Renormalization Group and Axial-Vector Current in Two-Dimensional Quantum Electrodynamics*

R. J. Crewther, Sun-Sheng Shei, and Tung-Mow Yan
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14850
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An exactly soluble model of the renormalization group for two-dimensional quantum electrodynamics is presented. The theory has a nontrivial eigenvalue at which it is scale-invariant and some operators acquire anomalous dimensions. The anomalous constant associated with the axial-vector current is independent of the coupling constant away from the eigenvalue, but vanishes at the eigenvalue.

I. INTRODUCTION

Now that we have experienced difficulty[1-3] in dealing with axial-vector currents for four-dimensional field theories in the Gell-Mann–Low limit,[4] it becomes worthwhile to check what happens in soluble or almost soluble models. Inevitably, theories of this type are two-dimensional in space-time; their simplicity can be directly attributed to properties peculiar to two-dimensional geometry. An important feature of these models is that, once an ordinary current J_μ has been found, there is a well-defined gauge-invariant axial-vector current $J_{5\mu}$ given by[5]

$$J_{5\mu}=\epsilon_{\mu\nu}J^{\nu}. \tag{1.1}$$

In contrast with the situation[3,6] for four-dimensional quantum electrodynamics, the scaling properties of J_μ and $J_{5\mu}$ are necessarily identical.

This paper contains an exactly soluble quantum field-theoretic model for the renormalization group, and analyzes the behavior of the anomaly associated with the corresponding current $J_{5\mu}$. Our example is based on quantum electrodynamics (QED) for massless fermions in two dimensions, for which explicit solutions were first given by Schwinger.[7] In Schwinger's model, a vector potential A_μ couples to the conserved current of a fermion field ψ according to the interaction

$$\mathfrak{L}_I=e_0J_\mu A^\mu, \quad J_\mu=\bar\psi\gamma_\mu\psi \tag{1.2}$$

where e_0 is the bare coupling constant. The exact unrenormalized photon propagator has the simple structure

$$e_0^2D'_{F\mu\nu}(q)=-\frac{g_{\mu\nu}e_0^2}{q^2-e_0^2/\pi}+\text{gauge terms}. \tag{1.3}$$

Since e_0 carries the dimension of mass, the theory is superrenormalizable. The only divergent graph, which occurs in the second-order correction to the vacuum polarization, becomes finite when gauge invariance is imposed. In the conventional approach to a superrenormalizable field theory, the bare coupling constant e_0 is held fixed at a finite value, so that Eq. (1.3) implies the existence of a massive vector particle with mass squared given by

$$\mu^2=e_0^2/\pi. \tag{1.4}$$

No coupling-constant renormalization is required. The short-distance properties of operator products are also trivial—they are those of a free-field theory.

However, Wilson[8] has proposed a renormalization scheme in which a superrenormalizable field theory is converted into a nontrivial renormalizable one by letting the bare coupling constant tend to infinity. Wilson has shown that the theory is exactly scale-invariant in this limit, and that it is possible to define a dimensionless renormalized coupling constant which satisfies a nontrivial Gell-Mann–Low eigenvalue condition. We shall apply Wilson's scheme to the Schwinger model. This results in renormalization-group equations which resemble those of four-dimensional quantum electrodynamics.

Section II contains an account of the renormalization-group program and the Gell-Mann–Low eigenvalue condition for two-dimensional electrodynamics. In Sec. III, the anomaly[9,10] of the corresponding axial-vector current is found to be independent of the coupling constant for all values of e_0 except at the eigenvalue ($e_0^2=\infty$) where it vanishes. In Sec. IV we observe that ψ and the composite operator $\bar\psi\psi$ possess anomalous dimensions at the eigenvalue, while the electromagnetic current retains its canonical dimension 1. Also, we test the assumption that the introduction of a scale-invariance-breaking mass term does not contribute significant corrections to the short-distance behavior of the theory. In other words, the $e_0^2\to\infty$ limit of the model can be regarded as the skeleton theory of a system of interacting massive particles. In particular, the analysis indicates that the correction to $\langle 0|J_\mu J_{5\nu}|0\rangle$ is too soft to produce an anomaly. In Sec. V there is a brief comparison of our model with four-dimensional quantum electrodynamics.

