PHYSICAL REVIEW D VOLUME 8, NUMBER 6 15 SEPTEMBER 1973

Renormalization and Gauge Independence in Spontaneously Broken Gauge Theories*

Thomas Appelquist, J. Carazzone,† T. Goldman,‡ and H. R. Quinn
The Physics Laboratories, Harvard University, Cambridge, Massachusetts 02138
(Received 19 March 1973)

We discuss the renormalization of spontaneously broken gauge theories in a large class of renormalizable gauges which includes the unitary gauge as a singular limit. Particular attention is paid to the constraints of gauge invariance on the renormalization program and to the gauge invariance and finiteness of the S matrix. Our intention is to supplement the formal discussions already in the literature by carrying out the renormalization program in an explicit and complete way (fixing counterterms and defining the physical parameters), and by restricting demonstrations of gauge invariance and finiteness to the one-loop level. The discussion is limited to an Abelian model for simplicity, and can easily be extended to more complicated gauge theories. All the essential features, however, are found already in the Abelian model.

I. INTRODUCTION

There are, by now, several papers in the literature dealing with the renormalization of spontaneously broken gauge (SBG) theories. The original work of 't Hooft[1] and Lee[2] has been extended and formulated in a large class of gauges[3,4] by Lee and Zinn-Justin,[5,6] 't Hooft and Veltman,[4] and others.[7] In addition to these formal discussions of renormalization and gauge invariance there have been explicit calculations exhibiting important features of these theories. Gauge invariance of several physical processes[3,8] has been demonstrated for models formulated in a general class of renormalizable gauges, and finiteness of some S-matrix elements has been shown when calculations are performed directly in the unitary gauge.[9]

In this paper we will discuss the renormalization of an Abelian SBG theory in an explicit and complete way in order to describe some of the interesting features of these theories that are not revealed in a formal treatment. We set up the renormalization program in a general class of gauges (the R_ξ gauge of Fujikawa, Lee, and Sanda[3]) which contains the unitary gauge as a singular limit. Our primary interest is in formulating the renormalization in a way that makes the gauge invariance of the S matrix easy to check (at least on the one-loop level) and the passage to the unitary gauge ($\xi \to 0$) easy to discuss. Thus, for example, we do subtractions at physical on-mass-shell points so that the renormalized parameters of the theory are defined in a gauge-invariant way. All proofs of gauge invariance and finiteness are restricted to the one-loop level. This paper is intended to supplement the formal treatments in the literature and to provide some theoretical foundation for the discussion of divergence cancellations in the Abelian theory.[9] It is intentionally written in a somewhat pedagogic style so that it could serve as an introduction to the renormalization of SBG theories. The discussion can readily be extended to more complicated gauge theories, but the Abelian model already has enough structure to show the essential features of any such renormalization program.

Two of these features should be pointed out here. The first is a consequence of the fact that there are more particle masses and vertices in the theory than there are free parameters. The simple relationships between measurable quantities which appear at the tree-graph level must be maintained by the renormalized theory (up to finite higher-order corrections) if the theory is to be renormalizable. This is because the tree-graph relations (or equivalently the original gauge symmetry of the Lagrangian) severely restricts the introduction of Lagrangian counterterms. This leads to finite relations in the renormalized theory involving the physical masses and physical coupling constants (decay rates). Such relations have already been pointed out in the literature in connection with mass-shift calculations[10] and μ-e universality in Weinberg's SU(2)×U(1) theory of leptons,[11] but they appear already in the Abelian model.

The other part of the renormalization program which is discussed in detail is the role of the scalar-meson tadpole contributions. This we treat carefully, pointing out that the tadpoles themselves are both gauge-dependent and divergent, canceling other gauge-dependent contributions to physical renormalization effects.

We begin by discussing quantization of the theory with a manifestly unitary choice of field variables (Sec. II A). The quantization in a class of gauges which are renormalizable by power-counting arguments is then given in Sec. II B. Section II C shows that, at the tree-graph level, the unitary-gauge S matrix is recovered in a simple way by taking the parameter ξ to zero. Section III gives a brief discussion of regularization. Section IV contains the renormalization program for general ξ. Finally in Sec. V the $\xi \to 0$ limit is discussed in the context of this program, and it is demonstrated that the results so obtained are identical to those which would be found for renormalized quantities, with the unitary choice of field variables and a straightforward renormalization procedure.

II. THE ABELIAN MODEL

The Abelian model which we analyze in this paper is described by the Lagrangian

$$\mathcal{L} = -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + |(\partial_\mu + igA_\mu)\Phi|^2 - \mu^2 \Phi^* \Phi - h(\Phi^*\Phi)^2 , \qquad (2.1)$$

where Φ is a complex scalar field. This Lagrangian is invariant under the gauge transformation

$$\begin{aligned} A_\mu(x) &\to A_\mu(x) + (1/g)\partial_\mu \Lambda(x) , \\ \Phi(x) &\to \Phi(x) e^{-i\Lambda(x)} . \end{aligned} \qquad (2.2)$$

When $\mu^2 < 0$, the Higgs mechanism[12,13] operates to give mass to the vector meson and avoid the presence of massless Goldstone bosons. We discuss the quantization of the theory first in the unitary gauge and then in a class of renormalizable gauges which include the unitary gauge as a singular limit.

A. The Unitary Gauge

The transformation of variables introduced by Higgs,[12]

$$\begin{aligned} \Phi(x) &= (1/\sqrt{2})[\lambda + \phi(x)]\, e^{-i\theta(x)/\lambda} , \\ A_\mu(x) &= B_\mu(x) + (1/\lambda g)\partial_\mu \theta(x) , \end{aligned} \qquad (2.3)$$

with $\langle \phi \rangle_0 = 0$, leads to the manifestly unitary Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I , \qquad (2.4)$$

where

$$\mathcal{L}_0 = -\tfrac{1}{4}(\partial_\mu B_\lambda - \partial_\lambda B_\mu)^2 + \tfrac{1}{2}(\lambda g)^2 B_\mu{}^2 + \tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2}(\mu^2 + 3h\lambda^2)\phi^2 \qquad (2.5)$$

and

$$\mathcal{L}_I = \tfrac{1}{2} g^2(\phi^2 + 2\lambda\phi) B_\mu{}^2 - \lambda(\mu^2 + \lambda^2 h)\phi - \tfrac{1}{4}h\phi^4 - h\lambda\phi^3 . \qquad (2.6)$$

