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# Generalization and Interpretation of Dirac's Positive-Energy Relativistic Wave Equation*

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We generalize Dirac's new equation so as to describe particles of mass $m$ and arbitrary spin $s$. The same remarkable properties are found: positive energy, non-negative density, a conserved four-vector current, and the impossibility of minimal electromagnetic interaction. We show that the particles described by a subset of these equations are composites of two subparticles interacting by a relativistic action-at-a-distance interaction characterized by two harmonic oscillators. For these composite particles we find a linear relation between the square of the mass and the spin. We emphasize that the essential content of the generalized new Dirac equation is that it constitutes an example of a convariant solution for two interacting particles, and provides an explicit example of a new quantal subdynamics distinct from the (classical) front relativistic dynamics of Dirac.

## I. INTRODUCTION

Dirac has given recently[1] a remarkable new relativistic equation which—though it superficially resembles the familiar Dirac equation for the electron—describes a spinless particle of mass $m$. What is surprising about this equation is that the particle energy is positive definite, yet there exists a conserved four-vector current, having a positive definite charge density. Most remarkable of all is the fact that the conserved particle current *cannot interact with the electromagnetic field* (using the minimal coupling $p_\mu \to \pi_\mu \equiv p_\mu - eA_\mu/c$) *without destroying the consistency of the defining structure*. (For the convenience of the reader, we summarize in Sec. II the essentials of Dirac's presentation of his new equation.)

We demonstrate in Sec. III how to generalize Dirac's new equation so as to describe particles which correspond to an irreducible representation (irrep) of the covering group of the Poincaré group with the invariant labels $m$ (mass) and $s$ (spin), where $s$ takes on the values $s=0, \frac{1}{2}, 1, \ldots$ . These generalized equations are based on the same internal space as used in Dirac's new equation, and share with this latter equation the same remarkable properties. We prove that they allow only positive-energy solutions, and have a conserved current with non-negative charge density. The most remarkable property of all, the impossibility of any electromagnetic interaction with minimal coupling, is proved in detail. (The methods by which these proofs are accomplished differ from those of Dirac, and are, perhaps, of independent interest since they generalize.) The relationship of Dirac's new equation to the Majorana[2] equation is also discussed in Sec. III.

In Sec. IV we develop what is possibly the principal result of our work. Here we demonstrate that the particles described by the new Dirac equation and its generalization *may be consistently viewed as composites of two subparticles interacting via action-at-a-distance forces*. We show, from Dirac's new equation, that such a possibility necessarily requires that the description be in the front frame.

In order to prove the consistency of such a view, it is necessary to reexamine the various possible methods of Hamiltonian relativistic mechanics.[3] We show that, for the front frame, there exists *a qualitative distinction between quantal and classical Hamiltonian relativistic mechanics*. In contrast to the (classical) Dirac prescription for the front frame,[3] we develop the "method of quantal front subdynamics."

From our point of view, the importance of the new Dirac equation derives from the fact that it provides a completely explicit example of a solution to a relativistic two-body problem which allows a verification of internal consistency through direct construction. It follows from this construction that for the two interacting subparticles of the new Dirac equation, the states of higher spin (belonging to the generalized equation) necessarily obey a mass-spin relation of the Chew-Frautschi form: $m^2 \propto s$. In a very real sense, the (generalized) new Dirac equation constitutes an explicit and precise solution to that much abused object, the relativistic harmonic oscillator.

A concluding section, Sec. V, contains further discussion and open questions.

## II. RESUMÉ OF DIRAC'S NEW EQUATION

The new relativistic wave equation[1,4] for particles of nonzero rest mass proposed by Dirac has a great formal similarity to the usual Dirac equation for the electron, but the physical significance is very different.

The internal degrees of freedom involve two harmonic oscillators. Let the dynamical variables describing these oscillators be $\xi_1$, $\xi_2$, $\pi_1$, and $\pi_2$; introduce the dimensionless variables

$$q_1=\left(\frac{\mu\omega}{\hbar}\right)^{1/2}\xi_1\,,\quad q_2=\left(\frac{\mu\omega}{\hbar}\right)^{1/2}\xi_2\,,\quad q_3=\frac{1}{(\mu\omega\hbar)^{1/2}}\pi_1\,, \tag{1}$$

and

$$q_4=\frac{1}{(\mu\omega\hbar)^{1/2}}\pi_2\,.$$

Denote by $Q$ the column vector

$$Q\equiv\begin{bmatrix}q_1\\q_2\\q_3\\q_4\end{bmatrix} \tag{2}$$

We then have the relations

$$[q_a,q_b]_-=i\beta_{ab}\,,\quad \beta=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix},\quad \beta^2=-I\,. \tag{3}$$

The new wave equation reads

$$\left(\frac{\partial}{\partial x_0}+\alpha_r\frac{\partial}{\partial x_r}+\frac{mc}{\hbar}\beta\right)Q\psi(x_\mu,\xi_i)=0\,. \tag{4}$$

The wave function $\psi(x_\mu,\xi_i)$ is a single-component function of two commuting variables $\xi_1$ and $\xi_2$, as well as the four $x$'s. With $\alpha_0=1$, the equation becomes

$$\left(\alpha_\mu\partial^\mu+\frac{mc}{\hbar}\beta\right)Q\psi=0\,. \tag{5}$$

Furthermore, if we let $\Gamma_\mu=\beta\alpha_\mu$, then we have

$$\left(\Gamma_\mu\partial^\mu-\frac{mc}{\hbar}\right)Q\psi=0\,. \tag{6}$$

In addition to the defining wave equations, the wave function $\psi(x,\xi)$ satisfies two important equations:

(a) A consistency condition: If a function $\psi$ exists which is a simultaneous solution to the four operators $P_a$,

$$P_a=\left(\alpha_\mu\partial^\mu+\frac{mc}{\hbar}\beta\right)_{ab}Q_b\,, \tag{7}$$

then it follows that

$$\left(\partial^\mu\partial_\mu+\frac{m^2c^2}{\hbar^2}\right)\psi=0\,. \tag{8}$$

(b) We must also have the quadratic equation [(6.9) in Ref. 1] (Pauli-Lubanski spin equation) holding

$$W_\sigma\psi=0,\quad W_\sigma=\tfrac{1}{8}\epsilon^{\mu\nu\rho\sigma}S_{\mu\nu}\partial_\rho\,. \tag{9}$$

These two equations then describe a particle with nonzero mass and zero spin.