II. RENORMALIZATION GROUP AND EIGENVALUE CONDITION

Simple modifications of the original discussion of Gell-Mann and Low[4] lead to equations for the renormalization group in two dimensions. We introduce a function $d(q^2, e_0^2)$ which is invariant under renormalization:

$$e_0^2 D'_{F\mu\nu} = g_{\mu\nu} d(q^2, e_0^2) + \text{gauge terms}. \tag{2.1}$$

Notice that the four-dimensional version of this definition contains an extra factor $1/q^2$. The difference arises because we want d to be dimensionless. From Eq. (1.3), we obtain

$$d(q^2, e_0^2) = -\frac{e_0^2}{q^2 - e_0^2/\pi} . \tag{2.2}$$

We also introduce a spacelike reference momentum λ, and a corresponding dimensionless renormalized coupling constant

$$e_\lambda^2 = d(\lambda^2, e_0^2). \tag{2.3}$$

The renormalized and unrenormalized coupling constants are related by

$$e_\lambda^2 = \frac{e_0^2}{-\lambda^2 + e_0^2/\pi} , \tag{2.4}$$

$$-\frac{e_0^2}{\lambda^2} = \frac{e_\lambda^2}{1 - e_\lambda^2/\pi} , \tag{2.5}$$

so that the function d can be expressed in terms of e_λ^2 as follows:

$$\bar{d}(q^2, \lambda^2, e_\lambda^2) = \bar{d}(q^2/\lambda^2, e_\lambda^2) = d(q^2, e_0^2), \tag{2.6}$$

$$e_\lambda^2 = \bar{d}(\lambda^2, \lambda^2, e_\lambda^2);$$

$$\bar{d}(q^2/\lambda^2, e_\lambda^2) = \frac{e_\lambda^2}{(1 - e_\lambda^2/\pi)q^2/\lambda^2 + e_\lambda^2/\pi} . \tag{2.7}$$

Since $\bar{d}(q^2/\lambda^2, e_\lambda^2)$ does not depend on the reference point λ, we have the formula

$$\bar{d}(q^2, \lambda_1^2, e_{\lambda_1}^2) = \bar{d}(q^2, \lambda_2^2, e_{\lambda_2}^2), \tag{2.8}$$

which, when combined with Eq. (2.6), yields the functional equation

$$\bar{d}(q^2/\lambda_1^2, e_{\lambda_1}^2) = \bar{d}(q^2/\lambda_2^2, \bar{d}(\lambda_2^2/\lambda_1^2, e_{\lambda_1}^2)). \tag{2.9}$$

The integrated version of (2.9) is the familiar expression

$$\ln(q^2/\lambda^2) = \int_{e_\lambda^2}^{\bar{d}(q^2/\lambda^2, e_\lambda^2)} \frac{dx}{\Psi(x)} , \tag{2.10}$$

where $\Psi(x)$, the Gell-Mann–Low function for our two-dimensional theory, is given by

$$\Psi(e_\lambda^2) = \frac{de_\lambda^2}{d\ln\lambda^2} = -e_\lambda^2(1 - e_\lambda^2/\pi). \tag{2.11}$$

The nontrivial zero of $\Psi(e_\lambda^2)$ at

$$e_\lambda^2 = \pi \tag{2.12}$$

is the eigenvalue for the renormalized coupling constant. Referring back to Eq. (2.5), we see that (2.12) corresponds to the limit $e_0^2 \to \infty$, as predicted by Wilson.[8] The scale invariance of the theory at the eigenvalue can be verified by consulting Schwinger's solutions for n-point functions. A few examples will be given in Sec. IV.

III. AXIAL-VECTOR CURRENT ANOMALY

In two dimensions, the low-energy definition of the axial-vector current anomaly S can be written

$$S = \tfrac{1}{2}i \int d^2x\, \epsilon^{\lambda\mu} x_\lambda T\langle 0|\partial^\nu J_{5\nu}(x) J_\mu(0)|0\rangle . \tag{3.1}$$

Shei[10] has checked the short-distance methods of Wilson[11] and Crewther[12] for this case. The result is that S is also given by R, the c-number part of the short-distance expansion

$$T\{J_{5\mu}(x) J_\nu(0)\} \sim \frac{R}{4\pi}\epsilon_{\mu\lambda}(\delta_\nu^\lambda \partial^2 - \partial^\lambda\partial_\nu) \times \ln(-x^2 + i\epsilon) + \cdots . \tag{3.2}$$

