed which were "superficially divergent," then no anomalous dimensions could ever help us to avoid divergences.

Generally speaking, what happens is this: Propagators in a theory with anomalous dimensions are more singular than free ones, but vertex functions are less singular (in some sense), and the better behavior of the vertex functions more than compensates the stronger singularity of the propagators. The theory is therefore more convergent in the ultraviolet region than canonical perturbation theory.

The same remark may be phrased in another way. Skeleton graph amplitudes can also be constructed from nonamputated 3-point functions and *inverse* propagators. The nonamputated 3-point functions (τ functions) are more singular here than in perturbation theory, but the improved behavior of the inverse propagators at large momentum p more than compensates this.

B. Removal of Infrared Cutoff

Let us consider the infrared cutoff amplitude $F_G(p,m)$ defined by Eq. (3.6), whose existence was shown in the last section. For dimensional reasons,

$$F_G(p_1\cdots p_n;\rho^{-1}m)=\rho^{-\nu}F_G(\rho p_1\cdots \rho p_n:m) \tag{3.15a}$$

ν depends on dimensions d and d' (cf. Ref. 1). Its actual value will not be needed in the following argument (but comes out of it).

The desired amplitude is then formally given by

$$\begin{aligned}F_G(p)&=\lim_{m\to 0}F_G(p_1\cdots p_n;m)\\&=\lim_{\rho\to\infty}\rho^{-\nu}F_G(\rho p_1\cdots \rho p_n;m).\end{aligned} \tag{3.15b}$$

We have to show that the limit exists. To demonstrate this we rely on power counting. (Similar techniques were employed by Symanzik[9] for dealing with more complicated problems.)

Let us define

$$-\Delta F_G(p;m)=m\frac{\partial}{\partial m}F_G(p,m). \tag{3.16}$$

Because of Eq. (3.15a) this is equivalent to

$$\left(\sum p_a\frac{\partial}{\partial p_a}-\nu\right)F_G(p_1\cdots p_n;m)=\Delta F_G(p_1\cdots p_n;m). \tag{3.17}$$

From definition (3.2) one has the identities

$$m\frac{\partial}{\partial m}\tilde{\Delta}_h^c(q;m)=-2m^2\tilde{\Delta}_h^c(q;m)(-q^2+m^2-i0)^{-1}. \tag{3.18}$$

Consequently, the amplitude ΔF_G may also be represented by a sum of generalized Feynman integrals,

$$\Delta F_G=-\sum_h\Delta_h F_G, \tag{3.19}$$

where summation is over all lines h of the Migdal graph G, and the operation Δ_h means that the propagator for line h in the generalized Feynman integral defining F_G should be replaced by the right-hand side of Eq. (3.18). Both amplitudes $F_G(p,m)$ and $\Delta_h F_G$ are given by convergent integrals which satisfy the hypotheses of Weinberg's power-counting theorem.[35] [Convergence was shown in Sec. III A for F_G and carries over to ΔF_G because of (3.16).]

Weinberg's power-counting theorem gives an estimate for the behavior of the amplitudes $F_G(\rho p_1\cdots\rho p_n;m)$ and $\Delta_h F_G(\rho p_1\cdots \rho p_n;m)$ as $\rho\to\infty$. In particular,

$$F_G(\rho p_1\cdots\rho p_n;m)\sim\rho^\alpha\log^\beta\rho\quad\text{as}\quad\rho\to\infty \tag{3.20a}$$

with some unspecified β, and

$$\alpha=\max_H\mathfrak{D}(H). \tag{3.20b}$$

Here, $\mathfrak{D}(H)$ is the "dimensionality" of the subgraph H, and the maximum is to be taken over all those subgraphs H of G whose set E_H of external lines includes all external lines of G. (For the precise definition of a "subgraph" as used here see Ref. 38.) $\mathfrak{D}(H)$ is to be computed by power counting: Each power $(-q^2)^{-\lambda}$ in the integrand contributes -2λ, a factor $\not{q}$ contributes 1, and each integration d^4q contributes 4. If all λ_h were equal to unity (ordinary Feynman integrals) one would have $\mathfrak{D}(H)=\mu_H$, the canonical superficial degree of divergence of H. In our more general case it follows from the arguments described in Sec. III A [compare Eq. (3.11)] that

$$\mathfrak{D}(H)=\mu_H+\sum_{\text{ext}}(\lambda_h-1). \tag{3.21}$$

[Condition (3.12) for ultraviolet convergence then reads $\mathfrak{D}(H)<0$ for all one-particle irreducible subgraphs H. This is the well-known statement of Dyson's power-counting theorem,[39] which we could have used in Sec. III A in place of Speer's results.]

Equation (3.21) shows that $\mathfrak{D}(H)$ depends only on the configuration of external lines of H. We shall now show that the maximum in (3.20b) is assumed for $H=G$, viz., $\alpha=\mathfrak{D}(G)$.

Let us denote the set of external lines of G and H by E_G and E_H, respectively; we must only consider H such that $E_G\subseteq E_H$, as explained after (3.20b). It is also obvious that $E_G=E_H$ is only possible for $G=H$. By Eqs. (3.4), (3.10), and (3.21), adding an extra line to $E_{\dots}$ increases the dimensionality $\mathfrak{D}(\cdots)$ by $-d'$, $\frac{1}{2}(d-4)$, and $-d$, $d' - \frac{1}{2}d - 2$ for undotted and dotted fermion lines, and undotted and dotted boson lines, respectively. All of these expressions are negative-definite for d, d' in the range (3.1). This proves that

$$\mathfrak{D}(H) < \mathfrak{D}(G) \quad \text{if} \quad H \neq G, \tag{3.22}$$

for subgraphs H as specified after Eq. (3.20b).

For the amplitudes $\Delta_h F_G$, Weinberg's power-counting theorem gives

$$\Delta_h F_G(\rho p_1 \cdots \rho p_n; m) \sim \rho^{\alpha'} \log^{\beta'} \rho \quad \text{as} \quad \rho \to \infty, \tag{3.23a}$$

where

$$\alpha' = \max \mathfrak{D}'(H) \tag{3.23b}$$

with $\mathfrak{D}'(H) = \mathfrak{D}(H)$ if H does not contain the line h, and $\mathfrak{D}'(H) = \mathfrak{D}(H) - 2$ otherwise. The extra -2 comes from power counting because the right-hand side of (3.18) involves -2 extra powers of q compared to $\tilde{\Delta}_h^c(q; m)$ itself. Since the whole graph G contains every line h, it follows from (3.22) that

$$\alpha' < \alpha.$$

This is true for every term in the sum (3.19); consequently, $\Delta F_G(\rho p_1 \cdots \rho p_n; m)$ *falls off faster* at large ρ than $F_G(\rho p_1 \cdots \rho p_n; m)$ itself by some power of ρ. It follows then from (3.17) that $F_G(\rho p, m)$ becomes a homogeneous function of ρ asymptotically at large ρ [viz., $\alpha = \nu$, $\beta = 0$ in (3.20a)]. Consequently, the limit (3.15b) exists for nonexceptional external momenta p.

Nonexceptional momenta are defined to be those for which Weinberg's power-counting theorem applies; this includes all Euclidean momenta $p = (p_1 \cdots p_n)$ such that no partial sum $\sum' p_j$ vanishes.[7,35]

C. Proof of Conformal Invariance; Manifestly Conformal-Invariant Generalized Feynman Rules

It remains to investigate whether the Green functions whose existence was proven in Secs. III A and III B are indeed conformal-invariant. This is especially important for the 3-point skeleton graphs that occur on the right-hand side of Migdal's bootstrap equation (4.5) below, for if conformal invariance were violated, that equation could not be solved by the conformal-invariant ansatz.

We consider n-point vertex functions in Euclidean space $\{x\}$. They are obtained from the vertex functions (i.e., full-propagator-amputated one-"particle"-irreducible Green functions) in Minkowski space by analytic continuation $x^0 \to ix^0 = x^4$ (cf. Sec. II A). Under this analytic continuation, the homogeneous Lorentz group goes over into SO(4), and the conformal group becomes $SO_0(5,1)$. We shall show that the vertex functions in Euclidean space are invariant under finite $SO_0(5,1)$ transformations. This implies that the Green functions *qua* analytic functions of the coordinates are invariant under infinitesimal conformal transformations. Thus they have the property of "weak conformal invariance" in the sense of Hortaçsu, Seiler, and Schroer,[26] which is sufficient for all our purposes.

Our proof will be based on rewriting the generalized Feynman rules for any skeleton graph in a manifestly conformal-invariant form.

The amplitude

$$F_G(p_1, \ldots, p_n)(2\pi)^4 \delta^4(\textstyle\sum p)$$

associated with some skeleton graph G_s is the Fourier transform of

$$F_G(x_1, \ldots, x_n) = i^n \int \cdots \int \left(\prod_{\text{int}} d^4 x_i\right)\left(\prod_V i\Gamma(x_{1v} x_{2v} x_{3v})\right)\left(\prod_l (1/i)\Delta_l^c(x_{i_l} - x_{f_l})\right), \tag{3.24}$$

where the product is over all dressed vertices v and propagator lines l of the skeleton graph G_s; Δ_l^c and S_d^c, of (2.24a) for boson and spinor lines, respectively. Integration is over Euclidean space. We know the integral exists if appropriately interpreted, i.e., one should first smear with a test function $\phi(x_1, \ldots, x_n)$, or one whose Fourier transformation vanishes at exceptional momenta if that were ever necessary.[10]

We use the *Euclidean version* of the projective coordinates ξ introduced in Appendix A, viz., $\xi = (\xi^1 \cdots \xi^6)$,

$$\begin{aligned} \xi^\mu &= \kappa x^\mu \quad (\mu = 1, \ldots, 4), \\ \xi^6 - \xi^5 &= \kappa, \\ \xi^2 &= g_{ab}\, \xi^a \xi^b \\ &= 0, \end{aligned} \tag{3.25}$$

where $\xi^6 > 0$ and $g_{ab} = \text{diag}(- - - -, - +)$. We will also make use of the six-dimensional Clifford algebra of 8×8 matrices β_a with defining property

$$\{\beta_a, \beta_b\} = 2 g_{ab}. \tag{3.26}$$

We shall occasionally use the explicit matrix representation given in Appendix A, with $\beta_4 = -i\beta_0$. There is also a conformal pseudoscalar

$$\beta_7 = -i\beta_1\beta_2\beta_3\beta_4\beta_5\beta_6 . \tag{3.27}$$

Matrices β_a transform as 6-vectors under $SO_0(5, 1)$ in the sense that

$$[\beta_a, s_{bc}] = i(g_{ab}\beta_c - g_{ac}\beta_b) \quad \text{for} \quad s_{bc} = \tfrac{1}{4}i[\beta_b, \beta_c] .$$

Finally we introduce a measure

$$d\mu_\eta(\xi) = 2d^6\xi\,\delta(\xi^2)\delta(\xi\cdot\eta - 1) , \tag{3.28}$$

where η is a positive "lightlike" 6-vector which will be chosen as

$$\eta^a = (0, \ldots, 0, 1, 1) . \tag{3.29}$$

As a result,

$$\int d\mu_\eta(\xi) f(\xi) = \int d^4x\, f(x, \kappa = 1) . \tag{3.30}$$

The manifestly conformal-covariant form of the n-point vertex function is a multispinor function $F_{\sigma_1 \cdots \dot{\sigma}_n}(\xi_1 \cdots \xi_n)$ whose spinor indices run through eight values each for every external fermion line. Let G be the Migdal graph obtained from the skeleton graph G_s by substituting the infraparticle representation Fig. 2 for the dressed vertex. The corresponding amplitude $F_G(\xi_1 \cdots \xi_n)$ is constructed according to the following rules [for (pseudoscalar) Yukawa theory]:

(i) For every vertex i of the Migdal graph G include a factor $\xi_i\beta$, and an extra factor β_7 for vertices with incident undotted pseudoscalar meson line.