The unitarity is manifest since there are no redundant degrees of freedom, and, as a result, the theory can be canonically quantized. The result is that the noncovariant part of the vector-meson propagator can be dropped if the effective interaction Hamiltonian in the interaction representation is taken to be[14]

$$\mathcal{H}_{I\,\mathrm{eff}} = -\mathcal{L}_I + i\,\delta^4(0)\ln(1 + \phi/\lambda) . \qquad (2.7)$$

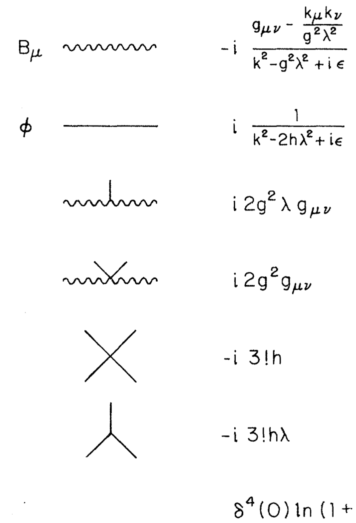

FIG. 1. Feynman rules in the unitary gauge.

A complete derivation of the $\delta^4(0)$ term in the context of canonical quantization has been given by Weinberg[15] for a large class of spontaneously broken gauge theories.

On the tree-graph level, the choice

$$\lambda^2 = -\mu^2/h \qquad (2.8)$$

will maintain the condition $\langle \phi \rangle_0 = 0$. What happens beyond the tree-graph level will concern us in Sec. IV. The unitary-gauge Feynman rules are shown in Fig. 1.

B. Renormalizable Gauges

Following Fujikawa *et al.*[3] we next quantize the theory in a class of renormalizable gauges. We introduce Cartesian components for Φ:

$$\Phi(x) = \frac{\lambda + \psi(x) + i\chi(x)}{\sqrt{2}} , \qquad (2.9)$$

again choosing $\lambda^2 = -\mu^2/h$. The gauge transformation (2.3) which leaves the Lagrangian (2.1) invariant involves a mixing of ψ and χ, which infinitesimally is

$$\begin{aligned} \psi &\to \psi + \chi\Lambda , \\ \chi &\to \chi - \psi\Lambda - \lambda\Lambda . \end{aligned} \qquad (2.10)$$

Quantization begins by adding a gauge-fixing term to the Lagrangian.[16] A convenient choice for our purposes is that of Fujikawa *et al.*,[3]

$$\mathcal{L}_c = -\tfrac{1}{2}\xi\left(\partial_\mu A^\mu - \frac{\lambda g}{\xi}\chi\right)^2 , \qquad (2.11)$$

which leads to their R_ξ gauge. With this choice of gauge, one must include Faddeev-Popov ghosts, and this we do by adding another term to the Lagrangian[17]:

$$\mathcal{L}_{\hat{\Phi}} = \hat{\Phi}^*\left(\partial^2 + \frac{(\lambda g)^2}{\xi} + \frac{\lambda g^2}{\xi}\,\psi\right)\hat{\Phi}\,. \tag{2.12}$$

The Faddeev-Popov (FP) ghosts appear only in closed loops and obey Fermi statistics. Feynman rules can now be derived from $\mathcal{L} + \mathcal{L}_c + \mathcal{L}_{\hat{\Phi}}$ and are shown in Fig. 2.

The condition (2.8) keeps $\langle\psi\rangle_0 = 0$ in the tree approximation. Using this condition, it can easily be seen from the Feynman rules of Fig. 2 that tree-graph contributions to S-matrix elements are independent of ξ. The pole in the A propagator at $k^2 = (\lambda g)^2/\xi$ is just canceled by the χ propagator pole at the same point.

C. The $\xi \to 0$ Limit

For S-matrix elements between in- and out-states consisting only of ψ and A particles, ξ independence is obvious. Furthermore, in the limit $\xi \to 0$, one recovers the unitary-gauge Feynman rules graph by graph. Graphs with χ propagators go to zero, and the A propagator takes the canonical form for a massive spin-one field.

In graphs with closed loops, the $\xi \to 0$ limit is singular because of the asymptotic behavior of the unitary-gauge vector propagator, and must be discussed carefully in the context of a renormalization program including a regularization procedure. This will be done in Sec. V. We merely point out here that formally the $\xi \to 0$ limit gives the unitary gauge theory in any order. Note in particular that Faddeev-Popov loops degenerate to quartically divergent contact terms in the $\xi \to 0$ limit. These are exactly the terms in the expansion of the logarithm in (2.8). The connection between these two objects is manifest in the path integral formalism where both arise from the Jacobian of a transformation of variables.

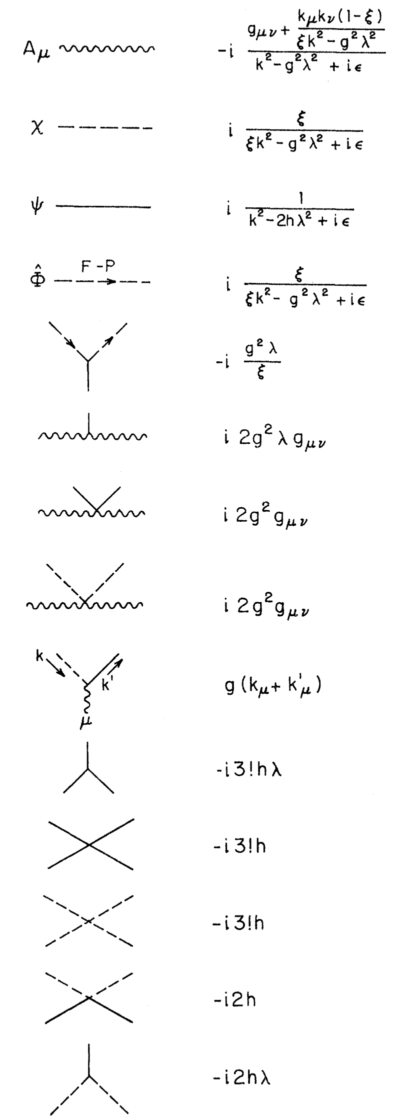

FIG. 2. Feynman rules in the R_ξ gauge. The dashed line with the arrow represents the Faddeev-Popov ghost and appears in closed loops only.

III. REGULARIZATION

In order to discuss renormalization in a general gauge we need a regularization technique which is gauge-invariant and also powerful enough to handle the highly divergent quantities which occur in the unitary gauge. The dimensional-continuation method of 't Hooft and Veltman[18] satisfies both these criteria and we have used it throughout our calculations. It is implicit in the discussion which follows that all divergent Feynman integrals are to be defined by this method.