Shei verified his result by explicit calculation, and observed that, in the Schwinger model, radiative corrections do not modify the lowest-order value for S:

$$S = R = -\pi^{-1}. \tag{3.3}$$

According to the point of view presented in Sec. II, there is an additional possibility, $e_\lambda^2 = \pi$, which remains to be investigated. The value of S at this point can be deduced from the exact expression

$$T\langle 0|J_\mu(x) J_\nu(0)|0\rangle = \frac{i}{\pi}(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\Delta_F(e_0^2/\pi, x) \quad (x \neq 0), \tag{3.4}$$

where $\Delta_F(\mu^2, x)$ is the propagator function in two dimensions:

$$\Delta_F(\mu^2, x) = (2\pi)^{-2}\int d^2p\, e^{ip\cdot x}(p^2 - \mu^2 + i\epsilon)^{-1}. \tag{3.5}$$

For finite values of e_0^2, the leading singularity of $\Delta_F(e_0^2/\pi, x)$ at short distance is

$$\Delta_F(e_0^2/\pi, x) \sim \frac{i}{4\pi}\ln(-x^2 + i\epsilon) + \cdots \quad (x \to 0,\ e_0^2 < \infty). \tag{3.6}$$

According to Eq. (1.1), we obtain $T\langle 0|J_5^\lambda(x)J_\nu(0)|0\rangle$ by multiplying (3.4) by $\epsilon^{\lambda\mu}$, and so Eq. (3.6) leads directly to the standard result (3.3). However, at the eigenvalue $e_\lambda{}^2=\pi$ where $e_0{}^2$ tends to infinity, Eq. (3.6) is no longer valid; instead, we obtain the expression

$$\lim_{e_0^2\to\infty} e_0{}^2\Delta_F(e_0{}^2/\pi, x) = -\pi\delta(x), \qquad (3.7)$$

which vanishes for $x\neq 0$. This means that the Green's function

$$T\langle 0|J_{5\mu}(x)J_\nu(0)|0\rangle = 0 \quad (x\neq 0) \qquad (3.8)$$

vanishes away from the origin, and therefore, the anomalous constant S vanishes at the eigenvalue:

$$S(e_\lambda{}^2=\pi)=0. \qquad (3.9)$$

The mechanism which causes this discontinuity in S is best appreciated in momentum space. The value of S is regulated by the high-momentum behavior of the propagator

$$\tilde\Delta_F(e_0{}^2/\pi, p^2) = (p^2 - e_0{}^2/\pi)^{-1}. \qquad (3.10)$$

Let us rewrite Eq. (3.10) in terms of the renormalized coupling constant $e_\lambda{}^2$:

$$\tilde\Delta_F(e_\lambda{}^2; p^2) = \frac{1-e_\lambda{}^2/\pi}{(1-e_\lambda{}^2/\pi)p^2+\lambda^2 e_\lambda{}^2/\pi}. \qquad (3.11)$$

Clearly, the $p^2\to\infty$ behavior of $\tilde\Delta_F$ depends on whether the eigenvalue condition is satisfied or not:

$$\tilde\Delta_F(e_\lambda{}^2; p^2)\sim 1/p^2 \quad (e_\lambda{}^2\neq\pi), \qquad \tilde\Delta_F(\pi; p^2)=0. \qquad (3.12)$$

This trivial mechanism permits nonrenormalization of S together with its vanishing at the eigenvalue.

It remains an open question whether an analogous discontinuity can occur in four dimensions. There, one deals with a renormalizable theory which, when summed, yields a finite coupling-constant renormalization factor Z_3 if there is an eigenvalue. Our two-dimensional model also exhibits a change in renormalizability. The difference is that our model is superrenormalizable in finite orders of perturbation theory; as a result, the short-distance behavior of the theory is $e_0{}^2$-independent for finite values of $e_0{}^2$. The short-distance behavior changes at the eigenvalue because the theory suddenly becomes renormalizable.

IV. ANOMALOUS DIMENSIONS AND SCALE-INVARIANCE BREAKING

Let us now give simple examples to show that the theory is scale-invariant and exhibits anomalous dimensions at the eigenvalue. The two- and four-point Green's functions of the fermion field ψ will be discussed. Consider the Lorentz gauge, for which the photon propagator is

$$D'_{F\mu\nu}(q) = -(g_{\mu\nu}-q_\mu q_\nu/q^2)(q^2-e_0{}^2/\pi)^{-1}. \qquad (4.1)$$