Dirac gives the general solution to his equation (rewritten in terms of dimensional variables) to be

$$\psi=\left(\frac{\mu\omega mc}{\hbar\pi(P_0+P_3)}\right)^{1/2}\exp\left\{-\frac{1}{2}\frac{mc(\mu\omega/\hbar)(\xi_1^2+\xi_2^2)+iP_1(\mu\omega/\hbar)(\xi_1^2-\xi_2^2)-2iP_2(\mu\omega/\hbar)\xi_1\xi_2}{P_0+P_3}\right\}\exp(-iP\cdot x)\,. \tag{10}$$

## III. GENERALIZATION OF DIRAC'S NEW EQUATION

The generalization of Dirac's new equation is based upon the fact that the 4×4 matrices $\Gamma_\mu$ and the four-dimensional $Q$ are simply related to the smallest totally symmetric irrep of Sp(4) that has a nonvanishing four-vector. The desired generalization then consists of replacing $\Gamma_\mu$ and $Q$ by a general Sp(4) irrep $(p,0)$ having $\binom{p+3}{3}$ dimensions.[5]

The group Sp(2, 2) ≅ SO(3, 2) (where ≅ denotes equivalence) is generated by the ten (Hermitian with respect to $\bar{a}_i|0\rangle=0$ metric) operators

$$\{\vec{J}\}:\quad J_+=a_1\bar{a}_2\,,\quad J_-=a_2\bar{a}_1,\quad J_3=\tfrac{1}{2}(a_1\bar{a}_1-a_2\bar{a}_2); \tag{11}$$

$$\{\vec{K}\}:\quad K_+=\tfrac{1}{2}(a_1^2-\bar{a}_2^2)\,,\quad K_-=\tfrac{1}{2}(\bar{a}_1^2-a_2^2)\,,\quad K_3=-\tfrac{1}{2}(a_1a_2+\bar{a}_1\bar{a}_2)\,; \tag{12}$$

$$\{\vec{V}\}:\ V_+ = -\tfrac{1}{2}i(a_1{}^2 + \bar{a}_2{}^2)\ ,$$
$$V_- = \tfrac{1}{2}i(\bar{a}_1{}^2 + a_2{}^2)\ , \tag{13}$$
$$V_3 = \tfrac{1}{2}i(a_1a_2 - \bar{a}_1\bar{a}_2);$$
$$\{V_0\}:\ V_0 = \tfrac{1}{2}(a_1\bar{a}_1 + \bar{a}_2a_2)\ . \tag{14}$$

The operators $\vec{J}$, $\vec{K}$, $\vec{V}$, and $V_0$ generate on the basis $\{a_1{}^k a_2{}^l|0\rangle\}$ two distinct (integer $J$, half-integer $J$) discrete unitary irreps of the Sp(2, 2) group; $\vec{J}$ and $\vec{K}$ generate the Lorentz subgroup; under the action of $\vec{J}$, $\vec{K}$ the operators $\vec{V}$, $V_0$ form a four-vector. (This is all very familiar from the Majorana equation, as will be discussed in a moment.)

The associated finite-dimensional irreps of Sp(4) are generated by the same $J$, $K$, and $V$ operators; let us denote these bases generically by $Q$. Then we have

four-dimensional irrep: $Q = (a_1, a_2, \bar{a}_2, -\bar{a}_1)$ , (15)

ten-dimensional irrep:

$$Q = \left(\frac{a_1{}^2}{\sqrt{2}}, a_1a_2, \frac{a_2{}^2}{\sqrt{2}}; -a_1\bar{a}_2, \tfrac{1}{2}(a_1\bar{a}_1 - a_2\bar{a}_2), a_2\bar{a}_1; \frac{\bar{a}_1{}^2}{\sqrt{2}}, -\bar{a}_1\bar{a}_2, \frac{\bar{a}_2{}^2}{\sqrt{2}}; \tfrac{1}{2}(a_1\bar{a}_1 + a_2\bar{a}_2)\right)\ . \tag{16}$$

The action of the generators on the basis $Q$ generates $N\times N$-dimensional matrices. For the four-dimensional case one finds

$$[\vec{J}, Q] = (\tfrac{1}{2}\vec{\sigma})Q\ ,$$
$$[\vec{K}, Q] = (\tfrac{1}{2}i\,\rho_2\vec{\sigma})Q\ ,$$
$$[\vec{V}, Q] = (-\tfrac{1}{2}i\,\rho_1\vec{\sigma})Q\ , \tag{17}$$
$$[V_0, Q] = (\tfrac{1}{2}\rho_3)Q\ .$$

Note that Eq. (17) defines a mapping: generator $\to N\times N$ matrix, which preserves the commutation relations of the generators[6] but not the Hermiticity character of the operators.

For the general case we define the $N\times N$ matrices by the rule

$$[\mathcal{O}, Q] = \tilde{\mathcal{O}}Q\ . \tag{18}$$

Now let us construct Dirac's new equation in this language. To do so, we use the $N$-dimensional Sp(4) irrep, with basis $Q$, and the $N\times N$-dimensional matrices $\tilde{V}_\mu$. Dirac's new equation then is

$$(p\cdot\tilde{V} - m)Q\psi = 0. \tag{19}$$

Let us demonstrate now that this equation is Lorentz-covariant. Consider the Lorentz generators $M_{\mu\nu}$ (Ref. 1):

$$M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}\ , \tag{20}$$

where $\{L_{\mu\nu}\}$ denotes the space-time Lorentz generators (acting on $x_\mu$ and $p_\mu$), and $\{S_{\mu\nu}\}$ denotes the Sp(2, 2) generators, $\vec{J}$ and $\vec{K}$. (The notation is $S_{ij} = \epsilon_{ijk}J_k$; $S_{0i} = K_i$.)

Consider then the commutator:

$$\mathcal{C} \equiv [M_{\mu\nu}, (p\cdot\tilde{V} - m)Q]\ . \tag{21}$$

Using the mapping $[V_\mu, Q] = \tilde{V}_\mu Q$, we may replace $p\cdot\tilde{V}Q$ by

$$p\cdot\tilde{V}Q = p\cdot[V, Q]$$
$$= [p\cdot V, Q],$$

since $[p^\mu, Q] = 0$.

Hence the commutator $\mathcal{C}$ becomes

$$\mathcal{C} = [M_{\mu\nu}, [p\cdot V, Q]] - m[M_{\mu\nu}, Q]\ . \tag{22}$$

Using now the Jacobi identity, we find that

$$[M_{\mu\nu}, [p\cdot V, Q]] = [p\cdot V, [M_{\mu\nu}, Q]]\ , \tag{23}$$

since the remaining term in the identity vanishes because $[M_{\mu\nu}, p\cdot V] = 0$.

But we can evaluate the action of $M_{\mu\nu}$ on the basis

$$[M_{\mu\nu}, Q] = [S_{\mu\nu}, Q]$$
$$= \tilde{S}_{\mu\nu}Q\ . \tag{24}$$

Thus the commutator $\mathcal{C}$ becomes

$$\mathcal{C} = [p\cdot V, [M_{\mu\nu}, Q]] - m[M_{\mu\nu}, Q]$$
$$= [p\cdot V, \tilde{S}_{\mu\nu}Q] - m\tilde{S}_{\mu\nu}Q$$
$$= \tilde{S}_{\mu\nu}([p\cdot V, Q] - mQ) \tag{25}$$

(since $\tilde{S}_{\mu\nu}$ are numerical and hence may be commuted through the operators $p\cdot V$).