In the context of the simple Abelian model here discussed, there are no problems in applying this method. One simply uses the usual prescription to express each loop in a Feynman diagram as an integral in momentum space, $\int d^n k\, f(k)$, $n=4$, and then evaluates this integral by continuing in n to a region such that it is finite. It is straightforward to generalize the necessary tensor algebra to dimension $n \neq 4$. In generalizing our treatment to processes involving fermions, we use dimensional continuation to define the coefficient of each spinor invariant. The only situation where any problem arises is in the case of Adler-Bell-Jackiw[19] triangle anomalies. Since theories containing uncanceled anomalies are not renormalizable,[20] this problem is not relevant to our discussion. For all renormalizable theories the prescription of 't Hooft and Veltman appears completely satisfactory as a regularization procedure.

A short description of the method and its application to our treatment of renormalization is given in the Appendix.

IV. RENORMALIZATION

Our starting point for formulating a renormalization program is the Lagrangian (2.1), which we now call $\mathcal{L}_{\rm inv}$. We generate Lagrangian counterterms by rescaling the fields and parameters according to

$$\begin{aligned}
&A_\mu \to (Z_3)^{1/2} A_\mu \ ,\\
&\Phi \to \sqrt{Z}\,\Phi\ ,\\
&g \to \frac{1}{(Z_3)^{1/2}}\, g\ ,\\
&\mu^2 \to \frac{Z_\mu}{Z}\,\mu^2,\\
&h \to \frac{Z_h}{Z^2}\, h\ .
\end{aligned} \tag{4.1}$$

We again write Φ in terms of its Cartesian components with the real part shifted by λ. Then

$$\begin{aligned}
\mathcal{L}_{\rm inv} = &-\tfrac{1}{4} Z_3 (\partial_\mu A_\nu - \partial_\nu A_\mu)^2\\
&+\tfrac{1}{2} Z\,|(\partial_\mu + igA_\mu)(\lambda+\psi+i\chi)|^2\\
&- Z_\mu \tfrac{1}{2}\mu^2 |\lambda+\psi+i\chi|^2 - Z_h \tfrac{1}{4} h |\lambda+\psi+i\chi|^4\ .
\end{aligned} \tag{4.2}$$

The parameters g, μ, and h are now renormalized parameters and μ^2 is taken to be negative. The parameter λ could be adjusted to give $\langle\psi\rangle_0 = 0$ to all orders of perturbation theory, that is, $\langle\Phi\rangle_0 = \lambda$ to all orders. This, as we shall see, would make λ both gauge-dependent and divergent even on the one-loop level. Instead, we simply set

$$\lambda^2 \equiv -\mu^2/h\ , \tag{4.3}$$

which makes $\langle\psi\rangle_0$ vanish in the tree-graph approximation but not beyond. The closed-loop contributions to $\langle\psi\rangle_0$ (tadpoles) are gauge-dependent and enter only as renormalization effects, canceling other gauge-dependent pieces. This we will show explicitly. We will express our results in terms of g, h, and λ, eliminating μ^2 by Eq. (4.3).

To quantize the theory, we again add the gauge-determining piece

$$\mathcal{L}_c = -\tfrac{1}{2}\xi\left(\partial_\mu A^\mu - \frac{\lambda g}{\xi}\chi\right)^2 \tag{4.4}$$

and the Faddeev-Popov term

$$\mathcal{L}_{\hat\Phi} = \hat\Phi^*\left(\partial^2 + \frac{\lambda^2 g^2}{\xi} + \frac{\lambda g^2}{\xi}\psi\right)\hat\Phi\ . \tag{4.5}$$

Note that the fields in (4.4) and (4.5) are renormalized fields. The result is a Lagrangian which can be divided into two pieces: One gives the Feynman rules of Fig. 2, and the other is a set of counterterms. These symmetric counterterms will be sufficient to remove the divergences of the theory.[21] This leaves a finite renormalization to perform if we wish to give g a simple physical definition. It is convenient to introduce such a further rescaling at this stage by letting

$$g \to (Z_1/Z)^{1/2} g. \tag{4.6}$$

Then the entire renormalization of g is given by $[Z_1/(Z_3 Z)]^{1/2}$, with Z_1 and Z relatively finite.

The entire counterterm Lagrangian is

$$\begin{aligned}
\mathcal{L}_{\rm ct} = &-\tfrac{1}{4}(Z_3-1)(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \tfrac{1}{2}(Z_1-1)\lambda^2 g^2 A_\mu{}^2 + \tfrac{1}{2}(Z-1)(\partial_\mu\psi)^2 + \tfrac{1}{2}(Z-1)(\partial_\mu\chi)^2\\
&+\tfrac{1}{2}(Z_1-1) g^2 A_\mu{}^2(\psi^2+\chi^2) + (Z_1-1) g^2 \lambda A_\mu{}^2 \psi\\
&+\left(\frac{Z_1}{Z}\right)^{1/2}(Z-1) g\lambda A_\mu \partial^\mu\chi - \left((ZZ_1)^{1/2}-1\right) g\left[(\partial_\mu\psi)A^\mu\chi - (\partial_\mu\chi)A^\mu\psi\right]\\
&-(Z_\mu - 1)\tfrac{1}{2} h\lambda^2 |\lambda+\psi+i\chi|^4 - (Z_h-1)\tfrac{1}{4} h |\lambda+\psi+i\chi|^4 - \tfrac{1}{2}\frac{\lambda^2 g^2}{\xi}\left(\frac{Z_1}{Z}-1\right)\chi^2\ .
\end{aligned} \tag{4.7}$$

There are two other finite counterterms associated with the FP ghost mass and coupling, but since we will not be going beyond the one-loop level, they do not enter our discussion.

The next step is to adjust the renormalization constants, thereby defining the renormalized parameters of the theory. There are five renormalization constants $(Z, Z_3, Z_\mu, Z_h, Z_1)$ which will be used to define the three parameters g, h, and λ in terms of physical masses and coupling constants and to do wave-function-renormalization subtractions for the fields A_μ and ψ which describe the physical particles. An important feature of our procedure is to do subtractions at on-shell points so that g, h, and λ are defined in a gauge-invariant way. This simplifies the discussion of gauge independence of the S matrix and the transition to the unitary gauge.