We introduce the symbols

$$H(x)=\Delta_F(0,x)-\Delta_F(e_0{}^2/\pi, x) \qquad (4.2)$$

and

$$G^0(x) = -(2\pi)^{-1}\gamma\cdot x/(x^2-i\epsilon), \qquad (4.3)$$

where G^0 is the Green's function for a free massless fermion. In this notation, Schwinger's explicit solutions[7] for the two- and four-point amplitudes are

$$G(x,y) = -iT\langle 0|\psi(x)\bar\psi(y)|0\rangle = G^0(x-y)\exp[-i\pi H(x-y)], \qquad (4.4)$$

$$\begin{aligned} G(x,x',y,y') &= -T\langle 0|\psi(x)\bar\psi(x')\psi(y)\bar\psi(y')|0\rangle \\ &= G^0(x-x')G^0(y-y')\exp\{i\pi[-H(x-x')+H(x-y)-H(x'-y)+H(x'-y')-H(y-y')-H(x-y')] \\ &\quad -i\pi(1-\alpha^x\alpha^y)[H(x-y)-H(x'-y)-H(x-y')+H(x'-y')]\} - \{x\leftrightarrow y\}. \end{aligned} \qquad (4.5)$$

The superscripts x, y on the symmetric matrix

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (4.6)$$

indicate that α^x is to be multiplied with $G^0(x-x')$, and α^y with $G^0(y-y')$. The Green's functions (4.4), (4.5) have been arbitrarily normalized by scaling out some infinite constants.

When $e_0{}^2$ is finite, the function $H(x-y)$ is not singular as x tends to y, and so the short-distance behavior of the complete Green's functions is that of a free-field theory. At the eigenvalue, $H(x-y)$ becomes singular:

$$H(x-y)\xrightarrow[e_0^2\to\infty]{} D_F(x-y) = \frac{i}{4\pi}\ln[-(x-y)^2+i\epsilon]. \qquad (4.7)$$

As a result, the corresponding Green's functions scale with anomalous dimensions. For example, we obtain

$$G(x-y) = G^0(x-y)\exp\{\tfrac14\ln[-(x-y)^2+i\epsilon]\} = [-(x-y)^2+i\epsilon]^{1/4}G^0(x-y) \qquad (4.8)$$

for the two-point function. This means that the fermion field ψ scales with dimension

$$d_\psi = \tfrac{1}{4} \tag{4.9}$$

instead of the canonical value $\frac{1}{2}$. However, it should be remembered that d_ψ is gauge-dependent, and therefore, it possesses no physical significance. Equation (4.9) is valid for the Lorentz gauge.

Similarly, $G(x, x', y, y')$ can be evaluated at the eigenvalue by combining Eqs. (4.5) and (4.7). The result is consistent with the scaling property (4.9). Now consider composite operators obtained by letting x and y' approach a common point z:

$$x \to z, \quad y' \to z. \tag{4.10}$$

As usual, we include an extra factor[7,13]

$$K(x-y') = \exp\left[+ie_0 \int_{y'}^{x} d\xi^\mu A_\mu(\xi)\right] \tag{4.11}$$

and average over orientations of the vector $(x-y')$ in order to ensure that the results are gauge-invariant. The mass operator u is given by the q-number contribution to the short-distance expansion

$$\mathrm{Tr}\{\psi(x)\overline{\psi}(y')K\} \sim \langle 0|\mathrm{Tr}\{\psi(x)\overline{\psi}(y')K\}|0\rangle + c[(x-y')^2]^{-1/4}u(z) + \cdots, \tag{4.12}$$

where c is a constant. The corresponding vertex function is

$$T\langle 0|\psi(x)u(z)\overline{\psi}(y)|0\rangle = \frac{G^0(x,z)G^0(z,y)[(x-z)^2(y-z)^2]^{1/2}}{[-(x-y)^2]^{1/4}}, \tag{4.13}$$

if an appropriate normalization convention for u is adopted. A matrix multiplication is understood for the product $G^0(x,z)G^0(z,y)$. It is immediately obvious from Eq. (4.13) that $u(z)$ has a dimension

$$d_u = 0, \tag{4.14}$$

which differs from the canonical value 1.