(ii) For every line h with initial (final) vertex i_h (f_h) write down a stripped propagator

$$(4\pi)^{-2}\Gamma(\delta_h)(\tfrac{1}{2}\xi_{f_h}\xi_{i_h})^{-\delta_h} , \tag{3.31}$$

where $\delta_h = d$ $(d' + \frac{1}{2})$ for undotted meson (nucleon) lines, and $\delta_h = \frac{1}{2}d - d' + 2$ $(-\frac{1}{2}d + \frac{5}{2})$ for dotted ones. For every fermion line, include an extra factor $\frac{1}{2}$.

(iii) Integrate over variables ξ associated with internal vertices, with measure $d\mu_\eta(\xi)$ given by Eq. (3.28).

(iv) For every dressed vertex subgraph shown in Fig. 2 include a factor $g/\Gamma(5 - \sum_{h\in\Delta}\delta_h)$, Δ = set of three lines in triangle (Fig. 2).

Note that all matrix factors are here associated with vertices, whereas the stripped propagators in rule (ii) lack them. This form of the generalized Feynman rules is possible because of certain factorization properties to be discussed below. The factors $\xi\beta$ have to be arranged in the same order as vertices are arranged along fermion lines, and traces are to be taken over internal fermion loops.

The rules (i)–(iv) can also be used to write down the left-hand side of Migdal's bootstrap equation, Fig. 10, i.e., the dressed vertex itself whose Migdal graph is Fig. 2. The result is equal to the right-hand side of Eq. (3.35) below.

By these rules, the amplitude is given by an integral whose integrand is manifestly conformal-invariant, since it only depends on invariant scalar products of 6-vectors. Thus it only remains to show that the integral is independent of the standard 6-vector η occurring in the definition (3.28) of the measure $d\mu_\eta(\xi)$. This is guaranteed by the fundamental covariance lemma.

Covariance lemma. Let $f(\xi)$ defined on the forward cone $\xi^2 = 0$, $\xi_6 > 0$, and such that $I = \int d\mu_\eta(\xi) f(\xi)$ exists (for some η). If $f(\rho\xi) = \rho^{-4} f(\xi)$ for $\rho > 0$, then I is independent of the positive lightlike 6-vector η in the measure (3.28).

A proof will be given in Appendix C. Let us note that the hypothesis of the lemma is fulfilled for the present application. Because of rule (ii) one has "conservation of dimension" at every vertex of G, viz.,

$$1 - \sum_{h\in S(i)} \delta_h = -4 . \tag{3.32}$$

Summation is over the lines h incident at the vertex i. As a consequence, the integrand is homogeneous of degree -4 in every integration variable ξ_i separately.

It remains to justify the conformal-invariant generalized Feynman rules. The x-space form $F(x_1, \ldots, x_n)$ of the amplitude is recovered from $i^B F_{\sigma_1 \cdots \dot{\sigma}_n}(\xi_1 \cdots \xi_n)$ by the following procedure:

(1) Substitute expression (3.25) for external coordinates ξ_a and put $\kappa_a = 1$.

(2a) To every ingoing fermion line incident at some external vertex a apply a boost matrix $T_{\tau_a\sigma_a}(x_a)$ acting on the corresponding spinor index σ_a.

(2b) Similarly, to every outgoing fermion line apply a boost $T^{-1}{}_{\dot{\sigma}_a\dot{\tau}_a}(x_a)$.

The result of this procedure will be nonvanishing only for four values of each spinor index, because of Eq. (3.36) below. The boost matrices are

$$T(x) = \exp[-ix^\mu(s_{5\mu} + s_{6\mu})] . \tag{3.33}$$

Apart from the application of these boost matrices, i.e., a change of basis in spinor space, the amplitude given by the above rules differs from expression (3.24) essentially only by a change of notation.

To see this, let us first inspect the dressed nucleon propagator which was found in Appendix B, viz.,

$$\tfrac{1}{2}(4\pi)^{-2}\xi_1\cdot\beta\xi_2\cdot\beta\Gamma(d'+\tfrac{1}{2})(\tfrac{1}{2}\xi_1\cdot\xi_2)^{-d'-1/2} = -iT^{-1}(x_1)S^c_{d'}(x_1-x_2)\tau_+T(x_2), \quad (3.34)$$

for $\kappa_1=\kappa_2=1$. We see that the matrix factor factorizes into $\xi_1\cdot\beta$ which may be assigned to initial vertex 1, and $\xi_2\cdot\beta$ which may be assigned to the final vertex 2. What remains is a stripped propagator in the sense of rule (ii). Thus the Dirac propagator accounts for matrix factors at internal vertices of G with an incident undotted fermion line, and for the stripped propagator of undotted fermion lines. The stripped propagator for undotted meson lines is provided by the dressed meson propagator $(4\pi)^{-2}\Gamma(d)(\frac{1}{2}\xi_1\cdot\xi_2)^{-d}$; see Eq. (B6). The matrix factors associated with vertices in G with incident undotted meson line [factor $\xi_3\cdot\beta$ in Eq. (3.35) below] and the stripped propagators of dotted lines are provided by the dressed vertex $\Gamma(x_3;x_1x_2)$, by virtue of the relation established in Appendix B,

$$-i\xi_1\cdot\beta T^{-1}(x_1)\Gamma(x_3;x_1x_2)\tau_-T(x_2)\xi_2\cdot\beta\equiv\Gamma_*(\xi_3;\xi_1\xi_2)$$
$$=g(4\pi)^{-6}\Gamma(5-\Sigma\delta_i)^{-1}\xi_1\cdot\beta\xi_3\cdot\beta\xi_2\cdot\beta\Gamma(\delta_3)(\tfrac{1}{2}\xi_1\cdot\xi_2)^{-\delta_3}\tfrac{1}{2}\Gamma(\delta_2)\times(\tfrac{1}{2}\xi_1\cdot\xi_3)^{-\delta_2}\tfrac{1}{2}\Gamma(\delta_1)(\tfrac{1}{2}\xi_2\cdot\xi_3)^{-\delta_1}, \quad (3.35)$$

with

$$\delta_1=\delta_2=-\tfrac{1}{2}d+\tfrac{5}{2},\quad \delta_3=\tfrac{1}{2}d-d'+2,\quad x_i^\mu=\xi_i^\mu,\quad \kappa_i=1\,.$$

The boost matrices $T(x_i)$ associated with internal vertices cancel out when dressed vertices and propagators are multiplied together. Integrations are the same as in (3.24), by virtue of Eq. (3.30). The only factors so far unaccounted for are matrices $\xi_a\cdot\beta$ associated with external vertices with incident external fermion lines. They are included for convenience. This can be done because

$$T(x)\beta\cdot\xi T^{-1}(x)=-i\tau_+ \quad\text{for}\quad x^\mu=\xi^\mu,\quad \kappa=1\,. \quad (3.36)$$

The manifestly conformal-invariant formalism described in this subsection is very convenient for carrying out reduction of spin terms. Traces may be evaluated with the help of Eq. (3.26). Let $\hat{\xi}\equiv\xi\cdot\beta$. One obtains

$$\mathrm{Tr}\hat{\xi}_1\hat{\xi}_2=8\xi_1\cdot\xi_2,$$
$$\mathrm{Tr}\hat{\xi}_1\hat{\xi}_2\hat{\xi}_3\hat{\xi}_4=8[(\xi_1\cdot\xi_2)(\xi_3\cdot\xi_4)-(\xi_1\cdot\xi_3)(\xi_2\cdot\xi_4)+(\xi_1\cdot\xi_4)(\xi_2\cdot\xi_3)]\,, \quad (3.37)$$
$$\mathrm{Tr}\hat{\xi}_1\cdots\hat{\xi}_6\beta_7=-8i\epsilon_{abcdef}\xi_1^a\xi_2^b\cdots\xi_6^f,$$

etc.,

with ϵ_{abcdef} the completely antisymmetric tensor in six dimensions.

The somewhat clumsy factors of $\frac{1}{2}$ and 4π in rule (ii) above could be dropped by adopting different normalization conventions. The rules can also be interpreted in Minkowski space, i.e., on sector (A7) of the cone $C_{2,4}$. Then $\xi\cdot\eta$ is to be read as $\xi\cdot\eta+i0$, otherwise the notation of Appendix A and (B37) applies. Extra factors of i coming from rotation of the paths of the ξ^4 integration must, however, be supplied.

IV. INTEGRAL EQUATIONS FOR 2- AND 3-POINT FUNCTIONS

To complete the analysis, one should give a discussion of the integral equations which are normally used to determine the 2- and 3-point functions (renormalized Schwinger-Dyson equations [6,15,16]). For consistency it should be shown that they can be solved by the conformal-invariant ansatz of Sec. III. Such a discussion has been given by Migdal[17] for the 3-point function. This will now be reviewed (Sec. IV A). The integral equation for the 2-point function is considered in Sec. IV B.

A. Migdal's Bootstrap Equation

The Schwinger-Dyson integral equation for the 3-point function is given diagrammatically in Fig. 7, where the right-hand side involves the Bethe-Salpeter kernel shown in Fig. 8.

If we consider amputated vertex functions, then the first term on the right-hand side of the equation in Fig. 7 is given by Fig. 9. In a theory with anomalous dimensions, the propagators are more singular at $x=0$ than in a free theory; therefore,[1,30]

$$Z_2=Z_3=0\,. \quad (4.1)$$

Observing this, Migdal argues that the first term on the right-hand side of the equation in Fig. 7 is zero, so that one obtains the homogeneous "bootstrap" equation[40] shown in Fig. 10. Actually, upon introducing the renormalized coupling con-

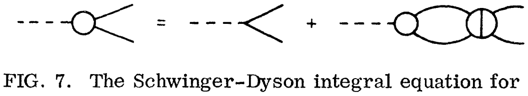

FIG. 7. The Schwinger-Dyson integral equation for the 3-point function.

stant $\bar{g} = Z_3 Z_2 Z_1^{-1} g_0$ one sees that the condition for the equation in Fig. 10 is $Z_1 = 0$.

Let us reformulate the argument without reference to the Z's. In a treatment of Lagrangian field theory based on renormalized integral equations[15] one uses not the equation in Fig. 7, but its derivative with respect to momentum flowing between two external legs. The first, constant term (bare vertex) disappears upon differentiating. The resulting renormalized Schwinger-Dyson (SD) integro-differential equation is exactly the same for all values of masses and coupling constants; it must therefore also hold for the conformal-invariant Gell-Mann–Low—(GML) limit theory (see Sec. IV C). It is evident that it will be satisfield, if the "bootstrap equation" (Fig. 10) is satisfied. On the other hand, in the more general solution (Fig. 7) the first term is to be interpreted as a constant of integration. In perturbation theory (with a cutoff, say) it would have to be determined by the usual normalization conditions, which have to be supplied in addition to the SD equation.[6,9,15] In the present (GML limit) theory, the requirement of dilatation invariance fixes this constant: It must be zero, because it would transform differently under dilatations from the second term in the equation in Fig. 7. Note that this is a meaningful statement, since we have proven that the right-hand side of the equation in Fig. 10 is not over-all divergent.

[More generally, logarithmic divergences are peculiar to canonical perturbation theory and are expected not to be present in the exact theory on the basis of the results of this paper. This statement applies perfectly well to a massive theory, since ultraviolet divergences are a problem of short-distance behavior (cf. Sec. IV C).]

Returning to Fig. 10, the important point is Migdal's observation that it is conformal-invariant, at least formally, even for anomalous dimensions in (2.1) and (2.2). He conjectures that there exists a regularization of the integrals on the right-hand side of the equation in Fig. 10, to be effected by analytic continuation in parameters d, d', which maintains the conformal invariance. If this is true then the bootstrap equation (Fig. 10) may be solved by the most general conformal-invariant ansatz. The results of the present paper show that, in fact, the right-hand side of the equation in Fig. 10 is well defined, conformal-invari-

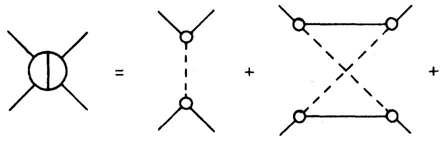

FIG. 8. The Bethe-Salpeter kernel.