Using the mapping once again, we find

$$\mathcal{C} = \tilde{S}_{\mu\nu}(p\cdot\tilde{V} - m)Q\ . \tag{26}$$

Thus the operator in Dirac's new equation is not an invariant but transforms covariantly such that if the equation is valid in one frame it is valid in all frames.

We must now examine what types of solutions exist for this equation. We know from the displacement invariance of the equation that $\vec{p}$ may be taken sharp. Assume that $p_\mu$ is spacelike: $p_\mu = (0, 0, p_3, 0)$. Then we find

$$(p_3\tilde{V}_3 - m)Q\psi = 0\ . \tag{27}$$

But $\tilde{V}_3$ is not Hermitian [e.g., see Eq. (17)]. Therefore we have a contradiction and conclude that spacelike momenta are not allowed. (Similarly lightlike momenta are excluded.)

Take $p_\mu$ to be timelike: $(p_0, \vec{0})$. Then we find

$$(p_0\tilde{V}_0 - m)Q\psi = 0\ . \tag{28}$$

Using the explicit diagonal (real) matrix for $\tilde{V}_0$ we find that

$$p_0 = m\,,$$

and (29)

$$\bar{a}_i\,\psi = 0\ .$$

Thus we find that timelike momenta are permitted only for $p_0 = m > 0$. (If $p_0 = -m$, one finds that $a_i\,\psi = 0$, which is not possible with the vacuum state $\bar{a}_i\,|0\rangle = 0$.)

The Pauli-Lubanski operator for this equation has the form

$$W_\mu = p_\alpha M_{\beta\gamma}\epsilon_{\alpha\beta\gamma\mu}\ . \tag{30}$$

In the rest frame we have the form

$$W_i = m J_i\ . \tag{31}$$

Hence for the solution $\bar{a}_i\,\psi = 0$, we find that $s = 0$. All solutions of Eq. (19), whatever the dimensionality of the matrices (except $N = 1$), have spin zero.

It is not difficult to overcome this difficulty and achieve general solutions. We simply replace $(p\cdot\tilde{V} - m)$ by a polynomial operator in $(p\cdot\tilde{V})/m$, choosing the particular operator

$$\mathcal{P}_s(p\cdot\tilde{V}/m) = \prod_{k=-s}^{s}\,[p\cdot\tilde{V}/m - (2k+1)]\ . \tag{32}$$

Then the generalized new Dirac equation becomes

$$\mathcal{P}_s(p\cdot\tilde{V}/m)Q\psi = 0\ . \tag{33}$$

To prove the Lorentz covariance, we proceed as before, commuting the operator $\mathcal{P}_s Q$ with $M_{\mu\nu}$. It is sufficient to consider only the generic term $(p\cdot\tilde{V})^n Q$:

$$\mathcal{C}^{(n)} \equiv [M_{\mu\nu}, (p\cdot\tilde{V})^n Q]\ . \tag{34}$$

Using the mapping, we easily find that

$$(p\cdot\tilde{V})^n Q = [p\cdot V, Q]_{(n)}\,, \tag{35}$$

where

$$[A, B]_{(n)} \overset{\text{def}}{\equiv} [A, [A, \ldots [A, B] \ldots]]\ , \tag{36}$$

where there are $n$ nested commutators on the right-hand side (that is, the $n$th multiple commutator).

Consider now the commutator of $M_{\mu\nu}$ with this multiple commutator. Since $M_{\mu\nu}$ commutes with $p\cdot V$, we find that

$$\begin{aligned}[M_{\mu\nu}, [p\cdot V, Q]_{(n)}] &= [p\cdot V, [M_{\mu\nu}, Q]]_{(n)}\\ &= [p\cdot V, \tilde{S}_{\mu\nu} Q]_{(n)}\\ &= \tilde{S}_{\mu\nu}[p\cdot V, Q]_{(n)}\ . \end{aligned} \tag{37}$$

(These steps use the same relations as in the earlier example.)

It follows that under commutation with $M_{\mu\nu}$, one obtains

$$[M_{\mu\nu}, \mathcal{P}_s(p\cdot\tilde{V}/m)Q] = \tilde{S}_{\mu\nu}\mathcal{P}_s(p\cdot\tilde{V}/m)Q\ . \tag{38}$$

Exactly as for Dirac's case, we conclude that the general new Dirac equation is valid in all frames if it is valid in any one frame.

It is now straightforward to verify that only timelike solutions with positive energy are allowed. In the rest frame one obtains

(a) $p_0 = m$,

(b) $(\bar{a}_1)^k(\bar{a}_2)^{s+1-k}\psi = 0$, for $k = 0, 1, \ldots, s+1$ . (39)

(This latter result uses the explicit $N\times N$ matrix $\tilde{V}_0$ and the particular form given for $\mathcal{P}_s$.) Using the Pauli-Lubanski operator, one sees that the rest-frame solutions $\psi$ correspond to spin $s$, and all lower spins. If we wish to have only spin $s$, then we must specify that (in the rest frame) $\psi$ contains precisely $s$ quanta, and that the $N\times N$ matrices have $N = \binom{s+3}{3}$.

It will not have escaped the reader that these results are all very closely related[7,8] to Majorana's equation[2]:

$$(p\cdot V - m)\phi = 0. \tag{40}$$

In a sense, the new Dirac equation is but a spin-projection of Majorana's equation.[9] While technically correct, this view is nonetheless quite misleading, since the real point of Dirac's new equation is the ingenious way in which the spacelike and lightlike Majorana solutions are eliminated. To appreciate this point, note that the operators $V_\mu$ are all Hermitian; hence we can obtain, for example, a spacelike solution to the equation $(p_3V_3 - m)\phi = 0$.

It is precisely as a consequence of the properties of the mapping: $V_\mu \to \tilde{V}_\mu$, (given by $[V_\mu, Q] = \tilde{V}_\mu Q$)—which preserves commutators, but not Hermiticity—that Dirac's construction eliminates these unwanted solutions for all spins and masses. [Majorana's equation relates $m$ and $s$ by $m = m_0/(s+\frac{1}{2})$.]

We have yet to discuss the remarkable property of the new Dirac equation that electromagnetic interactions, introduced by the minimal coupling prescription, are forbidden.[10]

Dirac[1] stated this result without explicit proof. Hence it is useful to begin with the example of the spin-0 equation, over 4×4 matrices. Let $p \to \Pi \equiv p - eA/c$. Then we wish to prove that the equation

$$(\Gamma\cdot\Pi - m)Q\psi = 0 \tag{41}$$

has no solutions for which $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ does not vanish.