It is convenient to begin by discussing the scalar-meson (ψ) two- and three-point functions. There

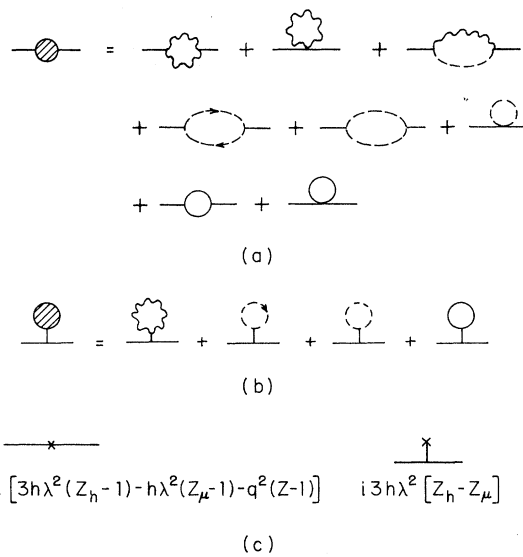

FIG. 3. One-loop graphs for the scalar-meson self-energy $\Pi(q^2)$.

are two types of graphs to consider in the two-point function: ordinary one-particle-irreducible (1PI) self-energy graphs and tadpole contributions. We denote the sum by $\Pi(q^2)$. The one-loop graphs of each type are shown in Figs. 3(a) and 3(b). Each set has a counterterm as shown and written explicitly in Fig. 3(c). They come directly from (4.7). We do *not* adjust the tadpole counterterm to cancel the tadpoles of Fig. 3(b). Rather, we proceed physically, adjusting the entire counterterm,

$$-i\,[\,2h\lambda^2(Z_\mu-1)-q^2(Z-1)] \tag{4.8}$$

(the sum of the two pieces) to effect conventional mass and wave-function subtractions on the entire $\Pi(q^2)$:

$$\begin{aligned} &i\,(Z_\mu - Z)2h\lambda^2 = \Pi(2h\lambda^2)\,, \\ &-i\,(Z-1) = \frac{\partial}{\partial q^2}\Pi(q^2)\bigg|_{q^2=2h\lambda^2}\,. \end{aligned} \tag{4.9}$$

This defines

$$\begin{aligned} 2h\lambda^2 &= M_\psi{}^2 \\ &= (\text{physical } \psi \text{ mass})^2 \end{aligned} \tag{4.10}$$

and fixes Z and Z_μ. Z will be gauge-dependent, but Z_μ/Z cannot be since it is the mass renormalization. In lowest order this means $Z_\mu - Z$ is gauge-independent, i.e., that $\Pi(2h\lambda^2)$ is gauge-independent. We have explicitly checked that this on-shell two-point function (both finite and cutoff-dependent parts) is gauge-independent on the one-loop level. Details are displayed in the Appendix.

The one-loop contributions to the scalar-meson three-point function $V(q_1{}^2, q_2{}^2, q_3{}^2)$ are also either

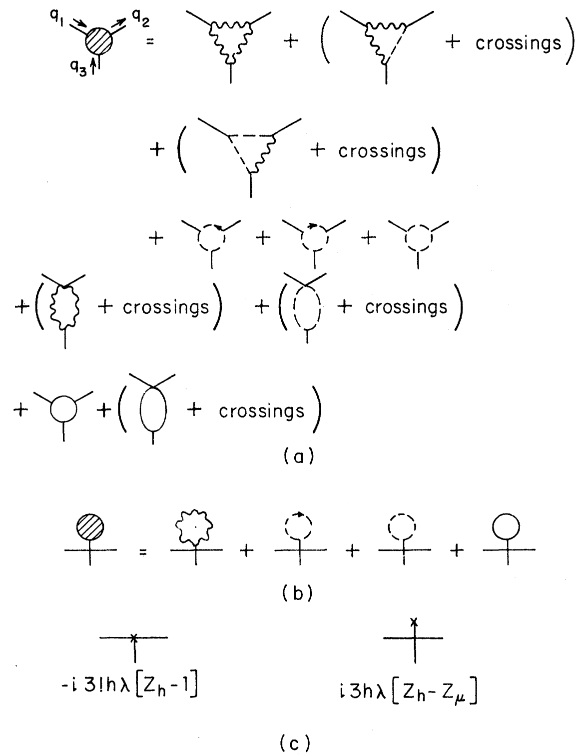

FIG. 4. One-loop graphs for the scalar three-point function $V(q_1{}^2, q_2{}^2, q_3{}^2)$.

1PI or tadpole as shown in Figs. 4(a) and 4(b). From Fig. 4(c), the complete counterterm is

$$-3ih\lambda[(Z_h-1)+(Z_\mu-1)]\,.$$

By adjusting Z_h so that

$$3ih\lambda[(Z_h-1)+(Z_\mu-1)] = V(M_\psi{}^2, M_\psi{}^2, M_\psi{}^2)\,, \tag{4.11}$$

the only contribution to the on-shell 3-ψ coupling constant g_ψ will be the tree graph. This defines

$$h\lambda \equiv g_\psi = \text{physical 3-}\psi\text{ coupling.} \tag{4.12}$$

This object must be gauge-independent since it is experimentally accessible by extrapolation to a pole in the unphysical region. Equations (4.10) and (4.12) completely fix h and λ in a gauge-independent way. Since Z_h/Z^2 is a coupling-constant renormalization, $(Z_h-1)-2(Z-1)$ should be gauge-independent on the one-loop level. Thus from (4.11) and the one-loop gauge independence of $Z_\mu - Z$,

$$V(M_\psi{}^2, M_\psi{}^2, M_\psi{}^2) = 9ih\lambda(Z-1) + \text{gauge-independent terms.} \tag{4.13}$$

The demonstration of this one-loop relation is outlined in the Appendix.