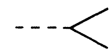 $= Z_3^{\frac{1}{2}} Z_2 g_0 \gamma_5$

FIG. 9. The bare vertex.

ant, and free from divergences for nonexceptional external momenta ($p_1, p_2, p_3 \neq 0$ in the Euclidean case). *No regularization*, by analytic continuation in d, d' or otherwise, *is necessary at all*. [This is important because such analytic continuations might destroy positivity (generalized unitarity) e.g., if one continues beyond singularity surfaces associated with subgraphs with some dotted external lines.] We remark that the general n-point function, and also the right-hand side of the equation in Fig. 10, depend only on the value of the vertex function at nonexceptional momenta, since they are given by convergent integrals and exceptional momenta form a set of measure zero.

Thus if we insert into the equation in Fig. 10 the most general conformal-invariant ansatz for the dressed vertex and propagators (the latter occur in the skeleton-graph expansion shown in Fig. 8 of the BS kernel) the ansatz will be reproduced by the right-hand side because of its conformal invariance, and one obtains an algebraic equation between the parameters in the ansatz, i.e., coupling constant(s) and field dimensions.

The possibility of imposing γ_5 invariance, and thus considering only the vertex (B42) [or (2.18)] can be demonstrated in the following way:

According to the results of Appendix B 4 the most general vertex function can be written in the form $\Gamma_1 + \Gamma_2$, where $\Gamma_1 \equiv \Gamma$ is γ_5-odd, that is

$$\gamma_5 \Gamma_1(x; y, \tilde{y}) \gamma_5 = -\Gamma_1(x; y, \tilde{y}),$$

while Γ_2 is γ_5-even,

$$\gamma_5 \Gamma_2(x; y, \tilde{y}) \gamma_5 = +\Gamma_2(x; y, \tilde{y}).$$

Therefore, in order to prove the consistency of the ansatz $g_2 = 0$, it is sufficient to verify that the right-hand side of the equation in Fig. 10 is γ_5-odd. Now, because no fermion line can end in the interior of a graph, the right-hand side of the equation in Fig. 10 will be of the form indicated in Fig. 11. Since, as noted in Sec. III A, all diagrams with an odd fermion loop vanish, there should be an even number of intermediate boson lines in Fig. 11, and hence an even number of fermion lines between them. From here (and from $\gamma_5^2 = -1$) it follows that the right-hand side of the equation in Fig. 10 is indeed γ_5-odd.

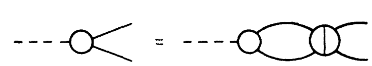

FIG. 10. The homogeneous "bootstrap" equation.

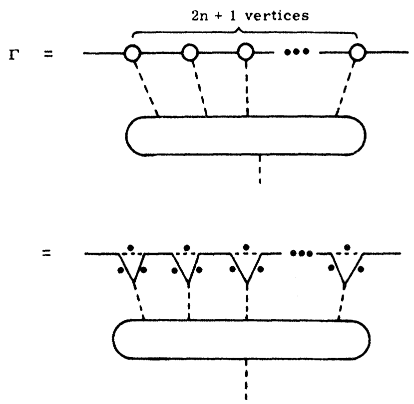

FIG. 11. Structure of graphs in Fig. 10.

Inserting (2.4) and (2.18) into the equation in Fig. 10 we therefore obtain one relation between g and d, d'. Thus at this stage the theory contains two free parameters.

B. The 2-Point Function

Finally we have to discuss the integral equations for the 2-point functions. One must show that they too can be solved by the conformal-invariant ansatz of Sec. II. As we shall see, these equations lead to two more algrbraic equations between parameters g, d, and d'. This remark was first made by Parisi and Peliti in the context of critical phenomena in statistical mechanics.[18] They use for propagator bootstrap the generalized 2-point unitarity relation in the form proposed by Polyakov.[41] This will now be discussed.

Because of the Källén-Lehmann representation, the 2-point functions are uniquely determined by their absorptive parts. (Subtractions in the Källén-Lehmann representation are not needed if $d<2$, and in general their values are uniquely determined by dilatation symmetry.)

The absorptive parts of the 2-point functions are determined in terms of the n-point functions with $n \geq 3$ by unitarity. As will be shown elsewhere,[14] use of such a unitarity relation is equivalent to imposing either the renormalized Schwinger-Dyson integral equations of Refs. 14 and 15, or the BS-equation analog to Fig. 10 for the stress tensor's vertex function.

Let us consider the meson propagator. In a finite-mass theory one can use the standard off-

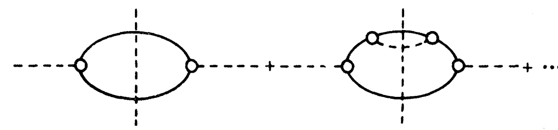

FIG. 13. Summation over cut self-energy subgraphs.

mass-shell unitarity relation given in Fig. 12, where the "blobs" stand for (amputated) $(n+1)$-point Green functions, a cut line indicates a factor

$$\Delta^+_{\text{free}}(p) = 2\pi\theta(p^0)\delta(p^2 - m^2)\ (\times\, d^4p),$$

i.e., the absorptive part of a free propagator (= phase space). Complex conjugation is understood on the right-hand side of the cut.

There exists a remarkable alternative form of Fig. 12 which involves the absorptive part $\Delta^+(p)$ of the dressed propagator. It is obtained by substituting into the equation in Fig. 12 the skeleton-graph expansion for the n-point Green function, and grouping together graphs with cuts through "self-energy subgraphs," for instance as shown in Fig. 13. That summation is similar to those considered in Symanzik's axiomatic many-particle structure analysis of Green functions.[42] The result is shown in Fig. 14. Here a cut line stands for the absorptive part $\Delta^+(p)$ of a *dressed* propagator ($\Rightarrow \Delta^+_d(p)$ in our conformal-invariant theory [cf. Eq. (2.4)]), and summation $\sum'$ is over all pairs Γ_1, Γ_2 of $(n+1)$-point skeleton graphs such that the combined graph Γ does not contain a proper "self-energy" subgraph. (By combined graph Γ we mean the result of juxtaposition of graphs Γ_1, Γ_2 with connecting lines, as in Fig. 14.)

In the partially summed form of Fig. 14, generalized unitarity continues to make sense in the zero-mass limit (Gell-Mann–Low limit, say) and is therefore appropriate for a zero-mass theory with infra-particles as we consider here.[41]

Let us now insert the conformal-invariant expressions for the $(n+1)$-point skeleton-graph amplitudes, $n \geq 2$, which were constructed in Secs. II and III. One finds that the right-hand side of the equation in Fig. 14 is finite and well defined. Indeed, constructing an absorptive part as in Fig. 14 can never lead to new ultraviolet divergences, since the integrations over the momenta of the cut lines have compact range due to energy conservation and spectrum condition. (Recall that phase space at fixed energy is compact.) As for infrared

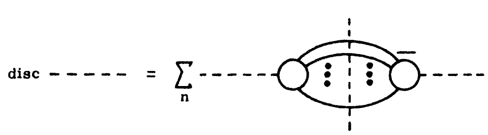

FIG. 12. Standard off-mass-shell unitarity relation.

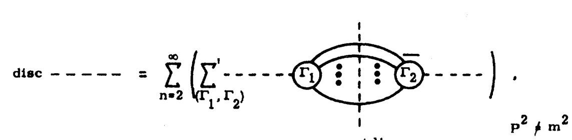

FIG. 14. Polyakov's "stream unitarity."

convergence it would, strictly speaking, need a separate argument. For in Sec. III B we used the Weinberg power counting theorem whose proof relies on the possibility of performing a Wick rotation to Euclidean space, while the unitarity relation given in Fig. 14 makes sense only in Minkowski space. However, using the above-mentioned equivalence of Fig. 14 to the BS equation (the analog of Fig. 10) of the stress tensor's vertex function, we can apply the argument of Sec. III B to the latter to imply its infrared convergence, and therefore also that of the relation in Fig. 14.

Because of dilatation symmetry of constituents, both sides of the equation in Fig. 14 have the same homogeneity in p^2, that is, they are proportional to $\theta(p^0)(p^2)_+^{d-2}$. Thus the equation in Fig. 14 is solved by the conformal-invariant ansatz, and we obtain an algebraic relation between the parameters g, d, and d' from it.

A second algebraic relation between g, d, and d' is obtained in the same fashion from the unitarity relation for the nucleon propagator.

Combining this with the relation derived in Sec. IV A we have a total of three equations for three parameters. They should in principle determine the three parameters uniquely, or at least up to a discrete set of solutions (g, d, d'). Such uniqueness may in fact be expected to hold if we want to relate our theory to the Gell-Mann–Low large momentum limit[19] of a finite-mass Yukawa theory in a nonredundant way. This will be discussed presently.

C. Discussion

To understand the last remark, let us first recall the result of Symanzik's analysis, Ref. 9, of ϕ^4 theory. The vertex functions (=amputated one-particle irreducible Green functions) $\Gamma(p_1\cdots p_{2n}; m^2, \bar{g})$ of the massive theory depend on one dimensionless coupling constant $\bar{g}$ and a mass m. Out of them one can construct in a first step vertex functions $\Gamma_{\rm as}(p_1\cdots p_{2n}; m^2, \bar{g})$ of a "pre-asymptotic" zero-mass theory. They can still be constructed by standard perturbation theory in the manner described in Appendix B of Ref. 8, and they are not dilatation-invariant. Rather, a change in renormalization mass m^2 can be compensated by a change in coupling constant $\bar{g}$ and a change of normalization. Thus there is a nontrivial dependence on one parameter left. However the (conformal-invariant[20]) Gell-Mann–Low limit theory has no dimensionless free parameter left, for its vertex functions are defined by

$$\Gamma_{\rm GML}(p_1\cdots p_{2n}; m^2, \bar{g}) = \Gamma_{\rm as}(p_1\cdots p_{2n}; m^2, \bar{g}_\infty)\,. \tag{4.2}$$

They exist if the Callan-Symanzik[43] function $\beta(\bar{g})$ has a (first) nontrivial zero at $\bar{g}=\bar{g}_\infty$ [with slope $\beta'(\bar{g}_\infty)<0$ if they are to describe a *large* momentum limit] and if $\Gamma_{\rm as}$ is continuous at $\bar{g}=\bar{g}_\infty$. $\Gamma_{\rm GML}$ has a trivial dependence on m^2 (an over-all factor of fractional power of m) because of dilatation symmetry. It follows in particular that the asymptotic large-momentum behavior of the massive theory is *independent* of its physical coupling constant $\bar{g}$, apart from some over-all factors.[44] Finite-mass pseudoscalar Yukawa theory has three dimensionless parameters (two coupling constants $g_{\pi N}$ and $g_{\pi\pi}$, and a mass ratio). The "pre-asymptotic" zero-mass theory constructed in analogy with the procedure of Appendix B of Ref. 9 still depends on one such parameter (besides dilatation freedom). However, the Gell-Mann–Low limit theory, with which we are effectively concerned here, contains no dimensionless free parameter if the "fixed point" case discussed in Wilson's renormalization-group analysis[19] is realized. The alternative discussed by Wilson, a "limit cycle," would not be consistent with conformal symmetry. Strictly speaking, though, "fixed point" and "limit cycle" do not exhaust all possibilities. It would be indicative of a more complicated behavior being consistent with conformal invariance if it should turn out that our three equations for g, d, and d' mentioned above are degenerate (i.e., possess a continuous family of solutions). We consider this a remote possibility.[45]

ACKNOWLEDGMENTS

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APPENDIX A: GENERALITIES ON THE CONFORMAL GROUP; MANIFESTLY COVARIANT FORMALISM

1. The Conformal Group and Its Lie Algebra

The conformal group of space-time can be defined as the set of local point transformations which preserve infinitesimal lightlike intervals: $dx^2=0 \to dx'^2=0$. Its connected component (containing the identity) is a 15-parameter continuous group which can be compounded from the following coordinate transformations.