We have restricted our attention to the 4×4 case in which the matrices $\Gamma$ are of the Dirac type. That is,

$$\tilde{V}_\mu \equiv \Gamma_\mu \underset{4\times4}{\longrightarrow} \begin{cases} -\frac{1}{2} i \rho_1 \vec{\sigma}, & \mu = 1, 2, 3 \\ \frac{1}{2} \rho_3, & \mu = 4. \end{cases}$$

For the 4×4 case these Γ's satisfy both commutation and anticommutation relations. Accordingly, we multiply Eq. (41) on the left-hand side by $(\Gamma \cdot \Pi + m)_{4\times4}$ and obtain

$$[(\Pi \cdot \Pi - m^2) + \tilde{S}_{\mu\nu} F_{\mu\nu}] Q\psi = 0. \tag{42}$$

It is a remarkable property of the present realization that there exist *no* bilinear operator realizations for the unitary group generators that do not belong to the symplectic subgroup.[5] Specializing to the case at hand $(n=2)$, we may state that bilinear operators in $\{a_i, \bar{a}_i\}$ corresponding to the six Dirac matrices 1, $\rho_3\vec{\sigma}$, $\rho_1$, and $\rho_2$ do not exist. We may exploit this fact by multiplying Eq. (42) on the left first by the 4×4 matrix $-i\rho_2$, and then by the adjoint vector $\bar{Q} \equiv (\bar{a}_1, \bar{a}_2, -a_2, \bar{a}_1)$. One obtains the result:

$$F_{\mu\nu}^{\text{dual}} S_{\mu\nu} \psi = 0 , \tag{43}$$

where $F_{\mu\nu}^{\text{dual}}$ is the dual electromagnetic field vector, and $S_{\mu\nu}$ is the bilinear operator (not the numerical matrix) defined in Eq. (14).

Similarly using the 4×4 matrices $\rho_1$ and $\rho_3\vec{\sigma}$, instead of $-i\rho_2$, we can in the same way derive the four relations:

$$F_{\mu\nu}^{\text{dual}} V_\nu \psi = 0. \tag{44}$$

From Eqs. (43) and (44) it follows that the vector $\psi$ is invariant to all transformations of the group Sp(2, 2). (This results from the fact that the given equations suffice to generate the group itself.)

The only such invariant-vector is the constant vector $\psi(x)$; that is, there is no $a_1, a_2$ dependence. But this result implies that Dirac's new equation does not exist. From this contradiction we deduce that $F_{\mu\nu}$ itself vanishes. This completes the proof that Eq. (41) has no solution if $F_{\mu\nu} \neq 0$.

It is natural now to ask if the more general $N\times N$ equation for (41) is also impossible with $F_{\mu\nu} \neq 0$. This result is correct; but it is simplest to proceed indirectly.

Let us instead note that the generalized equation

$$\mathcal{P}_s(\tilde{V} \cdot p/m) Q\psi = 0 \tag{45}$$

admits an alternative, but redundant, form:

$$\prod_{\otimes}^{2s+1} [(\tilde{V} \cdot p - m) Q]_{4\times4} \psi = 0. \tag{46}$$

[The equivalence of Eqs. (45) and (46) is most easily shown in the rest frame. The form (46) is redundant in that the outer product of $(2s+1)$ identical structures reduces to the irrep $(2s+1, 0)$ displayed explicitly in Eq. (45).]

The advantage of the form given in Eq. (46) is that we may go over to the minimal interaction $p \to \Pi$, and then repeat the previous proof. In detail, one multiplies Eq. (46) on the left by $(\tilde{V} \cdot \Pi - m)_{4\times4}$ and then multiplies by $\bar{Q}(-i\rho_2)$ on the left. One thus obtains

$$F_{\mu\nu}^{\text{dual}} S_{\mu\nu} \Phi \equiv (F_{\mu\nu}^{\text{dual}} S_{\mu\nu}) \prod_{\otimes}^{2s} [(\tilde{V} \cdot \Pi - m) Q]_{4\times4} \psi = 0, \tag{47}$$

and we once again conclude that if $F_{\mu\nu} \neq 0$, then $\Phi = 0$. But this establishes a recursion loop, whereby the existence of electromagnetic interactions for spin $s$ requires the existence for spin $s - \frac{1}{2}$. Since the case $s = 0$ has been shown to be impossible, this establishes that minimal electromagnetic interactions are impossible for all of the generalized new Dirac equations.

Now let us return to Eq. (41). The proof that minimal quantum electrodynamics (QED) is impossible has been established only for the 4×4 case. Conceivably the $N\times N$ case might escape this restriction. This, however, can be seen to be impossible by observing that if Eq. (41) is correct for $N\times N$ $(N > 4)$, then (by suitably multiplying the appropriate operator in from the left) we obtain the equation

$$(N\times N) \quad \mathcal{P}_s(\tilde{V} \cdot \Pi/m) Q\psi = 0 , \tag{48}$$

which is equivalent to $p \to \Pi$ in Eq. (42). This latter has been shown to be impossible; hence Eq. (41) has no solutions, for any $(N\times N)$ case, with $F_{\mu\nu} \neq 0$.

In the interest of completeness, let us now demonstrate that all of the generalized equations possess a conserved current with nonnegative density. In order to construct the conserved currents, we introduce

$$\mathcal{P}_{s\hat{j}}(p \cdot \tilde{V}/m) = \prod_{k=-s}^{j-1} \prod_{k=j+1}^{s} [p \cdot \tilde{V}/m - (2k+1)] \tag{49}$$

and define

$$\psi_{\hat{j}} = \mathcal{P}_{s\hat{j}} Q\psi . \tag{50}$$

Since the factors on the right-hand side of (32) commute, $\psi_{\hat{j}}$ satisfies

$$[i\partial \tilde{V}/m - (2j+1)] \psi_{\hat{j}} = 0 , \tag{51}$$

and also

$$\psi_{\hat{j}}^\dagger \tilde{V}^{0\dagger} [-i\overleftarrow{\partial} \cdot \tilde{V}/m - (2j+1)] = 0 . \tag{52}$$

Hence, a conserved current can be given for every $j$ by

$$J_{\hat{j}}^\mu = \iint d\xi_1 d\xi_2 \psi_{\hat{j}}^\dagger \tilde{V}^{0\dagger} \tilde{V}^\mu \psi_{\hat{j}}. \tag{53}$$

It is clear from Eq. (53) that the charge density is non-negative in each case. (It is of interest to note that one might just as well use linear combinations, with positive coefficients, of the different $J^{\mu}_{j}$. This freedom might possibly allow one to construct a current which has no contributions from lower-spin solutions.)

## IV. AN INTERPRETATION IN TERMS OF SUBSTRUCTURE

We seek in this section to develop an interpretation of Dirac's new equation in terms of a substructure, namely, that the set of generalized equations describes covariantly two interacting subparticles with a fixed mass-spin relationship, $m^2 \propto s$. In order to accomplish this analysis, we must first discuss the ancillary problem as to how quantal relativistic dynamics, for other than free particles, can be treated at all. This is a nontrivial problem in itself.