We complete the determination of the Z's and definition of the renormalized parameters by looking at the vector self-energy

$$\Pi_{\mu\nu}(p) = A(p^2) g_{\mu\nu} + B(p^2) p_\mu p_\nu .$$

The one-loop 1PI and tadpole contributions are shown in Figs. 5(a) and 5(b). The complete counterterm is the sum of the two pieces shown in Fig. 5(c). We adjust Z_1 and Z_3 to do conventional mass and wave-function subtractions eliminating the first two terms in the Taylor expansion of $A(p^2)$ about $p^2 = \lambda^2 g^2$. This fixes Z_1 and Z_3 and defines

$$\begin{aligned} \lambda^2 g^2 &\equiv M_A{}^2 \\ &= (\text{physical } A \text{ mass})^2. \end{aligned} \tag{4.14}$$

It can easily be checked that Z_1/Z is finite on the one-loop level. The quantity $Z_1/Z_3 Z$ which is coupling-constant renormalization must be gauge-independent. In fact Z_3 and Z_1/Z turn out to be separately gauge-independent in this model.[22] The parameters λ, g, and h are now all defined in a physical, gauge-independent way, and the counterterms are all fixed. It can now be checked that these counterterms remove the divergences in the Green's functions of the theory for nonzero ξ, and that physical processes are finite, gauge-independent functions of λ, g, and h.

Before looking at a few examples, a brief discussion of the tadpoles and their counterterm is in order. A simple calculation shows that

$$T - i\lambda^3 h (Z_h - Z_\mu) = 2\lambda^3 h \frac{g^2}{\xi} \int \frac{d^4 k}{k^4} + \text{finite terms}, \tag{4.15}$$

where T is the sum of the one-loop tadpole graphs. Thus the tadpole with its counterterm is both divergent and gauge-dependent, only becoming finite (on the one-loop level) in the Landau gauge ($\xi \to \infty$). The gauge dependence of this object which contributes solely to renormalization should not be surprising since, in each case, it is merely one part of a physical renormalization effect.

An alternate but completely equivalent treatment of the tadpole contributions can be achieved by introducing a further subtraction constant and requiring that the tadpole vanish identically. That is, in addition to the rescalings of Eq. (4.1) and (4.6) we could take

$$\lambda \to \left(\frac{Z_\lambda}{Z}\right)^{1/2} \lambda$$

in rewriting the Lagrangian, and again use

$$\lambda^2 = -\mu^2/h$$

as a relationship among renormalized quantities.

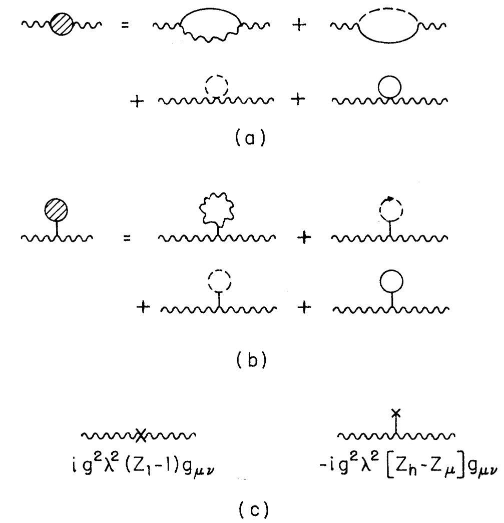

FIG. 5. One-loop graphs for the vector-meson self-energy $\Pi_{\mu\nu}(p^2)$.

The additional constant Z_λ is then fixed by the requirement that the tadpole contribution, consisting now of Eq. (4.15) plus a Z_λ-dependent counterterm, must vanish identically. Thus defined, Z_λ is divergent and gauge-dependent. At every point, where the previous procedure gave the tadpole contribution represented by (4.15), the modified procedure contains the same contribution, which now, however, enters by way of the Z_λ-dependent counterterm. This is simply a matter of choosing a slightly different method of bookkeeping. We note that this second method is the one used by two of us[11] for the Weinberg model in a calculation of μ decay in the unitary gauge. A similar method has also been used by Lee and Zinn-Justin[5] for their comparison of the $\xi = 0$ and $\xi = \infty$ gauges (U and R gauges of their paper).

Returning to the counterterms of (4.7), other physical processes can now be calculated. The simplest to discuss is the $\psi \to AA$ decay rate (taking $h > 2g^2$) which is given by the ψAA three-point function on shell. The one-loop graphs and counterterms are shown in Fig. 6. If we denote the graphs and counterterms of Fig. 6 by $\Lambda_{\mu\nu}(p, p')$, then the complete three-point function including the Born term is

$$\Gamma_{\mu\nu}(p, p') = 2ig^2 \lambda g_{\mu\nu} + \Lambda_{\mu\nu}(p, p') . \tag{4.16}$$

The counterterms of Fig. 6(c) (previously determined) are just sufficient to make $\Gamma_{\mu\nu}(p', p)$ finite for any value of p' and p with $\xi \neq 0$. Furthermore it is gauge-independent when put on shell and sandwiched between physical polarization vectors. Thus

$$\Gamma(\psi \to AA) \propto (g^2\lambda)^2 [1 + \text{finite, gauge-independent terms of order } g^2 \text{ and } h]. \tag{4.17}$$

This is a finite relation between the decay rates and masses of the theory and is a consequence of the small number of independent counterterms available. This feature of spontaneously broken gauge theories has been pointed out already in connection with mass-difference calculations[1,8] and the question of μ-e universality in Weinberg's theory,[9] but it is interesting to see it in this simplest of all such theories.

A few concluding comments might help put this section in perspective.

(1) First of all, it is not necessary to choose mass and wave-function renormalizations along with the triple scalar vertex to define the parameters g, h, and λ and to fix the subtraction constants. Any convenient choice which relates these parameters to physical quantities may be made.

(2) Once the counterterms are fixed, it can be checked that they remove the divergences of the remaining 1PI Green's functions (for $\xi \neq 0$). Thus, for example, the χ self-energy is rendered finite by its (previously fixed) counterterm. In the case of divergent Green's functions consisting only of external ψ and A lines, this result leads to finite higher-order relations such as (4.17).

(3) The fact that the counterterms already introduced make the remaining vertices finite can be proved by making use of the Ward identities of the theory. Such an analysis will be reported by one of us.[23] The Ward identities can also be used to show the gauge independence of the S matrix. This is the content of the papers by Lee and Zinn-Justin[6] and 't Hooft and Veltman.[4]

V. THE UNITARY GAUGE

With the program given in Sec. IV it is clear that the $\xi \to 0$ limit is easy to examine since particle masses, on-shell couplings, and S-matrix elements are explicitly gauge-independent. The only question which remains is whether the quantities so calculated correspond to those one would find by calculating with the manifestly unitary choice of fields. We turn now to this question.