(a) Poincaré transformations:

$$x'_\mu = \Lambda_\mu{}^\nu x_\nu + a_\mu,\ \Lambda \in L_+^\uparrow, \quad \mu=0,1,2,3.$$

(b) Dilatations:

$$x'_\mu = \rho x_\mu, \quad \rho>0.$$

(c) Special conformal transformations:

$$x'_\mu = (1-2c\cdot x + c^2x^2)^{-1}(x_\mu - c_\mu x^2) = (RT_cRx)_\mu, \tag{A1}$$

where

$$Rx = -x/x^2, \quad T_c x = x+c.$$

We note that, strictly speaking, Eq. (A1) does not define a coordinate transformation in Minkowski space since it is not defined on the cone

$$c^2\left(x-\frac{c}{c^2}\right)^2=0.$$

(For $c^2=0$ this singularity surface degenerates into a hyperplane.) If we insist on considering global conformal transformations we have to introduce a compactification of Minkowski space. Such a compactification can be defined by imbedding M_4 either into a set of "light" rays in six dimensions,[46]

$$\overline{M}_4 = C_{2,4}/R^1,$$

or (equivalently) into the manifold of all 2×2 unitary matrices.[47] We shall adopt instead the point of view that only infinitesimal special conformal transformations have a physical meaning and will consider (A1) only as local transformations.[24,48]

The generators of special conformal transformations, K_μ, and of dilatations, D, obey the following commutation relations among themselves and with the generators $M_{\mu\nu}$ and P_μ of the Poincaré group:

$$[D,M_{\mu\nu}]=0, \quad [D,P_\mu]=-iP_\mu, \quad [D,K_\mu]=iK_\mu,$$
$$[K_\mu,K_\nu]=0, \quad [K_\lambda,M_{\mu\nu}]=i(g_{\lambda\mu}K_\nu - g_{\lambda\nu}K_\mu), \tag{A2}$$
$$[P_\mu,K_\nu]=2i(Dg_{\mu\nu}-M_{\mu\nu}).$$

These are the commutation relations of the Lie algebra of $SO_0(4,2)$. Indeed, if we define the infinitesimal rotations J_{ab}, $a,b=0,1,2,3,5,6$ by the linear combinations

$$\begin{aligned} &J_{\mu\nu}=M_{\mu\nu},\\ &J_{65}=D,\\ &J_{5\mu}=\tfrac{1}{2}(P_\mu - K_\mu),\\ &J_{6\mu}=\tfrac{1}{2}(P_\mu + K_\mu), \end{aligned} \tag{A3}$$

then we get

$$[J_{ab},J_{cd}]=i(g_{ad}J_{bc}-g_{ac}J_{bd}+g_{bc}J_{ad}-g_{bd}J_{ac}), \tag{A4}$$

where

$$g_{aa}=(+---,-+) \text{ and } g_{ab}=0 \text{ for } a\neq b. \tag{A5}$$

The lowest-order faithful representation of this Lie algebra is four-dimensional. It is generated by the Dirac γ matrices and their traceless products satisfying the Hermiticity condition

$$\gamma_R^* A = A\gamma_R, \quad A=A^*,$$

where A defines the Dirac conjugation (for the usual choice of the basis $A=\gamma_0$).

It is possible[14] to define second-quantized generators of the conformal group in terms of the (improved[49]) energy-momentum tensor $\Theta_{\mu\nu}$ by

$$P_\mu = \int \Theta_{\mu 0}d^3x, \quad M_{\mu\nu}=\int (x_\mu\Theta_{\nu 0}-x_\nu\Theta_{\mu 0})d^3x,$$
$$D=\int x^\mu\Theta_{\mu 0}d^3x, \quad K_\mu = \int (2x_\mu x^\lambda - x^2\delta_\mu^\lambda)\Theta_{\lambda 0}d^3x.$$

2. Manifestly Covariant Formalism

The nonlinear character of special conformal transformations (A1) makes it somewhat intricate to exhibit the covariance of fields and Green functions in four-dimensional space-time. A straightforward manifestly conformal-covariant formalism[21] may be set up by imbedding Minkowski space in the "light-cone" $C_{2,4}$ in six dimensions:

$$\begin{aligned} C_{2,4}=\{\xi=(\xi_a,\ a=0,1,2,3,5,6)\,|\,\xi^2=g^{ab}\xi_a\xi_b\\ =\xi_0{}^2-\underline{\xi}^2-\xi_5{}^2+\xi_6{}^2=0\}. \end{aligned} \tag{A6}$$

Four-dimensional coordinates x_μ can be introduced as local coordinates on $C_{2,4}$ in the sector

$$\kappa=\xi_5+\xi_6>0 \tag{A7}$$

(along with $\kappa=\kappa_\xi$):

$$\xi_\mu=\kappa x_\mu, \quad \xi_5+\xi_6=\kappa, \quad \xi_5-\xi_6=\kappa x^2. \tag{A8}$$

Evidently x_μ is invariant under similarity transformations $\xi\to\rho\xi$, $\rho>0$. The conformal group acts on the manifold (A6) as the group of continuous pseudorotations $SO_0(4,2)$. Obviously its action commutes with the transformations $\xi\to\rho\xi$.

A conformal-covariant (quantized) field[50] $\chi(\xi)$ will be defined as a homogeneous vector valued distribution (operator) on the sector (A7) of $C_{2,4}$.

Homogeneity reads

$$\chi(\rho\xi)=\rho^s\chi(\xi) \quad (\rho>0,\ s \text{ real}). \tag{A9}$$

In general (for an arbitrary s), the manifestly covariant field $\chi(\xi)$ defined in the domain (A7) cannot be extended in a unique way for all ξ. [This fact is related to the difficulty in defining global conformal transformations for Green functions in Minkowski space (cf. Ref. 26).] The domain (A7) is however invariant under the 11-parameter group of Poincaré transformations and dilatations. [To see that it is translation-invariant, we note that translations are given in the ξ picture by

$$\xi'_\mu=\xi_\mu+a_\mu(\xi_5+\xi_6),$$
$$\xi'_5+\xi'_6=\xi_5+\xi_6,$$
$$\xi'_5-\xi'_6=\xi_5-\xi_6+2a\cdot\xi+a^2(\xi_5+\xi_6).]$$

It is also invariant (just as well as any open set on $C_{2,4}$) under infinitesimal special conformal transformations. Accordingly, we shall consider only this type of transformations for the field $\chi(\xi)$. The infinitesimal form of an arbitrary conformal transformation is

$$\delta\chi(\xi)=-i\xi^{ab}(L_{ab}+s_{ab})\chi(\xi) \tag{A10}$$

with

$$L_{ab}=i(\xi_a\partial_b-\xi_b\partial_a),\quad \partial_a=\partial/\partial\xi^a\,.$$

In terms of local coordinates (A8),

$$\begin{aligned}
&L_{\mu\nu}=i(x_\mu\partial_\nu-x_\nu\partial_\mu),\quad L_{5\mu}+L_{6\mu}=i\partial_\mu\,,\\
&L_{5\mu}-L_{6\mu}=i[2x_\mu\ \ \kappa\frac{\partial}{\partial\kappa}-x_\nu\partial^\nu\ \ +x^2\partial_\mu]\,,\\
&L_{65}=i\left(x_\nu\partial^\nu-\kappa\frac{\partial}{\partial\kappa}\right).
\end{aligned} \tag{A11}$$

The simplest example is given by a scalar field. In that case $s_{ab}=0$ and $\phi(\xi)$ is related to the $\phi(x)$ on Minkowski space by

$$\phi(x)=\kappa^d\phi(\xi)\quad (\kappa>0)\,. \tag{A12}$$

Thus it is homogeneous in ξ of degree $-d$.

3. The Dirac Field

We now turn to the discussion of spinor field $\chi(\xi)$ and its relation with the conventional Dirac field $\psi(x)$.

The covariant spinors over six-dimensional space are eight-dimensional (provided that space reflection is defined in the same spinor space). In analogy with the Dirac case one can start with the Clifford algebra spanned by units β satisfying

$$\{\beta_a,\beta_b\}_+=2g_{ab} \tag{A13}$$

with metric tensor given by (A5).

We shall use the following direct-product realization of the β matrices:

$$\beta_\mu=\tau_3\cdot\gamma_\mu,\quad \beta_5=i\tau_1\cdot\underline{1}\,,\quad \beta_6=\tau_2\cdot\underline{1}\,. \tag{A14}$$

Here $\underline{1}$ is the four-dimensional unit matrix; in the following it will be omitted in similar expressions. τ_i are the 2×2 Pauli matrices. The generators of the spinorial conformal transformations are given by

$$s_{ab}=\tfrac{1}{4}i[\beta_a,\beta_b]_- \tag{A15}$$

or, using (A14),

$$\begin{aligned}
&s_{\mu\nu}=\tfrac{1}{4}i[\gamma_\mu,\gamma_\nu]\,,\\
&s_{5\mu}\pm s_{6\mu}=\mp\tau_\mp\gamma_\mu\,,\\
&s_{65}=\tfrac{1}{2}i\tau_3\,,\\
&\tau_\pm=\tfrac{1}{2}(\tau_1\pm i\tau_2)\,.
\end{aligned} \tag{A16}$$

The matrices β_a form a 6-vector in the sense that

$$[\beta_a,s_{bc}]=i(g_{ab}\beta_c-g_{ac}\beta_b)\,.$$

The matrices s_{ab} generate a double-valued representation of $SO_0(4,2)$ which is a single-valued reducible representation of the pseudounitary group SU(2, 2). It becomes irreducible if we adjoin space reflections I_s to the group. There are two possible definitions of space reflections (Kastrup, Ref. 4); they differ by a reflection $\xi\to-\xi$. The physically interesting one is

$$(\xi_0,\underline{\xi},\xi_5,\xi_6)\to(\xi_0,\underline{\xi},\xi_5,\xi_6)\,.$$

Its spinor representation is

$$T(I_s)=\gamma_0\,. \tag{A17}$$

There exists a conformal pseudoscalar

$$\begin{aligned}
\beta_7&=-\beta_0\beta_1\beta_2\beta_3\beta_5\beta_6\\
&=\gamma_5\tau_3\,.
\end{aligned}$$

There are also invariant Hermitian-symmetric and skew-symmetric forms in the eight-dimensional space. The Hermitian metric tensor $\mathfrak{A}$ satisfies

$$\mathfrak{A}\beta_c=-\beta_c^*\mathfrak{A}\,,\quad \mathfrak{A}s_{bc}=s_{bc}^*\mathfrak{A}\,. \tag{A18}$$

For the choice (A14) of β matrices,

$$\mathfrak{A}=\tau_1 A \tag{A19}$$

with A defined after (A5), viz., $A=\gamma_0$ for the usual choice of basis. The symmetric metric tensor $\mathfrak{B}$ satisfies instead

$$\mathfrak{B}\beta_c=-{}^t\beta_c\mathfrak{B}\,,\quad \mathfrak{B}s_{bc}=-{}^ts_{bc}\mathfrak{B}\,,$$

where t denotes transposition. For the choice (A14) of β_c,

$$\mathfrak{B}=\tau_2 B,\quad \text{where}\quad B=\gamma_5 C\,,$$

with C the charge-conjugation matrix for an ordinary Dirac field. Finally, the skew-symmetric tensor $\mathcal{C}$ satisfies

$$\mathcal{C}\beta_c = {}^t\beta_c \mathcal{C}, \quad \mathcal{C}s_{bc} = -{}^t s_{bc}\mathcal{C} .$$

In the basis (A14), we have

$$\mathcal{C} = \tau_1 C .$$

The matrix $\mathcal{C}$ can serve to represent the action of charge conjugation on the 8-spinor field $\chi(\xi)$ which we are now going to define.