The problem of constructing a quantal relativistic dynamics is the problem of constructing a realization of the ten generators of the Poincaré group: $P_\mu$ and $M_{\mu\nu} = \{\vec{J}, \vec{K}\}$ (the total four-momentum and total angular momentum-boost operators, respectively). In a basic paper several years ago, Dirac[3] developed three possible approaches to Hamiltonian relativistic dynamics based on three subgroups of the Poincaré group $\mathcal{P}$:

(a) *Instant dynamics.* Subgroup E(3) generated by by $\{\vec{P}, \vec{J}\}$;

(b) *Point dynamics.* Subgroup $\mathcal{L}$ generated by $\{M_{\mu\nu}\}$;

(c) *Front dynamics.* Subgroup generated by

$$\{P_i, P_+, J_{12} = J_3, K_{-i}, M_{03}\}.$$

In each of these dynamics, the ten generators of $\mathcal{P}$ are divided into two classes: (a) the generators of the subgroup, and (b) the remaining generators (designated collectively as Hamiltonians). A solution to relativistic dynamics—according to Dirac—consists first of all of a solution to the subgroup generators by postulating a direct-product structure, followed by postulating interactions in the remaining generators (Hamiltonians) whose solutions are then the real problem to be solved.

Expressed differently, Dirac takes the subgroup generators to be simple, such that a given subgroup generator, say $P_1$, is merely the direct sum of the $P_1$ operators of each of the $n$ particles in the substructure. That is,

$$P_1 \overset{\text{def}}{\equiv} \sum_{i=1}^{n} P_1^{(i)} .$$

Expressed group-theoretically, this is equivalent to postulating a direct-product structure for the subgroup as a whole.

Dirac's procedure is by no means the only one possible, but it corresponds directly to physical intuition. This is particularly clear for the instant and point forms; in these two cases the subgroups, E(3) and $\mathcal{L}$, respectively, have the physical significance of kinematical symmetries. In the instant form especially, the subgroup E(3) appears also as a basic symmetry of nonrelativistic (quantal or Newtonian) mechanics; the kinematic nature of this symmetry is apparent in the standard procedures wherein these momenta are, by definition, the sum of the individual momenta of the constituents. Kinematical independence implies commuting operators and a direct product group structure.

At this point we come to a fundamental distinction between classical and quantal dynamics, a distinction which is basic to the existence of our proposed interpretation.

Let us first reconsider classical dynamics in the instant form. The subspace defined by the given instant ($t=0$) is a three-dimensional surface which intersects the world line of every particle once and only once. A Hamiltonian dynamics consists then of determining in the next instant the next intersection point on each world line. The question as to whether there is a "dynamics" within the (three-dimensional) subspace is clearly absurd: one deals with three-dimensional events in this subspace, not particles—there is no vestige of "world lines" within the subspace and hence no dynamics. Quantum mechanically, this situation still obtains, since the three momenta conjugate to the position $(x, y, z)$ have unbounded spectra, $-\infty$ to $+\infty$, and there exista a well-defined (Newton-Wigner) position operator[11] localizing the particles. Thus in both the classical and quantum-mechanical approaches, the subspace defined at an instant has a kinematic (and not a dynamic) character; this leads directly to Dirac's prescription.[12-14]

The situation for the point form of dynamics is qualitatively similar.

The situation for the front form is very different, but only for the quantal case. Classically, the three-dimensional subspace, tangent to the light cone, once again intersects every particle world line in a single point. Once again, any question as to a dynamics of these three-dimensional events within the subspace is absurd—the points of intersection simply exist kinematically, and that ends the description.

Quantum mechanically (for a massive particle $P^2 = m^2$), the situation changes, for now the momenta $P_+$ and $P_-$ have spectra confined to the (open) half-line: $0 < P_\pm < \infty$. In consequence, the conjugate variables $x_\pm$ are timelike in character, and cannot be localized. One variable, say $x_-$, defines the front: $x_-=0$. The other variable $x_+$ is the time within the front subspace, and as a time variable defines a world line for a particle within the front subspace. The two position variables, $x_1$ and $x_2$ have momenta $P_1, P_2$ with unbounded spectra: there exists a well-defined (Newton-Wigner) position operator to localize these coordinates.

We conclude from this discussion that a kinematical description of points within the front subspace is not possible quantum mechanically, and that the correct description within the front actually requires considerations of a dynamical nature. This implies that the operator $P_-$ must be adjoined to the seven generators of Dirac's classical front dynamics.

It is quite remarkable that the Poincaré subgroup generated by this set of eight operators has an immediate and clear physical significance: nonrelativistic (2+1 Galilean) quantum mechanics for the subsystem. The possibility of a relativistic form of dynamics based upon regarding this subgroup, not as kinematical, but as an interactive, dynamical subsystem in its own right is indeed both suggestive and natural.[15]

In order to make this clear, let us consider this subgroup in detail. To do so we change the metric so as to adapt it to the front form. First introduce the indices + and − and then denote

$$x_\pm=(x_0\pm x_3) \quad (\text{where } x_0=ct) . \tag{54}$$

The metric, using the indices 1, 2, +, − becomes

$$g_{ij}=\delta_{ij} \quad (i,j=1,2), \tag{55}$$

$$\begin{aligned} g_{+-}&=g_{-+}\\ &=\tfrac{1}{2},\\ g_{++}&=g_{--}\\ &=0 . \end{aligned} \tag{56}$$

The subgroup algebra $\mathfrak{F}$ for the front form is then generated by the eight operators $(i=1,2)$

$$\mathfrak{F}=\{P_i, P_-, J_{12}=J_3, K_{-i}; M_{03}; P_+\} ,$$

which obey the commutation rules:

*(a) (2+1) Galilei group generators:* $\mathcal{G}$, where

$$\begin{aligned} &[J_3,P_i]=i\epsilon_{3ij}P_j , && [K_{-i},K_{-j}]=0,\\ &[J_3,K_{-i}]=j\epsilon_{3ij}K_{-j} , && [K_{-i},P_j]=i\delta_{ij}P_+ ,\\ &[J_3,P_-]=0 , && [K_{ij},P_-]=2iP_i ,\\ &[P_i,P_j]=0 ,\\ &[P_i,P_-]=0 . \end{aligned} \tag{57a}$$

*(b) Mass generator:* $P_+$, where

$$[P_+,\mathcal{G}]=0 . \tag{57b}$$

*(c) Scaling generator:* $M_{03}$, where

$$\begin{aligned} &[M_{03},J_3]=0 ,\\ &[M_{03},K_{-i}]=iK_{-i} ,\\ &[M_{03},P_i]=0,\\ &[M_{03},P_\pm]=\pm iP_\pm . \end{aligned} \tag{57c}$$

These commutation rules can be recognized as the commutation relations of the Galilei group $\mathcal{G}$ (in 2+1 dimensions) together with a dilation (scaling) operator $D\equiv M_{03}$. For this interpretation one must identify the operator $\frac{1}{2}P_-$ as the Galilei-group Hamiltonian $H_\mathcal{G}$ and the operator $P_+$ as the mass $M_\mathcal{G}$ for the Galilei group.