The starting point is the renormalized Lagrangian (4.2), expressed in terms of the unitary gauge variables (2.3). This, together with the further rescaling $g \to (Z_1/Z)^{1/2}\, g$, gives a Lagrangian with five independent counterterms which can be canonically quantized. The choice $\lambda^2 = -\mu^2/h$ can again be made to make $\langle\phi\rangle_0 = 0$ on the tree-graph level. Quantization leads to an effective interaction Hamiltonian given by the negative of the interaction

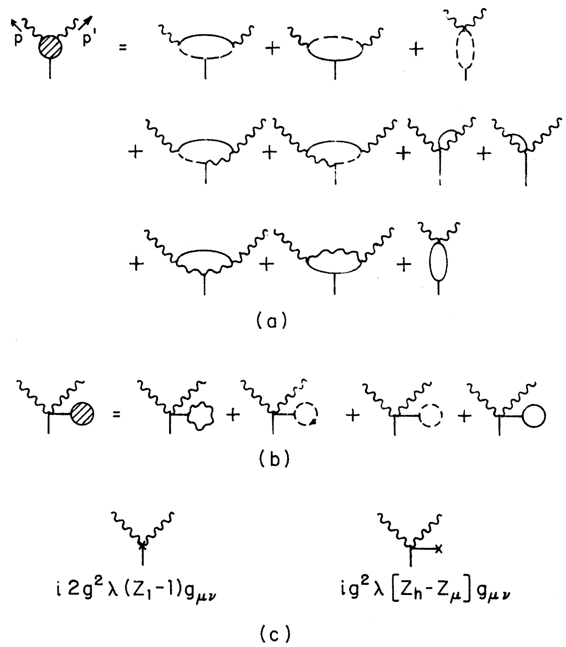

FIG. 6. One-loop graphs for the one-scalar–two-vector three-point function.

Lagrangian (including counterterms) along with the $\delta^4(0)$ term of (2.7). The dimensional continuation method can again be used to regulate divergent integrals, and the renormalization program of Sec. IV can be carried out. The difference now is that individual graphs and the gauge-dependent renormalization constants become much more divergent as the regulator is removed. Green's functions now contain divergences even after renormalization, and only after calculating the on-shell S matrix may the regulator be removed ($n \to 4$). In this context, it is worth pointing out that in the framework of the 't Hooft-Veltman regulator, the integral of any polynomial is *defined* to be zero.[18] In particular, $\delta^4(0) \propto \int d^4k \equiv 0$.

The entire unitary gauge formalism can be obtained, graph by graph, from the R_ξ formalism by taking the limit $\xi \to 0$ in the presence of the regulator. The vector-meson propagator takes its unitary-gauge form, and all graphs with χ propagators go to zero provided that the space-time dimension n is less than 2. The $\xi \to 0$ limit of the FP ghost loops is more delicate, but if the limit is taken with $n < 0$ they go to zero, in agreement with the above result. This has already been pointed out by 't Hooft and Veltman.[4] There are also terms of this form (formally quartically divergent but vanishing when regulated) in the $\xi \to 0$ limit of closed loops consisting only of vector mesons. If the vector-meson loops and the FP ghost loops are added together before taking the $\xi \to 0$ limit, the would-be quartic divergences cancel, and it is sufficient to take $n<2$ for the entire calculation.

It is clear from this discussion that for all physical quantities a U-gauge treatment yields the same results as the $\xi\to 0$ limit of the R_ξ-gauge method. Thus the two are equivalent.

VI. SUMMARY

The following are the essential features of our renormalization program

(1) The independent renormalization constants are adjusted to effect subtractions at on-mass-shell points. This leads to gauge-independent definitions of the renormalized parameters of the theory. Although we have worked in a restricted class of gauges, the program clearly applies to any gauge.

(2) The renormalized scalar field is shifted by an amount equal to its tree-graph vacuum expectation value. A further shift to include the one-loop and higher contributions to the vacuum expectation value in the shift parameter is superfluous and confusing since these tadpole contributions are both gauge-dependent and divergent. Thus it is not useful to speak of higher-order corrections to the relation (4.3). The tadpole graphs must be included, and they contribute only to renormalization effects canceling other gauge-dependent pieces.

(3) The limited number of independent subtraction constants leads to finite, gauge-independent, higher-order relations among the physical masses and on-shell vertices of the theory.

ACKNOWLEDGMENTS

One of us (T.A.) acknowledges several helpful conversations on the renormalization of gauge theories with Professor M. Veltman. Discussions with Professor B. W. Lee and our colleagues in Cambridge are also gratefully acknowledged.

APPENDIX

This appendix presents the details involved in examining the gauge-dependent portions of our renormalization procedure. A brief discussion of gauge-covariant regularization is included along with the explicit formulas used to verify relation (4.13). Besides exhibiting the gauge dependence of the renormalization constants, it is also worthwhile writing down their divergent, gauge-independent pieces. This is necessary in order to check finiteness relations such as (4.17). We have, in fact, checked (4.17) in this way, but the details are not presented here.

As remarked in the text, all divergent integrals which arise in our calculations are to be regarded as defined by the 't Hooft-Veltman regulation procedure, which consists of an analytic continuation in the number of space-time dimensions.[18] We note that in the absence of fermions, our theory contains no objects, such as $\epsilon_{\mu\nu\alpha\beta}$, which are peculiar to 4-dimensional space, and the continuation encounters no difficulty. This method is an extremely attractive one to use for gauge theories because dimensional continuation leaves the form of Ward identities unchanged, thus ensuring the validity of the formal arguments based upon them. Indeed, it is precisely this property of the regulation procedure which also makes the method very convenient; vector algebra is left formally unaltered by continuation in the number of space-time dimensions. This means that one is free to perform shifts in integration variables so long as divergent integrals are regularized by dimensional continuation, and the results presented in this Appendix were obtained by making repeated use of this property.