The 8-spinor field $\chi(\eta)$ is a homogeneous function of degree $-d'+\frac{1}{2}$,

$$\chi(\rho\xi) = \rho^{-d'+1/2}\chi(\xi), \quad \rho > 0, d' \text{ real.} \tag{A20}$$

The Dirac field $\psi(x)$ is recovered from $\chi(\xi)$ by

$$\psi(x) = \kappa^{d'-1/2}T(x)\chi(\xi) \tag{A21}$$

in local coordinates (A6) with $\kappa > 0$. The matrix $T(x)$ is chosen in such a way that a trivial index transformation of $\psi(x)$ under translations P_μ is ensured. This gives

$$\begin{aligned} T(x) &= e^{-i(s_{5\mu}+s_{6\mu})x^\mu} \\ &= 1 + ix^\mu \tau_- \cdot \gamma_\mu . \end{aligned} \tag{A22}$$

The field (A21) is still 8-component. In order to obtain a 4-component spinor field we shall impose a conformal-invariant subsidiary condition[51]

$$\beta \cdot \xi\chi(\xi) = 0 . \tag{A23a}$$

This reads for $\psi(x)$

$$(s_{5\mu} - s_{6\mu})\psi(x) = 0 \quad \text{or} \quad \tau_+\psi(x) = 0 . \tag{A23b}$$

To verify the equivalence of (A23a) and (A23b) one uses the identity

$$T(x)\beta \cdot \xi T(x)^{-1} = -i\kappa\tau_+ . \tag{A24}$$

Finally, the adjoint field is defined by

$$\begin{aligned} \tilde{\chi}(\xi) &= \chi^*(\xi)\mathfrak{A} \\ &= \chi^*\gamma_0\tau_1 . \end{aligned} \tag{A25}$$

The transformation law of fields $\phi(x)$ and $\psi(x)$ over Minkowski space under infinitesimal conformal transformations may be computed from the transformation law (A10) in ξ-space by inserting definitions (A12) and (A21). With the help of identity (A11) and making use of the subsidiary condition (A23b), one finds that they transform according to Eqs. (2.1) and (2.2) of Sec. II (cf. Ref. 3).

4. The Fields as Operator-Valued Distributions; Hermitian Nonintegrable Representations of the Conformal Lie Algebra

The objective of this section is to study in some more detail the mathematical nature of the representation of the conformal Lie algebra involved in the field transformation law, and in particular to give an elementary proof (using test functions) that the 2-point Wightman function, considered as a distribution, is invariant under infinitesimal special conformal transformations. The proof is valid for field dimensions $d_\psi < 2$, but the result is more general. Further discussion of the intricacies involved in conformal transformations in Minkowski space can be found in Ref. 26.

In accord with the general framework of quantum field theory (cf. Ref. 28) the fields are defined as operator-valued tempered distributions. Let $f(x) = \{f_\alpha(x)\}$, $f_\alpha(x) \in \mathcal{S}(\mathbb{R}^4)$ be a spin-tensor test function such that the unbounded operator

$$\psi(f) = \int \psi^\alpha(x) f_\alpha(x) d^4x \tag{A26}$$

be invariant under (global) Poincaré transformations and dilatation and under infinitesimal spacial conformal transformations. This means that $\psi(x)$ and $f(x)$ should transform according to dual representations of the (infinitesimal) conformal group. In particular, if the transformation laws for ψ and f under dilatation are

$$U(\rho)\psi(x)U^{-1}(\rho) = \rho^{d_\psi}\psi(\rho x), \tag{A27}$$

$$[V(\rho)f](x) = \rho^{-d_f} f(\rho^{-1}x), \tag{A28}$$

then the invariance condition

$$U(\rho)\psi(f)U^{-1}(\rho) = \psi(V(\rho)f) \tag{A29}$$

implies

$$d_f + d_\psi = 4 . \tag{A30}$$

The test functions $f(x)$ also play the role of "ψ-particle" wave functions, provided that their Fourier transforms

$$\tilde{f}(p) = \int e^{ipx} f(x) d^4x$$

vanish outside the forward light cone ($p^0 \geq |\underline{p}|$).

The physical scalar product in the space of positive-energy wave functions is defined by

$$\begin{aligned} (f, g) &= \langle 0|\psi(f)^*\psi(g)|0\rangle \\ &= \iint \bar{f}(x) F_{\psi*\psi}(x-y) g(y) d^4x d^4y \\ &= \int \bar{\tilde{f}}(p) \tilde{F}_{\psi*\psi}(p) \tilde{g}(p) \frac{d^4p}{(2\pi)^4} , \end{aligned} \tag{A31}$$

where

$$F_{\psi*\psi}(x-y) = \langle 0|\psi^*(x)\psi(y)|0\rangle \tag{A32}$$

is the 2-point Wightman function. It is assumed to satisfy the positivity condition

$$\tilde{F}_{\psi*\psi}(p) = \int F_{\psi*\psi}(x) e^{ipx} d^4x \geq 0 . \tag{A33}$$

We have seen (Sec. IIA) that this condition restricts the range of the dimension of the fields [see (2.10)–(2.13)]. On the other hand, the spectral condition implies that

$$\tilde{F}_{\psi^*\psi}(p)=0 \quad \text{for } p^0<|\underline{p}| \ . \tag{A34}$$

Clearly the 2-point Wightman function, considered as a distribution, will be invariant under infinitesimal conformal transformations if and only if the scalar product (g, f) of test functions defined by Eq. (A31) will be invariant thereunder. Because of (A34), the Wightman function $F_{\psi^*\psi}(x)$ can be represented as the boundary value of an analytic function $F(z)$ holomorphic in the backward tube[28]

$$T_-=\{z=x+iy\,|\,x\in \mathbb{R}^4,\ y_0<-|\underline{y}|\}:$$

$$F_{\psi^*\psi}(x_0-y_0,\underline{x}-\underline{y})=\lim_{\epsilon\downarrow 0}F(x_0-y_0-i\epsilon,\underline{x}-\underline{y})\ . \tag{A35}$$

For $x-y$ timelike, the limit in the right-hand side of (A35) depends on the sign of x_0-y_0 which is *not* invariant under global conformal transformations. It is only invariant under the 11-parameter subgroup of SO(4, 2) consisting of Poincaré transformations and dilatations (in accord with the result of Zeeman[52]).

We shall prove, however, that at least under certain restrictions on the dimension d_ψ, the scalar product (A31) is invariant under infinitesimal special conformal transformations, that is,

$$(g, K_\mu f)-(K_\mu g, f)=0\,, \tag{A36}$$

where K_μ is given by (2.2) with $d_f=d_g=4-d_\psi$. Away from its singularity on the light cone $(x-y)^2=0$ the Wightman function $F^{\dot\alpha\beta}_{\psi^*\psi}(x-y)$ satisfies the differential conformal invariance condition

$$K'_\mu(x)^{\dot\alpha}_{\dot\alpha'}F^{\dot\alpha'\beta}_{\psi^*\psi}(x-y)+K_\mu(y)^{\beta}_{\beta'}F^{\dot\alpha\beta'}_{\psi^*\psi}(x-y)=0\,, \tag{A37}$$

where K'_μ differs from K_μ in that $s_{\mu\nu}$ is replaced by $-\bar{s}_{\mu\nu}$. Therefore, we only need to investigate the neighborhood of the light cone. To this end we write the scalar product (A31) as the limit

$$(g,f)=\lim_{\epsilon\downarrow 0}\int\int_{|(x-y)^2|\geq\epsilon}\bar{g}(x)F_{\psi^*\psi}(x-y)f(y)d^4x\,d^4y\ . \tag{A38}$$

Taking into account that in the domain of integration (A38) F satisfies (A37), and using (A30), we can write the left-hand side of (A36) as an integral over a total divergence:

$$\begin{aligned}(g,K_\mu f)-(K_\mu g,f)&=\lim_{\epsilon\downarrow 0}\iint_{|(x-y)^2|\geq\epsilon}\left[\frac{\partial}{\partial x_\nu}R_{\mu\nu}(x)-\frac{\partial}{\partial y_\nu}R_{\mu\nu}(y)\right]\bar{g}(x)F_{\psi^*\psi}(x-y)f(y)d^4x\,d^4y\\&=\lim_{\epsilon\downarrow 0}\int_{|(x-y)^2|=\epsilon}ds^\nu\int d^4\frac{x+y}{2}[R_{\mu\nu}(x)-R_{\mu\nu}(y)]\bar{g}(x)F_{\psi^*\psi}(x-y)f(y)\,.\end{aligned} \tag{A39}$$

Here

$$R_{\mu\nu}(x)=2x_\mu x_\nu-x^2g_{\mu\nu}\,, \tag{A40}$$

and ds^ν is the surface element on the pair of hyperboloids $|(x-y)^2|=\epsilon$ which is proportional to $\epsilon^{3/2}$. Noting further that for $\epsilon\to 0$

$$R_{\mu\nu}(x)-R_{\mu\nu}(y)\sim\epsilon^{1/2},\quad F_{\psi^*\psi}(x-y)\sim\epsilon^{-d_\psi}\,,$$

we conclude that for $2-d_\psi>0$ the limit of the right-hand side of (A39) is zero.

This completes our proof of infinitesimal conformal invariance of the scalar product.

APPENDIX B: GENERAL FORM OF THE COVARIANT 2- AND 3-POINT FUNCTIONS

1. The 2-Point Functions for a (Pseudo) Scalar Field

Consider the 2-point functions

$$\Delta^Z(x-y)=i\langle 0|Z(\phi(x)\phi^*(y))|0\rangle, \tag{B1}$$

where Z stands for any of the different types of products (ordinary, time-ordered, retarded, etc.). We are looking for the general conformal-invariant expression for Δ^Z. It can be found right away by using invariance under Aut$\mathcal{P}$ alone, i.e., Poincaré and scale invariance. We shall, however, use instead the manifestly covariant technique of Appendix A which has the advantage of also applying the Green functions of a spinor field as well as to the vertex function.

Thus, we start with the auxiliary problem of finding all SO(4, 2)-invariant distributions $\mathfrak{D}(\xi,\eta)$ ($\xi,\eta\in C_{2,4}$) homogeneous of degree $-d$ with respect to each argument:

$$\begin{aligned}\mathfrak{D}(\rho\xi,\eta)&=\mathfrak{D}(\xi,\rho\eta)\\&=\rho^{-d}\mathfrak{D}(\xi,\eta)\quad\text{for }\rho>0\ .\end{aligned} \tag{B2}$$

It follows from the SO(4, 2) invariance that $\mathfrak{D}(\xi,\eta)$ is a function of the only nonvanishing scalar product

$$-(\xi-\eta)^2=2\xi\eta$$
$$=-\kappa_\xi\kappa_\eta(x-y)^2\,. \tag{B3}$$

Combining this with the homogeneity property (B2) we find just two independent distributions [53]

$$(\xi\eta\pm i0)^{-d}=(\xi\cdot\eta)_+^{-d}+e^{\mp i\pi d}(\xi\cdot\eta)_-^{-d}\,. \tag{B4}$$

In order to derive from here the expressions for Δ^Z we have to restrict our consideration to the domain (A7), i.e., $\kappa_\xi>0$, $\kappa_\eta>0$. It is only in this do-that we have an unambiguous relation between $\mathfrak{D}(\xi,\eta)$ and $\Delta(x-y)$ through (A12), viz.,

$$\Delta(x-y)=(\kappa_\xi\kappa_\eta)^d\mathfrak{D}(\xi,\eta) \tag{B5}$$

with x, ξ (y,η) related through (A8) which implies (B3). The different Δ^Z are then specified by their known analytic and support properties which follow from locality and spectrum condition [e.g., the Fourier transform of the Wightman function must vanish for momentum p outside the forward light cone, as is verified explicitly in Eq. (2.6) of Sec. II]. In particular, the time-ordered propagator Δ^c is obtained from

$$\mathfrak{D}_d^c(\xi,\eta)=C_d(\xi\cdot\eta+i0)^{-d}$$
$$=C_d(\xi\cdot\eta+i0\kappa_\xi\kappa_\eta)^{-d}\quad\text{for}\quad\kappa_\xi,\kappa_\eta>0\,. \tag{B6}$$

In order to obtain a simple expression for the propagator in momentum space [see Eq. (2.5′)], we fix the normalization factor to

$$C_d=i2^d(4\pi)^{-2}\Gamma(d)\,. \tag{B7}$$

We remark that the choice of normalization of the 2-point function is a matter of convention; it merely fixes the normalization of the fields. Since these fields do not satisfy canonical equal-time commutation relations on the one hand, and the Källén-Lehmann spectral function receives no discrete δ-function contribution on the other hand (see Sec. II), there is no "canonical" choice of normalization.