We will show that the explicit solutions to Dirac's new equation are consistent with this proposed interpretation (and in fact suggested it).

We are therefore led to postulate a quantal subdynamics of interacting point particles and accordingly identify the generator $\frac{1}{2}P_-$ with the Hamiltonian:

$$\tfrac{1}{2}P_-\to H_\mathcal{G}=H_{\text{c.m.}}+H_{\text{int}} . \tag{58}$$

We will arbitrarily impose the requirement that both the center-of-mass Hamiltonian $H_{\text{c.m.}}$ and the "internal" Hamiltonian $H_{\text{int}}$ are separately Galilean-invariant, and scale-covariant, under the seven generators of the front form. (This requirement will turn out to be consistent with our proposed interpretation of Dirac's new equation, but for more general dynamical situations, only $H_\mathcal{G}$ itself need obey this symmetry.)

It is a remarkable property of the Galilei group that the possible forms of (Galilean-invariant) interactions are limited exclusively to the gauge-invariant form[16]:

$$H_{\text{int}}=\frac{(P_i-A_i)^2}{2\mu}+V , \tag{59}$$

where $A_i$ and $V$ are arbitrary functions of the coordinates of the coordinates of the Galilei space.

This limitation, combined with the scale invariance (dilation operator $D$), severely restricts the possible interactions. For the explicit realization of scale invariance implied by Dirac's new equation, we are essentially limited—as will be shown—to harmonic oscillator interactions alone.

To proceed further let us consider once again the explicit operators that enter into Dirac's new equation. It is clear that the oscillator variables are linked to the Minkowski variables through the operators $M_{\mu\nu}=L_{\mu\nu}^{\text{Minkowski}}+S_{\mu\nu}$; a choice of a subgroup of $\mathcal{P}$ fixes a set of operators $S_{\mu\nu}$. But for any given set of the $S_{\mu\nu}$, the specific realization requires also a specification of the oscillator variables to be taken diagonally; for example, one might take $a_1$ and $a_2$ as the sharp variables; or one might take $\xi_1$ and $\xi_2$ as the variables. Each choice of two oscillator variables fixes a certain subspace; the operators leaving this subspace-invariant determine a set of $S_{\mu\nu}$ which may, or may not, agree with chosen subgroup of $\mathcal{P}$. The two choices (sharp oscillator variables and $\mathcal{P}$ subgroup) must therefore be compatible.

In order to interpret the internal space as the (two-dimensional) relative coordinates between two (2+1, Galilean) point particles, we are forced to identify $\xi_1$ and $\xi_2$ as position variables within the front. This choice implies that the proper set of operators[1] be $S_{12}$, $S_{03}$, $S_{-1}$, and $S_{-2}$. This set of operators is compatible only with the front form.[17] Hence, from a very different consideration, we are once again led to conclude that if an interpretation in terms of subparticles for Dirac's equation exists, then we are forced to consider the (quantal) front form of the dynamics.

The most critical part of the interpretation concerns the scaling behavior. Let us now consider this explicitly.

The scaling is produced by the action of the operator $M_{03}$ and obeys the commutation rules given in Eq. (57). Since $M_{03}=L_{03}+S_{03}$, we may separate the scaling into two parts, calling $S_{03}$ the internal scaling operator.

Under the operator $L_{03}$ we see that $P_+$ and $H_{\mathcal{G}} = \frac{1}{2}P_-$ both scale, but in opposite senses (to keep $P_+P_-$ invariant). The momenta $P_i$ and the $x_i$ as well are scale-invariant.

By contrast the momenta $\pi_i$ and the coordinates $\xi_i$ scale under the operator $S_{03}$ (in opposite senses, so that $[\pi, \xi]=$invariant).

We interpret $P_+$ now as the total mass of the Galilean system; this implies that $P_+ = m_1 + m_2$, where $m_i$ are the masses of our Galilean subparticles. Accordingly, we are forced to conclude that $m_i$ scale under $L_{03}$, and that the reduced mass $\mu = m_1 m_2/(m_1+m_2)$ must also scale under $L_{03}$. It follows that $\omega$ must scale (under $L_{03}$) oppositely to $\mu$, in order that the dimensionless variables $(\mu\omega/\hbar)^{1/2}\,\xi_i$ scale properly under $M_{03}$.

The most economical procedure now is to present our proposed subdynamics and demonstrate that it leads uniquely to Dirac's equation.

We assume that we deal with two particles: mass $m_1$ and position $\vec{R}_1$ and $(m_2, \vec{R}_2)$, defined in a (2+1) Galilean world. Introducing center-of-mass and relative (internal) coordinates, we take the Hamiltonians to be

$$H_{\text{total}} = H_{\text{c.m.}} + H_{\text{int}}\,, \tag{60}$$

$$H_{\text{c.m.}} = \frac{P_1^{\,2}+P_2^{\,2}}{2(m_1+m_2)}\,, \tag{61}$$

$$H_{\text{int}} = \frac{1}{2\mu}\sum_{i=1,2}(\pi_i - A_i)^2 + \tfrac{1}{2}\mu\omega^2(\xi_1^{\,2}+\xi_2^{\,2})\,, \tag{62}$$

with

$$A_1 = \left(\frac{\mu\omega}{mc}\right)(P_2\xi_2 - P_1\xi_1)\,, \tag{63}$$

$$A_2 = \left(\frac{\mu\omega}{mc}\right)(P_1\xi_2 + P_2\xi_1)\,. \tag{64}$$

Here $\omega$ and $m$ are fixed numerical constants, so that we may vary both the frequency of the harmonic-oscillator interaction and the "vector potential" $A_i$ independently. [Note that we have used the full freedom allowed by Galilean symmetry; cf. Eq. (59).]

The eigenenergies of these Hamiltonians are immediate:

$$H_{\text{c.m.}} \to \frac{P_1^{\,2}+P_2^{\,2}}{2(m_1+m_2)}\,, \tag{65}$$

$$H_{\text{int}} \to (N+1)\hbar\omega\,. \tag{66}$$

The corresponding wave functions are equally easy. For the ground state ($N=0$) we have

$$\begin{aligned}\psi &= \psi_{\text{c.m.}}\,\psi_{\text{int}} \\ &= \exp\left[i(P_1x_1+P_2x_2)/\hbar - i\left(\frac{P_1^{\,2}+P_2^{\,2}}{2(m_1+m_2)}\right)x_+/\hbar\right]\left(\frac{\mu\omega}{\hbar}\right)^{1/2}\exp\left[-\tfrac{1}{2}\left(\frac{\mu\omega}{\hbar}\right)(\xi_1^{\,2}+\xi_2^{\,2})\right] \\ &\quad\times\exp\left\{-i\left(\frac{\mu\omega}{2\hbar mc}\right)[P_1(\xi_1^{\,2}-\xi_2^{\,2}) - 2P_2\xi_1\xi_2]\right\}e^{-i\omega x_+}\exp\left[-i\,\frac{(m_1+m_2)c}{2\hbar}\,x_-\right].\end{aligned} \tag{67}$$

Now let us consider the scaling properties of these solutions. Under the internal scaling operator $S_{03}$, we see that the energy eigenvalue is invariant as it should be, but that the explicit solution is not scale-invariant in form.