The Feynman integrals presented below can all be evaluated by use of the 't Hooft-Veltman formula[18]

$$i\int\frac{d^nk}{(2\pi)^n}\,\frac{1}{(k^2+2k\cdot p+m^2)^\alpha}=\frac{i\pi^{n/2}}{(2\pi)^n(m^2-p^2)^{\alpha-n/2}}\,\frac{\Gamma(\alpha-\frac{1}{2}n)}{\Gamma(\alpha)}\,, \tag{A1}$$

where the integration is carried out over an n-dimensional Euclidean momentum space, and p_μ is a Euclidean vector. Using this formula one can obtain the following parametric forms for the Feynman integrals which arise in the calculation. As n tends to 4 we have

$$\int\frac{d^nk}{(2\pi)^n}\,\frac{1}{k^2-m^2+i\epsilon}=\frac{i}{(4\pi)^2}\,\frac{m^2}{(2-\frac{1}{2}n)}\,,$$

$$\int\frac{d^nk}{(2\pi)^n}\,\frac{1}{(k^2-m_1^2+i\epsilon)}\,\frac{1}{[(k+p)^2-m_2^2+i\epsilon]}=\frac{i}{(4\pi)^2}\left[\frac{1}{(2-\frac{1}{2}n)}-\int_0^1 dx\,dy\,\delta(1-x-y)\{\ln(m_1^2x+m_2^2y+p^2xy)+\text{const}\}\right],$$

$$\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{(k^2-m_1^2+i\epsilon)}\,\frac{1}{[(k+p)^2-m_2^2+i\epsilon]}\,\frac{1}{[(k+q)^2-m_3^2+i\epsilon]} \tag{A2}$$

$$=\frac{i}{(4\pi)^2}\int_0^1 dx\,dydz\,\delta(1-x-y-z)\,\frac{1}{p^2xy+q^2xz+(p+q)^2yz-m_1^2x-m_2^2y-m_3^2z}\,,$$

where these are conventional momentum-space integrals.

We present below the explicit gauge dependence of the vector-meson self-energy, the scalar-meson self-energy, and the scalar-meson three-point function.

To simplify notation, define

$$\begin{aligned} &A(k+q)=(k+q)^2-g^2\lambda^2+i\epsilon\,, \quad A=A(k)\\ &A(j)=A(k+q_j), \quad j=1,2,3\\ &B(k+q)=\xi(k+q)^2-g^2\lambda^2+i\epsilon\,, \quad B=B(k)\\ &B(j)=B(k+q_j)\,, \quad j=1,2,3 \end{aligned} \tag{A3}$$

where the q_j are the external momenta of the scalar-meson three-point function $V(q_1^2, q_2^2, q_3^2)$ as defined in Fig. 4(a).

The gauge-dependent part of the vector meson self-energy, as calculated from the diagrams of Figs. 5(a) and 5(b), is

$$\Pi_{\mu\nu}(p^2)=p_\mu p_\nu\left\{\frac{g^2}{p^2}\left(p^2+\frac{2g^2\lambda^2}{\xi}-4h\lambda^2\right)\int_0^1 dx\left[\ln\left(p^2x(1-x)+2h\lambda^2(1-x)+\frac{xg^2\lambda^2}{\xi}\right)+\text{const}\right] + \text{gauge-independent terms (GIT)}\right\}+g_{\mu\nu}\times\text{GIT}\,. \tag{A4}$$

Note that only the longitudinal part is gauge-dependent and that this part becomes divergent in the limit $\xi\to 0$ (unitary gauge). The gauge independence of the coefficient of $g_{\mu\nu}$ implies that the vector-meson mass and wave-function renormalization are both gauge-independent.

The scalar meson self-energy calculated from Figs. 3(a) and 3(b) is

$$\Pi(q^2)=(q^2-2h\lambda^2)\frac{\xi}{\lambda^2}\int\frac{d^nk}{(2\pi)^n}\,\frac{1}{B(k)}-(q^4-4h^2\lambda^4)\frac{\xi^2}{2\lambda^2}\int\frac{d^nk}{(2\pi)^n}\,\frac{1}{B(k)B(k+q)}+\text{GIT}\,. \tag{A5}$$

Note that the gauge-dependent parts vanish on shell ($q^2=2h\lambda^2$), so that mass renormalization is gauge-independent as expected. The formulas given above for the regulated integrals can be used to compute the scalar-meson wave-function renormalization constant as

$$(Z-1)=-\frac{2h}{(4\pi)^2}\int_0^1 dx\left[\ln\left(x(x-1)2h\lambda^2+\frac{g^2\lambda^2}{\xi}\right)+\text{const}\right]-\frac{g^2}{\xi(4\pi)^2}\,\frac{1}{(2-n/2)}+\text{GIT}\,. \tag{A6}$$

For $V(q_1^2, q_2^2, q_3^2)$, the diagrams of Fig. 4(a) give the contributions

$$\begin{aligned} &\frac{1}{\lambda^3}\int\frac{d^nk}{(2\pi)^n}\,\frac{1}{BB(2)B(3)}\{g^4\lambda^4\xi(q_1^2+q_2^2+q_3^2-6h\lambda^2)+2g^2\lambda^2\xi^2[2h\lambda^2(q_1^2+q_2^2+q_3^2)-q_1^2q_2^2-q_2^2q_3^2-q_3^2q_1^2]\\ &\qquad +2\xi^3[q_1^2q_2^2q_3^2-h\lambda^2(q_1^2q_2^2+q_2^2q_3^2+q_3^2q_1^2)+4h^3\lambda^6]\}\\ &\quad+\frac{g^4}{\lambda^3}\left\{\int\frac{d^nk}{(2\pi)^n}\,\frac{(q_1^2-2h\lambda^2)}{AB(2)B(3)}[-(1-\xi)^2\lambda^4+\xi(1+\xi)\lambda^2(q_2^2+q_3^2)-\xi(2q_1^2\lambda^2+q_2^2q_3^2)]+\text{cyclic permutations of } q_j\right\}\\ &\quad+\frac{g^6}{\lambda^3}(1+\xi)\left[\int\frac{d^nk}{(2\pi)^n}\,\frac{q_1^2(q_3^2-q_2^2)}{AA(3)B(2)}+\text{cyc. perm. }(j)\right]\\ &\quad+\frac{g^2}{\lambda^3}\left\{\int\frac{d^nk}{(2\pi)^n}\,\frac{[\lambda^2(1-\xi)-\xi q_1^2](4h\lambda^2-q_2^2-q_3^2)}{AB(1)}+\text{cyc. perm. }(j)\right\}\\ &\quad+\frac{1}{\lambda^3}\left(\int\frac{d^nk}{(2\pi)^n}\,\frac{1}{BB(1)}\{g^2\lambda^2\xi(q_2^2+q_3^2-4h\lambda^2)\right.\\ &\qquad\left.+\xi^2[2h^2\lambda^4+g^2\lambda^2(2h\lambda^2-q_1^2)+\tfrac{1}{2}(8h\lambda^2-q_2^2-q_3^2)q_1^2]\}+\text{cyc. perm. }(j)\right)\\ &\quad-\frac{6h\xi}{\lambda}\int\frac{d^nk}{(2\pi)^n}\,\frac{1}{B}+\text{GIT}\,. \end{aligned} \tag{A7}$$