With the choice (B7) we obtain from (B5) for the time-ordered and Wightman function of a scalar field ϕ the expressions given in Eqs. (2.4c) and (2.4d) of Sec. IIA, and for the retarded function

$$\Delta_d^R(x)=i\langle 0|T(\phi(x)\phi^*(0)|0\rangle$$
$$=\Delta_d^c(x)-\Delta_d^+(-x)$$
$$=\frac{\theta(x^0)}{8\pi\Gamma(1-d)}\left(\frac{x^2}{4}\right)_+^{-d}\quad, \tag{B8}$$

while

$$\Delta_d(x)=i\langle 0|[\phi(x),\phi^*(0)]|0\rangle$$
$$=\Delta_d^+(x)-\Delta_d^+(-x)$$
$$=\frac{\epsilon(x_0)}{8\pi\Gamma(1-d)}\left(\frac{x^2}{4}\right)_+^{-d} \tag{B9}$$

[In the derivation of (B8) and (B9) we have used the identity $\Gamma(d)\Gamma(1-d)\sin\pi d=\pi$.] We note that when d approaches its canonical value $d=1$ these expressions approach the 2-point functions of a free massless scalar field with conventional normalization. To see this we note the identity

$$\lim_{d\to 1}\frac{1}{\Gamma(1-d)}(x^2)_+^{-d}=\delta(x^2)\,. \tag{B10}$$

This formula is a special case of the equation

$$\lim_{\lambda\to -n}\frac{x_+^\lambda}{\Gamma(\lambda+1)}=\delta^{(n-1)}(x)$$

proven in Ref. 29, Chap. 1, Sec. 3.5.

It is easy to verify formally that the functions (2.4c), (2.4d), (B8), (B9) are invariant, as expected, under infinitesimal conformal transformations, i.e., that they satisfy the differential equation

$$[K_\mu(x)+K_\mu(y)]\Delta^Z(x-y)=0\,,\quad Z=c,+,R,\ldots \tag{B11}$$

[cf. (A37)], where K_μ is given by (2.2). This formal verification is justified for $(x-y)^2\neq 0$. In order to prove that $\Delta^Z(x-y)$ satisfy (B11) as distributions, one uses the well-known fact that identities like, e.g.,

$$x^2\partial_\mu(-x^2+i0)^{-d}=-2dx_\mu(-x^2+i0)^{-d}\,,\quad d\neq\text{integer} \tag{B12}$$

are entirely correct in the sense of distribution theory. They can be derived by analytic continuation from the results of Appendix A 4 (cf. Ref. 29 where the technique of analytic continuation of distributions is expounded).

The 2-point functions are determined uniquely (up to normalization) from conventional axioms [28] and the imposed invariance conditions. In the case of a spinless field, considered here, they are in fact already determined by dilatation symmetry,

$$\rho^{2d}\Delta^Z(\rho x-\rho y)=\Delta^Z(x-y)\,.$$

2. The 2-Point Functions for a Dirac Field

The 2-point functions $\langle 0|Z(\psi(x)\tilde\psi(y))|0\rangle$ for a Dirac field $\psi(x)$ can also be determined from the manifestly covariant formalism of Appendix A.

An invariant 8×8 matrix $\mathcal{S}_{d'}(\xi,\eta)$ will be a function of the scalar products of the 6-vectors ξ_a, η_a, and β_a. We are looking for the general form of $\mathcal{S}_{d'}$ satisfying the homogeneity property

$$\mathcal{S}_{d'}(\rho\xi,\eta)=\mathcal{S}_{d'}(\xi,\rho\eta) =\rho^{1/2-d'}\mathcal{S}_{d'}(\xi,\eta), \quad \rho>0 \tag{B13}$$

and the subsidiary condition coming from (A23a),

$$\beta\cdot\xi\mathcal{S}_{d'}(\xi,\eta)=\mathcal{S}_{d'}(\xi,\eta)\beta\cdot\eta =0\,. \tag{B14}$$

Taking into account that, according to (A13), $(\beta\xi)^2=\xi^2=0$, we find

$$\mathcal{S}^c_{d'}(\xi,\eta)=\tfrac{1}{2}C_{d'+1/2}\xi\cdot\beta(\xi\cdot\eta+i0\kappa_\xi\kappa_\eta)^{-d'-1/2}\eta\cdot\beta\,. \tag{B15}$$

Using (A21) we can express the causal propagator of a spinor field in Minkowski space by

$$(\kappa_\xi\kappa_\eta)^{d'-1/2}T(x)\mathcal{S}^c_{d'}(\xi,\eta)T^{-1}(y)=S^c_{d'}(x-y)\tau_+\,. \tag{B16}$$

The appearance of a τ_+ factor in the direct product on the right-hand side of (B16) could have been guessed from the form (A23b) of the subsidiary condition. To find the explicit expressions for S^c and the other S functions, we use (A24) and

$$T(x)T(y)^{-1}=T(x-y) =1+i(\not x-\not y)\tau_-\,, \tag{B17}$$

$$\tau_+\tau_-\tau_+=\tau_+\,.$$

With the choice of normalization indicated in (B15), we obtain

$$\begin{aligned} S^c_{d'}(x)&=i\langle 0|T(\psi(x)\bar\psi(0))|0\rangle\\ &=-\frac{\Gamma(d'+\frac12)}{(4\pi)^2}\,\frac{\not x}{2}\left(i0-\frac{x^2}{4}\right)^{-d'-1/2}\\ &=-\frac{\Gamma(d'-\frac12)}{(4\pi)^2}\,\not\partial\left(i0-\frac{x^2}{4}\right)^{-d'+1/2}\\ &=i\not\partial\Delta^c_{d'-1/2}(x), \end{aligned} \tag{B18}$$

$$\begin{aligned} S^+_{d'}(x)&=\frac{\Gamma(d'+\frac12)}{(4\pi)^2}\,\frac{\not x}{2}\left(i0x_0-\frac{x^2}{4}\right)^{-d'-1/2}\\ &=i\not\partial\Delta^+_{d'-1/2}(x), \end{aligned} \tag{B19}$$

and analogous expressions for S^R and S [cf. (B8) and (B9)].

3. The 3-Point Functions

The manifestly covariant technique is particularly useful in the derivation of the general form of the conformal-invariant 3-point functions

$$G_Z(x;\,y,\tilde y)=\langle 0|Z(\phi(x)\psi(y)\bar\psi(\tilde y))|0\rangle,$$

where Z stands as before for the different types of products. (We will be mainly interested in the case $Z=T$ and we shall write $G_T=\tau$.)

We shall determine the general form of the manifestly covariant 8×8 matrix $\mathcal{G}(\xi;\eta,\tilde\eta)$ satisfying (a) the homogeneity requirements,

$$\begin{aligned} \mathcal{G}(\xi;\,\rho\eta,\tilde\eta)&=\mathcal{G}(\xi;\,\eta,\rho\tilde\eta) =\rho^{-d'+1/2}\mathcal{G}(\xi;\eta,\tilde\eta), \\ \mathcal{G}(\rho\xi;\,\eta,\tilde\eta)&=\rho^{-d}\mathcal{G}(\xi;\,\eta,\tilde\eta),\quad \rho>0, \end{aligned} \tag{B20}$$

(b) the subsidiary condition

$$\beta\cdot\eta\mathcal{G}(\xi;\,\eta,\tilde\eta)=\mathcal{G}(\xi;\,\eta,\tilde\eta)\beta\cdot\tilde\eta=0\,, \tag{B21}$$

(c) covariance under space reflection. We shall assume for the sake of definiteness that the field $\phi(\xi)=\kappa_\xi{}^{-d}\phi(x)$ is pseudoscalar. Then we would have

$$\gamma_0\mathcal{G}(I_S\xi;\,I_S\eta,I_S\tilde\eta)\gamma_0=-\mathcal{G}(\xi;\,\eta,\tilde\eta)\,, \tag{B22}$$

where

$$I_S\xi=(\xi_0,\,-\underline{\xi};\,\xi_5,\,\xi_6)\,.$$

For ξ, η, and $\tilde\eta$ in the domain (A7) $\mathcal{G}(\xi;\,\eta,\tilde\eta)$ can be defined as

$$\mathcal{G}(\xi;\eta,\tilde\eta)=\langle 0|Z(\phi(\xi)\chi(\eta)\bar\chi(\tilde\eta))|0\rangle. \tag{B23}$$

In the domain in which all three scalar products $\xi\cdot\eta$, $\xi\cdot\tilde\eta$, and $\eta\cdot\tilde\eta$ are positive (that is, when the intervals $x-y$, $x-\tilde y$, and $y-\tilde y$ in Minkowski space are spacelike) the Wightman function and the τ function coincide and $\mathcal{G}$ with the above properties will have the general form

$$\mathcal{G}(\xi;\,\eta,\tilde\eta)=g_1\eta\cdot\beta\xi\cdot\beta\beta_7\tilde\eta\cdot\beta(2\eta\cdot\tilde\eta)^{-d'+d/2}[(2\xi\cdot\eta)(2\xi\cdot\tilde\eta)]^{-d/2-1/2}+g_2\eta\cdot\beta\beta_7\tilde\eta\cdot\beta(2\eta\cdot\tilde\eta)^{d/2-d'-1/2}[(2\xi\cdot\eta)(2\xi\cdot\tilde\eta)]^{-d/2}. \tag{B24}$$

The x-space expression G is related to $\mathcal{G}$ by

$$\kappa^d_\xi(\kappa_\eta\kappa_{\tilde\eta})^{d'-1/2}T(y)\mathcal{G}(\xi;\,\eta,\tilde\eta)T(\tilde y)^{-1}=G(x;\,y,\tilde y)\tau_+\,. \tag{B25}$$

[We assume here all κ positive in accord with (A7).] Inserting (B24) in (B25) gives

$$\begin{aligned} G(x;\,y,\tilde y)&=g_1G_1(x;\,y,\tilde y)+g_2G_2(x;\,y,\tilde y),\\ G_1(x;\,y,\tilde y)&=i(\not y-\not x)\gamma_5(\not x-\tilde{\not y})[-(y-\tilde y)^2]^{d/2-d'}\{[-(x-y)^2][-(x-\tilde y)^2]\}^{-d/2-1/2},\\ G_2(x;\,y,\tilde y)&=-i(y-\tilde y)\gamma_5[-(y-\tilde y)^2]^{d/2-d'-1/2}\{[-(x-y)^2][-(x-\tilde y)^2]\}^{-d/2}\,. \end{aligned} \tag{B26}$$

Here we have again used (A24) and (B17) as well as the identities

$$T(x)\beta_7 T(x)^{-1} = \beta_7 = \tau_3\gamma_5,$$

$$\tau_-\tau_3 = -\tau_3\tau_- = \tau_-.$$

In order to obtain the 3-point Wightman function

$$w(x;\, y,\tilde{y}) = \langle 0|\phi(x)\psi(y)\tilde{\psi}(\tilde{y})|0\rangle, \tag{B27}$$

we note that it is characterized by its property of being the boundary value of an analytic function regular in the backward tube

$$-\mathrm{Im}(x-y)\in V_+, \quad -\mathrm{Im}(y-\tilde{y})\in V_+,$$

V_+ being the forward light cone. This implies that $w(x;\, y,\tilde{y})$ can be obtained from (B26) through analytic continuation from the spacelike region. The resulting expression will be related to G by the substitution

$$-(x-y)^2 \to i(x_0-y_0)0-(x-y)^2 \tag{B28}$$

for each of the squared intervals on the right-hand side of (B26).