Let us now exploit this freedom of internal scale to choose a particular scale:

$$\xi_i \to \xi_i' \equiv \left(\frac{mc}{m_1+m_2}\right)^{1/2}\xi_i\,, \tag{68}$$

$$\pi_i \to \pi_i' \equiv \left(\frac{m_1+m_2}{mc}\right)^{1/2} \pi_i . \tag{69}$$

The particular merit of this choice is that under the combined scaling $M_{03}=L_{03}+S_{03}$, the new choices are invariant, and now are analogous to the variables $P_i$, $x_i$ which are invariant under $M_{03}$.

The (normalized) new ground-state wave function then assumes the form

$$\text{scaled}\,\psi = \left|\frac{\mu\omega}{\hbar}\ \frac{m}{(m_1+m_2)}\right|^{1/2} \exp\left[i\left(\frac{P_1x_1+P_2x_2}{\hbar}\right)\right]\exp\left[-\tfrac{1}{2}\frac{\mu\omega}{\hbar}\ \frac{m}{(m_1+m_2)}\ (\xi_1{}^2+\xi_2{}^2)\right]$$
$$\times\exp\left\{-i\frac{\mu\omega}{2\hbar(m_1+m_2)}\ [P_1(\xi_1{}^2-\xi_2{}^2)-2P_2\xi_1\xi_2]\right\}\exp\left\{-i\left[\frac{P_1{}^2+P_2{}^2}{2(m_1+m_2)}+\hbar\omega\right]\frac{x_+}{\hbar}\right\}\exp\left[-i\frac{(m_1+m_2)c}{2\hbar}\ x_-\right] . \tag{70}$$

Under the combined scaling $M_{03}$ this wave function is now scale-invariant.

To be consistent, however, with the requirement that $P_0{}^2=\vec{P}^2+m^2$, or in the front form $P_+P_-=P_1^2+P_2{}^2+m^2$, we see that we must require that

$$\tfrac{1}{2}P_- \to H_S=\frac{P_1{}^2+P_2{}^2}{2(m_1+m_2)}+\hbar\omega$$

(from two-particle solution), (71)

$$P_-=\frac{P_1^2+P_2^2+m^2c^2}{(P_+/c)} \quad (\text{from } \vec{P}^2=m^2). \tag{72}$$

This is consistent if we take

$$m^2c^2=2(m_1+m_2)\hbar\omega, \tag{73}$$

since

$$P_+=(m_1+m_2)c$$

is already required. (It is gratifying to note that in this last expression $m^2$ is scale-invariant.)

It remains only to remark that the scaled wave function given by (70)—together with Eq. (73) for $m^2$—is precisely the wave function for the new Dirac equation. Let us note that in making this comparison, the momentum eigenstate is expressed in front form[18]:

$$e^{iP\cdot x/\hbar}=\exp\left[i\left(\frac{P_1x_1+P_2x_2}{\hbar}\right)\right]\exp\left(-i\frac{P_-x_+}{2\hbar}\right)$$
$$\times\exp\left(-i\frac{P_+x_-}{2\hbar}\right). \tag{74}$$

We conclude that *the scale-invariant Galilean subdynamics of two point particles interacting via action-at-a-distance forces given by Eq. (70), leads to a wave function that agrees exactly with the Dirac solution.*

Since we already have in this Galilean subdynamics the eight generators $\mathfrak{F}$, we may simply adjoin the remaining two generators [using Eq. (57)] and conclude that the system properly extends to a Poincaré world, satisfying Dirac's new equation.

For higher states of excitation of the system, we remark that we have, in Eq. (70), actually imposed a mass-spin restriction. Since the spin is given by $N\hbar$, we see that Eq. (73) becomes

$$m^2c^2=2(m_1+m_2)(N+1)\hbar\omega, \tag{75}$$

or

$$m^2=\left[\frac{2(m_1+m_2)\hbar\omega}{c^2}\right](2s+1) . \tag{76}$$

The mass-spin relation is of the form $M^2\propto J$, that is, of the Chew-Frautschi form, experimentally indicated for hadrons.

Before concluding this section it may be useful to answer a question that must surely have arisen: Granted, say, everything we have claimed about the subdynamics is true why do we feel that this hypothetical Galilean subsystem has anything to do with two genuine (Poincaré) particles? To put the question differently, how can two (Poincaré) particles each with four (Minkowski) coordinates ever be related to the (Dirac) system having but six $(x_\mu, \xi_1, \xi_2)$ coordinates?

To answer these questions, let us consider our Galilean subdynamics in the limit that the interaction vanishes. The two-particle Galilean system then takes on the structure of two free particles, since

$$H_{\text{c.m.}} \to (P_1{}^2+P_2{}^2)/2m_1$$

and (77)

$$H_{\text{int}} \to (\pi_1{}^2+\pi_2{}^2)/2m_2 .$$

We may go from the internal and c.m. coordinates back to the original coordinates. The key point is that we may now follow the world line of either particle separately into the Poincaré world, and each of the two world lines would appear as a free particle. It is in this precise limiting sense that we assert that we have, in the absence of any Galilean interaction, two independent particles.

(Let us note explicitly, however, that such a limiting procedure is not allowed within the frame-work of the new Dirac equation, since the limit is singular in the Dirac system. To see this, note that in the harmonic oscillator the kinetic and potential energies are, on the average, equal. Thus for vanishing interaction the interparticle separation becomes arbitrarily large. But the Dirac equation is predicated on the fact that $P_0 > 0$ vs $P_0 < 0$ is distinguished precisely by the behavior at infinity; this distinction is lost in the limit and the Dirac equation acquires, discontinuously, solutions with $P_0 < 0$. Hence the limit is singular and forbidden.)

The fact that, in the front frame, the particles share two time coordinates ($x_\pm$) is a kinematically correct Galilean statement, as we have previously discussed. This situation sharply contrasts with the instant and point forms, where there exists but one time coordinate.

The fact that the interaction seems to reduce the number of independent coordinates (when integrating from the Galilean subdynamics to the Poincaré world) is surprising, but not without precedent, since the concept of "frozen out" degrees of freedom is a familiar physical example of such an effect.