The diagrams of Fig. 4(b) contribute

$$-\frac{3h\xi}{\lambda}\int\frac{d^nk}{(2\pi)^n}\,\frac{1}{B}+\text{GIT}\,. \tag{A8}$$

The renormalization procedure has been set up so that external-line corrections vanish on shell. Thus, when $q_j^{\,2}=2h\lambda^2$, the sum of the above terms is

$$V(M_\psi^{\,2},M_\psi^{\,2},M_\psi^{\,2})=\mathrm{GIT}-\frac{9h\xi}{\lambda}\int\frac{d^nk}{(2\pi)^n}\frac{1}{B}+6h^2\lambda\xi^2\int\frac{d^nk}{(2\pi)^n}\frac{1}{B}\left(\frac{1}{B(1)}+\frac{1}{B(2)}+\frac{1}{B(3)}\right)$$

$$=-\frac{9h\xi}{\lambda}\int\frac{d^nk}{(2\pi)^n}\frac{1}{B}+18h^2\lambda\xi^2\int\frac{d^nk}{(2\pi)^n}\frac{1}{BB(1)}+\mathrm{GIT}\,, \tag{A9}$$

since the three terms of the second integral are all equal (each term is evaluated on the same mass shell). Converting to Feynman-parameter form, we verify Eq. (4.13) of the text:

$$V(M_\psi^{\,2},M_\psi^{\,2},M_\psi^{\,2})=9ih\lambda(Z-1)+\mathrm{GIT}\,. \tag{A10}$$

This ensures that the relation between the bare and physical three-scalar coupling, arising from both 1PI vertex corrections and external-line wave-function renormalization, is gauge-independent.

---

*Work supported in part by the Air Force Office of Scientific Research under contract No. F44620-70-C-0030.

†National Science Foundation Predoctoral Fellow.

‡National Research Council (of Canada) Postgraduate Fellow. Present address: Stanford Linear Accelerator Center, Stanford University, Stanford, California 94355.

[1]G. 't Hooft, Nucl. Phys. B35, 167 (1971).

[2]B. W. Lee, Phys. Rev. D 5, 823 (1972).

[3]K. Fujikawa, B. W. Lee, and A. I. Sanda, Phys. Rev. D **6**, 2923 (1972).

[4]G. 't Hooft and M. Veltman, Nucl. Phys. B50, 318 (1972); CERN Report No. Th-1571, 1972 (unpublished).

[5]B. W. Lee and J. Zinn-Justin, Phys. Rev. D **5**, 3121 (1972); Phys. Rev. D 5, 3137 (1972); Phys. Rev. D **5**, 3155 (1972).

[6]B. W. Lee and J. Zinn-Justin, Phys. Rev. D **7**, 1049 (1973).

[7]Y.-P. Yao, Phys. Rev. D 7, 1647 (1973); K. Shizuya, University of Tokyo-Komaba Report No. 72-17 (unpublished).

[8]S. Weinberg, Phys. Rev. D **7**, 2887 (1973); J. R. Primack and H. R. Quinn, Phys. Rev. D **6**, 3171 (1972).

[9]T. W. Appelquist and H. R. Quinn, Phys. Lett. **39B**, 229 (1972).

[10]S. Weinberg, Phys. Rev. Lett. **29**, 388 (1972); H. Georgi and S. L. Glashow, Phys. Rev. D **6**, 2977 (1972).

[11]T. W. Appelquist, J. R. Primack, and H. R. Quinn, Phys. Rev. D **6**, 2998 (1972); Phys. Rev. D **7**, 2998 (1973).

[12]P. W. Higgs, Phys. Rev. Lett. **12**, 132 (1964); Phys. Rev. Lett. **13**, 508 (1964); Phys. Rev. **145**, 1156 (1966); F. Englert and R. Brout, Phys. Rev. Lett. **13**, 321 (1964); G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble, Phys. Rev. Lett. 13, 585 (1965).

[13]T. W. B. Kibble, Phys. Rev. **155**, 1554 (1967).

[14]T. D. Lee and C. N. Yang, Phys. Rev. **128**, 885 (1962); I. S. Gerstein, R. Jackiw, B. W. Lee, and S. Weinberg, Phys. Rev. D **3**, 2486 (1971).

[15]S. Weinberg, Phys. Rev. D **7**, 1068 (1973).

[16]The necessity of adding a gauge-fixing term in order to derive Feynman rules is well known. Discussions of the range of possibilities for this term can be found in both papers of Ref. 4.

[17]This entire discussion is to be understood in the context of the path-integral method for deriving the Feynman rules. The ghost loops were first derived in this way for the Yang-Mills field by I. Faddeev and V. N. Popov [Phys. Lett. **25B**, 29 (1967)]. Their work was generalized to a large class of gauges by G. 't Hooft [Nucl. Phys. **B33**, 173 (1971)], who expressed the ghost contribution as an additional term $\mathcal{L}_{\hat\Phi}$ in the effective Lagrangian. A prescription for writing down the $\mathcal{L}_{\hat\Phi}$ corresponding to any gauge-fixing term is given in the papers of Ref. 4. For the gauge-fixing term (2.11), this prescription leads to a nonzero $\mathcal{L}_{\hat\Phi}$; thus Faddeev-Popov ghosts appear even in the Abelian theory.

[18]G. 't Hooft and M. Veltman, Nucl. Phys. **B44**, 189 (1972).

[19]S. L. Adler, Phys. Rev. **177**, 2426 (1969); J. S. Bell and R. Jackiw, Nuovo Cimento **60A**, 47 (1969). See also S. L. Adler, in *Lectures on Elementary Particles and Quantum Field Theory*, edited by S. Deser *et al.* (MIT Press, Cambridge, Mass., 1971).

[20]C. Bouchiat, J. Iliopoulos, and P. Meyer, Phys. Lett. **36B**, 519 (1972); and D. Gross and R. Jackiw, Phys. Rev. D **6**, 477 (1972).

[21]The situation here is similar to that in the σ models, discussed by B. W. Lee [Nucl. Phys. **B9**, 649 (1969)], and by J.-L. Gervais and B. W. Lee [Nucl. Phys. **B12**, 627 (1969)]. See also Ref. 6.

[22]The gauge independence of Z_3 is a special property of this very simple theory. In general in a non-Abelian theory the vector wave-function renormalization is not gauge-independent. This has been noted also by Lee and Zinn-Justin in Ref. 5.

[23]J. Carazzone (unpublished).