The time-ordered function

$$\tau(x;\, y,\tilde{y}) = \langle 0|T(\phi(x)\psi(y)\tilde{\psi}(\tilde{y}))|0\rangle \tag{B29}$$

can be constructed from the Wightman function in standard fashion. As a result, each bracket of the type $[-(x-y)^2]$ in (B26) has to be replaced by

$$i0-(x-y)^2. \tag{B30}$$

We can also define the time-ordered function on the part (A7) of the cone $C_{2,4}$ using the substitution

$$2\xi\cdot\eta \to 2\xi\cdot\eta + i0 = 2\xi\cdot\eta + i0\kappa_\xi\kappa_\eta \tag{B31}$$

in (B24).

Finally, we mention that if the theory is assumed to be γ_5-invariant,[54] that is, if the Green functions are invariant under the discrete transformation (2.3), then g_2 must vanish and we are left with only one conformal-invariant 3-point function. Indeed, the γ_5 invariance condition

$$-\gamma_5 G(x;\, y,\tilde{y})\gamma_5 = G(x;y,\tilde{y}) \tag{B32}$$

implies that

$$G(x;\, y,\tilde{y}) = g_1 G_1(x;\, y,\tilde{y}). \tag{B33}$$

4. The Vertex Function

In perturbative computations it is convenient to work with "amputated" vertex functions (which do not contain propagators corresponding to the external lines). Such a vertex function Γ is related to the time-ordered Green function τ by

$$\tau(x;\, y,\tilde{y}) = i\int\!\!\int\!\!\int d^4x'\,d^4y'\,d^4\tilde{y}'\,\Delta_d^c(x-x')S_{d'}^c(y-y')\Gamma(x';\, y',\tilde{y}')S_{d'}^c(\tilde{y}'-\tilde{y}), \tag{B34}$$

or in a manifestly covariant form

$$\mathcal{G}^c(\xi;\eta,\tilde{\eta}) = i\int\cdots\int d\mu_\zeta(\xi')\,d\mu_\zeta(\eta')\,d\mu_\zeta(\tilde{\eta}')\mathfrak{D}^c(\xi,\xi')\mathcal{S}^c(\eta,\eta')\,\mathcal{T}(\xi';\,\eta',\tilde{\eta})\mathcal{S}^c(\tilde{\eta}',\tilde{\eta}), \tag{B35}$$

where

$$d\mu_\zeta(\xi) = 2\delta(\xi^2)\delta(\xi\cdot\zeta-1)d^6\xi \tag{B36}$$

and ζ is any fixed vector on $C_{2,4}$ with $\kappa_\zeta > 0$. In the sequel we shall choose $\zeta = (0,\underline{0};\, 1, 1)$ in order to have $\xi\cdot\zeta = \kappa_\xi$ [cf. Eq. (3.29)]. The covariant vertex function cannot be determined uniquely from (B35) since the fermion propagators $\mathcal{S}^c$ contain nilpotent factors of type $\beta\cdot\eta$. For this reason it is appropriate to introduce the generalized vertex

$$i\Gamma_*(\xi;\,\eta,\tilde{\eta}) = \beta\cdot\eta\,\mathcal{T}(\xi;\,\eta,\tilde{\eta})\beta\cdot\tilde{\eta}. \tag{B37}$$

According to (B35) Γ_* is a manifestly covariant homogeneous distribution

$$\begin{aligned}\Gamma_*(\rho\xi;\,\eta,\tilde{\eta}) &= \rho^{d-4}\Gamma_*(\xi;\,\eta,\tilde{\eta}),\\ \Gamma_*(\xi;\,\rho\eta,\tilde{\eta}) &= \Gamma_*(\xi;\,\eta,\rho\tilde{\eta})\\ &= \rho^{d'+1/2-4}\Gamma_*(\xi;\,\eta,\tilde{\eta}),\quad \rho>0,\end{aligned} \tag{B38}$$

satisfying the subsidiary condition [cf. (B37)]

$$\beta\cdot\eta\Gamma_*(\xi;\,\eta,\tilde{\eta}) = \Gamma_*(\xi;\,\eta,\tilde{\eta})\beta\cdot\tilde{\eta} = 0. \tag{B39}$$

It is determined from these properties uniquely (up to two constants):

$$\Gamma_*(\xi;\,\eta,\tilde{\eta}) = g\beta\cdot\eta[a_1\mathfrak{D}^c_{d/2-d'+2}(\eta,\tilde{\eta})\mathfrak{D}^c_{5/2-d/2}(\eta,\xi)\beta\cdot\xi\mathfrak{D}^c_{5/2-d/2}(\xi,\tilde{\eta}) + a_2\mathfrak{D}^c_{d/2-d'+5/2}(\eta,\tilde{\eta})\mathfrak{D}^c_{2-d/2}(\eta,\xi)\mathfrak{D}^c_{2-d/2}(\xi,\tilde{\eta})]\beta_7\beta\cdot\tilde{\eta}\,, \tag{B40}$$

where $\mathfrak{D}^c_\delta$ is given by (B6), and ga_1 and ga_2 are proportional to g_1 and g_2 of (B24). The vertex function $\Gamma(x;\,y,\tilde{y})$ in Minkowski space can be obtained from Γ_* by (B25), with Γ_* substituted for $\mathcal{G}$ and Γ for G. An equivalent formula is given in (3.5) of Sec. III C [cf. Eq. (A24)]. The reader is advised to verify for himself as an exercise that the manifestly covariant amputation formula (B35) is equivalent to (B34) if Γ_* and Γ are so related. In this, Eq. (B37) is to be used, and $\mathcal{G}$ and G are of course related by the original Eq. (B25). For the γ_5-odd part of Γ we find, with a suitable choice of normalization (which, together with the normalization of the propagators, defines g),

$$\Gamma(x;\,y,\tilde{y}) = g\Gamma(d'+\tfrac{1}{2}d-2)^{-1}S^c_{2-d/2}(y-x)\gamma_5 S^c_{2-d/2}(x-\tilde{y})\Delta^c_{2+d/2-d'}(y-\tilde{y})\,. \tag{B41}$$

APPENDIX C: PROOF OF THE COVARIANCE LEMMA

The group $SO_0(5,1)$ acts transitively on the forward cone (3.25). Thus every positive lightlike vector η' may be obtained from the standard vector $\eta^a = (0,\ldots,0,1,1)$ as

$$\eta' = \Lambda\eta \tag{C1}$$

for suitable $\Lambda \in SO_0(5,1)$. Matrices Λ which do not leave η invariant may be factorized as

$$\Lambda = E_1 V B E_2\,, \tag{C2}$$

where V is a rotation by π in the 1–5 plane, $E_{1,2}$ are in the little group of η, and B is a boost in the 5–6 plane. That is,

$$E_{1,2}\eta = \eta\,;\quad B\eta = \rho\eta\,,\quad \rho > 0.$$

An analogous decomposition was proven for $SO(3,1)$ in Ref. 55. We can use that result to obtain (C2) as a corollary by noting that every transformation may be composed from an $SO(4)$ transformation acting on components $(\xi^1\cdots\xi^4)$ and therefore contained in the little group of η, and an $SO(3,1)$ transformation acting on components $(\xi^1\xi^2\xi^5,\xi^6)$

The main piece of work is to show that the integral I is invariant under $\eta\to\eta' = V\eta = (0,\ldots,0,-1,1)$. Let $d\xi \equiv 2d^6\xi\delta(\xi^2)$. We have $\eta'\xi = \xi^6 + \xi^5 = \kappa|x|^2$ in coordinates (3.25). Thus,

$$I' = \int d\mu_{V_\eta}(\xi)f(\xi)$$

$$= \int d\xi\delta(\kappa|x|^2 - 1)f(\xi)\,.$$

We introduce new variables $\xi' = \xi|x|^2$, so that $d\xi = d\xi'|x|^{-8}$. Hence

$$I' = \int d\xi'|x|^{-8}\delta(\kappa' - 1)f(|x|^{-2}\xi')$$

$$= \int d\xi\delta(\kappa - 1)f(\xi)$$

by homogeneity. The last integral is equal to I since $\eta\xi = \kappa$.

Let us now return to general transformations Λ of the form (C2). We have

$$I' = \int d\mu_{\Lambda_\eta}(\xi)f(\xi)$$

$$= \int d\xi\delta(V_\eta\rho E_1^{-1}\xi - 1)f(\xi)$$

$$= \int d\xi\delta(V_\eta\xi - 1)f(E_1\xi)$$

by introducing new variables and using homogeneity of f. We now apply our previous result which guarantees invariance under $\eta \to V_\eta$. Hence,

$$I' = \int d\xi\delta(\eta\xi - 1)f(E_1\xi)$$

$$= \int d\xi\delta(\eta E_1^{-1}\xi - 1)f(\xi)$$

$$= \int d\xi\delta(\eta\xi - 1)f(\xi)\,,$$

since $E_1\eta = \eta$. QED.

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B. Klaiber, in *Lectures in Theoretical Physics* (Gordon and Breach, New York, 1968), Vol. X A, pp. 141–176. Operator product expansions in the Thirring model were recently considered in J. H. Lowenstein and B. Schroer, Phys. Rev. D 3, 1981 (1971); G. F. Dell'Antonio, Y. Frishman, and D. Zwanziger, *ibid.* 6, 988 (1972); S. Ferrara, R. Gatto, and A. F. Grillo, Nuovo Cimento 12A, 959 (1972).

[3]G. Mack and Abdus Salam, Ann. Phys. (N.Y.) 53, 174 (1969), and references therein.

[4]H. A. Kastrup, Nucl. Phys. 58, 561 (1964); Phys. Rev. 150, 1189 (1966), and references therein; J. E. Wess, Nuovo Cimento 18, 1086 (1960); D. Gross and J. E. Wess, Phys. Rev. D 2, 753 (1970); D. G. Boulware, L. S. Brown, and R. D. Peccei, *ibid.* 2, 293 (1970); C. J. Isham, Abdus Salam, and J. Strathdee, Phys. Letters 31B, 300 (1970); P. Carruthers, Phys. Reports 1C, 1 (1971); V. A. Matveev, R. M. Muradyan, and A. N. Tavkhelidze, Particles Nucl. 2, 7 (1971).

[5]A. M. Polyakov, ZhETF Pis. Red. 12, 538 (1970) [JETP Lett. 12, 381 (1970)]; A. A. Migdal, Phys. Letters 37B, 98 (1971). For 3-point functions involving currents and tensor fields see: E. Schreier, Phys. Rev. D 3, 980 (1971); S. Ferrara, R. Gatto, and A. F. Grillo, in Springer Tracts in Modern Physics (Springer, Heidelberg, 1973), Vol. 67.

[6]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965); K. Symanzik, DESY Report No. DESY T-71/1 (University of Islamabad Lectures, 1968, unpublished).

[7]Euclidean external momenta are nonexceptional except when a partial sum of them vanishes (Sec. III B). Lightlike Minkowski momenta are usually exceptional. For further discussion see Ref. 9.

[8]M. Baker and K. Johnson, Phys. Rev. 183, 1292 (1969); Phys. Rev. D 3, 2516 (1971); 3, 2541 (1971); S. L. Adler and W. A. Bardeen, Phys. Rev. D 4, 3045 (1971); 6, 734(E) (1972); S. L. Adler, *ibid.* 5, 3012 (1972).

[9]K. Symanzik, Commun. Math. Phys. 23, 49 (1971).

[10]For the applications envisaged here, one needs the Green functions of the zero-mass theory as functions at nonexceptional momenta, not as distributions over all of momentum space. This matches with our results. However, our Green functions are in fact also well-defined distributions; this follows from the result of Ref. 11.

[11]K. Symanzik, Lett. Nuovo Cimento 3, 734 (1972).