Let us now make one further remark on scaling. We saw that the internal Hamiltonian must scale as $(m_1+m_2)^{-1}$ under the $M_{03}$ generator, but that $H_{\text{int}}$ was scale-invariant under $S_{03}$. This is a severe requirement and, for power-law potentials, limits the interaction to the harmonic oscillator alone (with the freedom of adding a vector potential). At first glance this is surprising, since the free Hamiltonian (no interaction) must surely be allowed. The point here is that we have also required that the momenta $\pi_i$ and coordinates $\xi_i$ scale under $S_{03}$. It is consistent to drop this requirement, and then one finds $H_{\text{int}} = \pi^2/2\mu$ is allowed. This scales properly under $L_{03}$, but then $S_{03}$ no longer exists. The scaling requirements under $S_{03}$ were chosen to agree with the desired result (the new Dirac equation) and this choice (plus Galilean invariance) forces a unique answer. [The peculiar vector potential that was found is necessary to implement the required symmetry, so that in the final answer (extended to the Poincaré world by adjoining the remaining generators) the final system does not distinguish any particular front, tangent to the light cone, but displays full Poincaré symmetry.]

The crucial importance of scaling may be seen from a very different aspect. One knows that the Galilean group possesses a continuum of superselection subspaces, distinguished by the mass. How can such a structure be compatible with an embedding in the Poincaré group, which possesses only two superselection spaces? The answer, clearly, is that the embedding implies the existence of a scaling operator, $M_{03}$, which ties together the mass superselection subspaces, leaving only the integer vs half-integer splitting common to both groups.

## V. FURTHER DISCUSSION

Viewed as a possible model for hadron dynamics, the new Dirac equation (and its generalization) has severe disadvantages, the most striking being the inability to introduce consistently conventional QED. However suggestive the deduction of the hadronic Regge trajectory rule, $m^2 \propto J$, may be, it is clear that drastic modifications in the equation are necessary. It is premature to speculate on such modifications here, although it is very likely that $CPT$ can be put in "by hand" and thus rescued.

To concentrate on this aspect of the new Dirac equation would, we feel, be to miss the main point. In our view the (generalized) new Dirac equation should be viewed as the first relativistic (covariant) solution of a two-body problem in which one may obtain two distinct answers: (a) for no interaction, two free (Poincaré) particles, and (b) with interaction, a complete set of Poincaré $(m, s)$ solutions in which $m^2 = (\text{constant})(2s+1)$. From this point of view the generalized new Dirac equation is no more and no less than the solution for a relativistic harmonic oscillator; the separate equations may be viewed as covariant statements that the oscillator possesses $N$ (or fewer) quanta.

More importantly, our interpretation of the new Dirac equation suggests a very different attitude toward quantal relativistic dynamics in which one uses the front form to define an interacting subdynamics which is then extended into the Poincaré world.[19,20] Such a procedure is—as we discussed in Sec. IV—not possible for classical physics. To distinguish this procedure from Dirac's (classical) front form of dynamics, we might call it the "method of quantal Galilean subdynamics."

From the point of view of group theory such a method is most natural. One defines representations of a subgroup ($\mathcal{G}+\mathfrak{D}$ here) and extends these representations to the full group ($\mathcal{P}$) by the Mackey-Wigner-Frobenius technique of induced representations.[21] Group theoretically, the primary problem was very different. Given an irrep of the subgroup, when is the induced representation irreducible? This is solved by the familiar Wigner little group (stability group) construction. For dynamical problems the situation is different; one expects that the solution will be a set of Poincaré irreps with the mass, spin, and multiplicities characterizing the interaction. It is quite fortunate that the Galilean subdynamics is not only the largest subgroup, but also the subgroup for which one has a full grasp of the quantal subdynamics.[22]

Considered pragmatically, one might view progress in the last few years in understanding hadron physics as ingenious usage of nonrelativistic quantum mechanics far outside its proper domain; the success of the nonrelativistic quark model is the most obvious example. To take a speculative, but clearcut example, we found that for interacting (Schwinger) dyons,[23] to avoid an enormous nucleonic electric dipole moment, one had to postulate nonrelativistic dynamics as absolute (not approximate).

Similarly the recent emphasis on the "infinite-momentum frame" (IMF) as, for example, in Feynman's parton model,[24] may be viewed as an approach to nonrelativistic dynamics (eliminating backward going diagrams). We would like to emphasize that the IMF approach, though very similar in actual results, is in fact quite distinct logically from front subdynamics. The IMF in mathematical discussions[25] is viewed as a contraction limit (and not a subgroup) to the Poincaré group [just as the limit $c\to\infty$ is a contraction limit yielding $(3+1)$ Galilean mechanics]. As discussed by Feynman, one considers an active transformation[26] in which a massive physical system takes on large, but nonetheless finite, momentum. By contrast, the front subdynamics is a passive redescription,[26] valid without restriction to large physical momenta, and applicable, in principle, to any physical system. Feynman's (parton) discussion[27] is, from this view, a brilliant intuitive justification for the way in which instant dynamics takes on the character of front subdynamics for systems colliding with very large relative momenta.

Let us point out that the Regge trajectory relation $m^2 \propto J$ has also been obtained recently[28] by using Gell-Mann's $SL(3,R)$ symmetry as applicable to hadrons.[29] This result was obtained by an explicit realization of the $SL(3,R)$ symmetry on a two-boson structure. Since the new Dirac equation similarly is a realization on two bosons, the question immediately occurs: Can these two models be united into a single (covariant) model? More precisely, is there a larger group containing both $SL(3,R)$ and the $SO(3,2)$ group used in the Dirac construction? Both $SL(3,R)$ and $SO(3,2)$ do indeed fit into noncompact versions of the $A_3$ algebra, but (unfortunately) the two subgroup realizations are not simultaneously compatible.[30]

Let us conclude by discussing a possible significance to the fact that Dirac's new equation does not allow a QED coupling. Let us assume, notwithstanding, that such a coupling does exist; we will indicate a "Gedankenexperimental" contradiction. From our interpretation, we must assume that one or both of the two subparticles will bear electric charge. Now let us assume that, somehow, the interaction is turned off, and we get two free particles. By means of electromagnetic interactions we would thus have the possibility of experimentally observing a system having half-integer spin become a system having only integer spin. This contradicts the Wick-Wightman-Wigner superselection rule.[31] We conclude that one or more of our assumptions (of this paragraph) is wrong. The new Dirac equation chooses to deny the possibility of an electromagnetic (minimal) coupling, so that it is the experimental observability which is denied. One might also conclude that the system does not permit the interaction to be removed (and hence the spin of the constituents to be determined). Precisely these two properties (lack of external electromagnetic coupling to the constituents, and finite separation distance), underlay our kinematic model for quarks leading to the concept of finite-size spinors.[32] The two models differ, however; in particular, the new Dirac equation has yet to be generalized to incorporate SU(3). Hopefully this is possible; it constitutes a challenge.

*Note added in proof.* The material in Sec. IV has been revised in order to eliminate the confusion between $P_+$ and $P_-$ that existed in our original version. We wish to acknowledge the aid of, and thank, Professor L. P. Horwitz for helpful discussions.

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