[12]G. Mack, Phys. Rev. Letters 25, 400 (1970); Nucl. Phys. B35, 592 (1971), especially Sec. III. For background see J. D. Bjorken, Phys. Rev. 179, 1547 (1969). Conformal-invariant operator product expansions have been extensively studied by S. Ferrara, R. Gatto, A. F. Grillo, and G. Parisi, Lett. Nuovo Cimento 2, 1363 (1971); 4, 115 (1972); Nucl. Phys. B49, 77 (1972). See also S. Ferrara, R. Gatto, and A. F. Grillo, Nucl. Phys. B34, 349 (1971); Phys. Rev. D 5, 3102 (1972); and Ref. 5.

[13]R. Gatto, Riv. Nuovo Cimento 1, 514 (1969).

[14]G. Mack and K. Symanzik, Commun. Math. Phys. 27, 247 (1972).

[15]K. Symanzik, in *Lectures in High Energy Physics*, edited by B. Jakšić (Gordon and Breach, New York, 1965).

[16]Another formulation of the renormalized Schwinger-Dyson equations is given in: E. S. Fradkin, Zh. Eksp. Teor. Fiz. 29, 121 (1955) [Sov. Phys. JETP 2, 148 (1956)]; I. Białynicki-Birula, Bull. Acad. Polon. Sci. 13, 499 (1965).

[17]A. A. Migdal, Phys. Letters 37B, 386 (1971).

[18]G. Parisi and G. Peliti, Nuovo Cimento Lett. 2, 627 (1971).

[19]M. Gell-Mann and F. E. Low, Phys. Rev. 95, 1300 (1954); K. Wilson, Phys. Rev. D 3, 1818 (1971).

[20]B. Schroer, Lett. Nuovo Cimento 2, 867 (1971).

[21]P. A. M. Dirac, Ann. Math. 37, 429 (1936); Mack and Salam, Ref. 3, and references contained therein to the work of Murai, Hepner, and Kastrup.

[22]Infrared divergences of the "catastrophic kind" would prevent the existence of Green functions at any external momenta. They could arise from divergences of integrals at large x. Absence of this type of divergences in perturbation theory is the nontrivial starting point of the renormalization group approach. See Gell-Mann and Low, Ref. 19; N. N. Bogolubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience, New York, 1959), Chap. VIII. On the other hand, there will of course be infrared singularities at some exceptional momenta (as in QED—see Ref. 9). A nontrivial conformal-invariant QFT is necessarily an infraparticle theory (see Ref. 23). There are no asymptotic free-particle states and hence no on-shell S matrix (see Ref. 1).

[23]The concept of infraparticle is due to B. Schroer, Fortschr. Physik 11, 1 (1963).

[24]When speaking about conformal invariance we shall have in mind invariance under (global) Poincaré transformations and dilatations, and under infinitesimal special conformal transformations, or the (equivalent) concept of "weak conformal invariance" of Hortaçsu, Seiler, and Schroer (see Ref. 26). The reason is that, first, global conformal transformations are not defined on the whole Minkowski space (see Appendix A) and, second, they do not preserve either time ordering or even the sign of $(x-y)^2$. The relevance of infinitesimal conformal transformations was stressed by M. Flato and D. Sternheimer, Compt. Rend. Acad. Sc. Paris 263, A935 (1963). Global conformal transformations may be considered after analytic continuation to the Euclidean region (x^0 imaginary), this will be used in Sec. III C. The conformal group then becomes SO(5,1). K. Johnson (private communication); see also footnote 48.

[25]G. Mack and I. T. Todorov, J. Math. Phys. 10, 2078 (1969).

[26]M. Hortaçsu, R. Seiler, and B. Schroer, Phys. Rev. D 5, 2519 (1972).

[27]E. Speer, *Generalized Feynman Amplitudes, Annals of Math Studies No. 62* (Princeton Univ. Press, Princeton, New Jersey, 1969). See also, E. Speer, J. Math. Phys. 9, 1404 (1968); Commun. Math. Phys. 23, 23 (1971); E. Speer and J. Westwater, Ann. Inst. Henri Poincaré 14A, 1 (1971). The techniques of analytic regularization were also used earlier, see, e.g., T. Gustafson, Arkiv. Mat. Astron. Fysik 34A, No. 2 (1947); C. G. Bollini, J. J. Giambiagi, and A. Gonzalez Dominguez, Nuovo Cimento 31, 550 (1964).

[28]R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics and All That* (Benjamin, New York, 1964).

[29]I. M. Gel'fand and G. E. Shilov, *Generalized Functions* (Academic, New York, 1964), Vol. 1.

[30]G. Källén, Helv. Phys. Acta 25, 417 (1952); H. Lehmann, Nuovo Cimento 11, 342 (1954).

[31]If $d>2$, the Källén-Lehmann representation (2.10) needs a subtraction. It is most conveniently made at $p=0$; the subtraction constant is then zero by dilatation symmetry.

[32]K. Pohlmeyer, Commun. Math. Phys. 12, 204 (1969). Pohlmeyer's theorem extends earlier results of Federbush and Johnson and Jost and Schroer (Ref. 33). The possibility that $d=2$ cannot occur either for an interacting fundamental scalar field, because a conformal-invariant time-ordered propagator with $d=2$ would have a vanishing absorptive part, which is incompatible with the propagator bootstrap condition (4.6) whose right-hand side is positive definite. This remark applied only to fundamental fields whose propagators enter in the skeleton graph expansions. In all other cases, invariance of only the 2-point Wightman function would be acceptable. This allows for canonical dimensionality [cf. Ref. 14, Eq. (2.14) and subsequent discussion].

[33]P. G. Federbush and K. A. Johnson, Phys. Rev. 120, 1926 (1960); R. Jost and B. Schroer, see R. Jost, in *Lectures on Field Theory and the Many-Body Problem,* edited by E. R. Caianello (Academic, New York, 1961).

[34]J. Schwinger, Ann. Phys. (N.Y.) 2, 407 (1957); M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960).

[35]S. Weinberg, Phys. Rev. 118, 838 (1960). Weinberg's power-counting theorem is valid for arbitrary, not necessarily integer, asymptotic power behavior of integrands.

[36]Skeleton graph expansions which hold for a massive theory constructed by canonical perturbation theory must also hold for its conformal-invariant Gell-Mann–Low limit theory (see Sec. IV C). The possibility of "dissolving" in addition the mesonic 4-point function is at least self-consistent (Sec. III) and, of course, compatible with the axioms (cf. Ref. 14).

[37]Migdal, Ref. 5.

[38]Weinberg, Ref. 35, bottom of p. 847.

[39]F. J. Dyson, Phys. Rev. 75, 1736 (1949); Abdus Salam, *ibid.* 82, 217 (1951); 84, 426 (1951).

[40]Exploitation of this equation has often been tried, but without using conformal symmetry. See, e.g., S. F. Edwards, Phys. Rev. 90, 284 (1953).

[41]A. M. Polyakov, Zh. Eksp. Teor. Fiz. 59, 542 (1970) [Sov. Phys. JETP 32, 296 (1971)]; M. Veltman, Physica 29, 186 (1963), especially Secs. 3, 4.

[42]K. Symanzik, J. Math. Phys. 1, 249 (1960).

[43]C. G. Callan, Phys. Rev. D 2, 1451 (1970); K. Symanzik, Commun. Math. Phys. 18, 227 (1970). Theories with several coupling constants were considered by K. Symanzik (unpublished).

[44]In particular, therefore, predictions for the physical coupling constant $\bar{g}$ (or $g_{\pi N}$, $g_{\pi\pi}$) are possible in principle on the basis of extra assumptions only: The GML large-momentum asymptote is approached fastest (Ref. 9) if $\bar{g}=\bar{g}_\infty$. Thus, assuming some form of "precocious scaling" would fix physical coupling constant $\bar{g}$ to $\bar{g}=\bar{g}_\infty$. [The term "precocious asymptopia" was coined by R. A. Brandt and G. Preparata, in *Broken Scale Invariance and the Light Cone,* 1971 Coral Gables Conference on Fundamental Interactions at High Energy, edited by M. Dal Cin, G. J. Iverson, and A. Perlmutter (Gordon and Breach, New York, 1971), Vol. 2, pp. 43–60.] The proposal made by one of us that dilatation symmetry-breaking effects should be purely $(3,3^*)\oplus(3^*,3)$ under chiral $SU(3)\times SU(3)$ (or anything without singlet, to lowest order) would fix the physical coupling constants to the same values, because $\Delta_1\Gamma$ in the notation of Ref. 14 (Appendix C) will contain a singlet piece, but is zero for $\bar{g}=\bar{g}_\infty$ because $c(\bar{g})\propto\bar{g}-\bar{g}_\infty$ in Eq. (C6) of Ref. 14. [See G. Mack, Nucl. Phys. B5, 499 (1968).] $\bar{g}_\infty$ is related to our parameter g (but with a factor that can be evaluated only in the framework of the massive theory). The physical coupling constants are defined by the value of the vertices at certain finite momenta. Hence they will be strongly affected by symmetry breaking due to rest masses, etc.

[45]It is an entirely different question whether or not there could exist continuous families of conformal-invariant QFT's which are not GML limits of perturbation-theoretically renormalizable theories, and which, in particular, do not admit of any skeleton graph expansions. While validity of such expansions is sufficient (modulo convergence problems of the series involved) to ensure the axioms, they are by no means necessary: They are only iterative solutions of integral equations which are specializations of axiomatically valid ones (Ref. 15). (The canonical counterexample is Bogoliubov-Parasiuk-Hepp renormalized perturbative solution of a "nonrenormalizable" theory.)

[46]See, e.g., W. Kopczynski and L. S. Woronowicz, Rep. Math. Phys. 2, 35 (1971).

[47]A. Uhlmann, Acta Phys. Polonica 24, 295 (1963).

[48]Global conformal transformations could also be considered if one works with fields φ that live on a manifold M which is an ∞-sheeted covering of the cone $C_{2,4}$ (i.e., M/R_1 is the universal covering of $\overline{M}_4$). The values $\varphi(\xi)$ for arguments $\xi\in M$ over the same point of $C_{2,4}$ would then differ by a phase factor. According to Segal, the (noncompact) manifold M/R_1 admits a causal orientation. [I. Segal, Bull. Am. Math. Soc. 77, 6 (1971).]

[49]C. G. Callan, S. Coleman, and R. Jackiw, Ann. Phys. (N.Y.) 59, 42 (1970). The symmetric traceless tensor $\Theta_{\mu\nu}$ which is coupled to the gravitational field $g_{\mu\nu}$ was considered earlier by N. A. Chernikov and E. A. Tagirov, Ann. Inst. H. Poincaré 9, 109 (1968).

[50]In this and the following appendixes we could have avoided the use of the auxiliary fields $\chi(\xi)$ and introduced the manifestly covariant formalism directly for Green's functions.

[51]There is an alternative possibility (discussed in Ref. 3): One may identify physical fields as equivalence classes of 8-spinor fields, two fields $\chi(\xi)$ being called equivalent if they differ by a new kind of additive gauge transformation $\chi(\xi)\rightarrow\chi(\xi)+\xi\cdot\beta\Lambda(\xi)$, where $\Lambda(\xi)$ is also an (arbitrary homogeneous) 8-spinor. In this scheme $\chi(\xi)$ has to be homogeneous in ξ of degree $-d'-\frac{1}{2}$.

[52]E. C. Zeeman, J. Math. Phys. 5, 490 (1964); L. Michel, in *Applications of Mathematics to Problems in Theoretical Physics,* edited by F. Lurçat (Gordon and Breach, New York, 1967), pp. 409–452.

[53]Cf. Chap. 3 of Ref. 29, where it is shown that for noninteger d the distributions (B4) are well defined and exhaust the list of homogeneous distributions of $\xi\cdot\eta$.

[54]Cf. Adbus Salam, Nuovo Cimento 5, 298 (1957).

[55]G. J. Iverson and G. Mack, J. Math. Phys. 11, 1581 (1970).