We can also consider $\mathrm{Tr}\{\psi\overline{\psi}\gamma_\mu K\}$, and hence derive the vertex function of the electromagnetic current:

$$T\langle 0|\psi(x)\overline{\psi}(y)J_\mu(z)|0\rangle = i(g_{\mu\nu}+\epsilon_{\mu\nu}\gamma^5)\partial^\nu[D_F(z-x)-D_F(z-y)]G(x-y). \tag{4.15}$$

This expression satisfies the usual Ward identity

$$\frac{\partial}{\partial z^\mu} T\langle 0|\psi(x)\overline{\psi}(y)J_\mu(z)|0\rangle = -i[\delta(x-z)-\delta(y-z)]G(x-y). \tag{4.16}$$

From Eq. (4.15) we conclude that J_μ has canonical dimension

$$d_J = 1. \tag{4.17}$$

To conclude this section, we consider the possibility that the introduction of a scale-invariance-breaking perturbation

$$\mathcal{L}_\kappa = \kappa u \tag{4.18}$$

generates a new massive theory with the same short-distance behavior as in our model. Wilson[11,14] has given a general analysis of this situation: Given operators A, B, O_n with dimensions d_A, d_B, d_n, the coefficient functions $C_n(x,\kappa)$ in

$$A(x)B(0) \sim \sum_n C_n(x,\kappa)O_n(0) \tag{4.19}$$

possess κ expansions of the form

$$C_n(x,\kappa) \sim x^{d_n - d_A - d_B}\left[c_0 + \sum_{k=1} c_k(x)(\kappa x^{d-d_u})^k\right], \tag{4.20}$$

where d is the dimension of space-time, and the dependence of the c_k on x is not stronger than logarithmic. In our model, $d_u = 0$ is less than $d=2$; therefore, the κ-dependent terms are softer at short distances, which is consistent with the presumed existence of a massive theory. For example, the leading correction to $\langle 0|J_\mu(x)J_\nu(0)|0\rangle$ is of order $\kappa(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)x^2$, which is too soft at short distances to produce an anomaly. We conclude that the anomaly continues to vanish in the massive theory.

V. COMPARISON WITH FOUR-DIMENSIONAL QED

We conclude with some additional remarks about differences between the two- and four-dimensional theories.

Let us return to the Gell-Mann–Low equation (2.10):

$$\ln(q^2/\lambda^2) = \int_{e_\lambda{}^2}^{\overline{d}(q^2/\lambda^2, e_\lambda{}^2)} \frac{dx}{\Psi(x)}.$$

Although this formula holds for both two- and four-dimensional QED, the presence of a zero in $\Psi(x)$ leads to different conclusions. Consider the region $q^2/\lambda^2 \ll 1$ for the two-dimensional case:

$$\ln(q^2/\lambda^2) \to -\infty \quad (q^2/\lambda^2 \ll 1). \tag{5.1}$$

From the explicit formula (2.7), it is easy to see that the Gell-Mann–Low integral tends to $-\infty$ in this limit because of the zero of $\Psi(x)$ at $x=\pi$:

$$\Psi(\pi) = 0, \qquad \overline{d}(q^2/\lambda^2, e_\lambda{}^2) \to \pi \quad (q^2/\lambda^2 \ll 1). \tag{5.2}$$

In other words, the nontrivial eigenvalue controls the *low*-momentum behavior of the photon propagator,[15] in complete contrast with the situation for the four-dimensional theory discussed by Gell-Mann and Low.[4]

In our model, the eigenvalue $x=\pi$ is a simple zero of $\Psi(x)$. This is related to the vanishing of radiative corrections to the basic one-loop contribution to the proper part of $\langle 0|J_\alpha(x)J_\beta(0)|0\rangle$ in Schwinger's model.[16] Our analysis has no bearing on the possible presence of infinite-order zeros in Adler's version[17] of QED or in similar theories, despite the fact that Adler's argument is based on equations

$$\langle 0|J_{\mu_1}(x_1)\cdots J_{\mu_n}(x_n)|0\rangle_{\text{conn}}=0 \qquad (\text{fermion mass}=0), \quad (5.3)$$

which are also valid at the eigenvalue of our model.[18] In contrast with Adler's theory, the one-fermion loop contribution $-e_0^2/\pi q^2$ to the proper self-energy of the photon diverges at the eigenvalue.

Our main interest in this model is that it is the first example of an exactly soluble theory of the renormalization group in which the anomaly of the axial-vector current can be studied.[19] We have not tried to draw strong analogies with the four-dimensional case, because the absence of difficulties in our model is probably connected with properties peculiar to two-dimensional space-time.

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[15]A similar phenomenon was discussed by Wilson in statistical mechanics. He showed that the eigenvalue (or "fixed point") may control the strength of the coupling among the long-wavelength fluctuations of a system; see K. G. Wilson and J. B. Kogut, Ref. 8. The discussion in Sec. III F of Ref. 14 is also relevant.

[16]This can be verified by explicit calculation of the second-order contribution $\pi_2(q^2)$ to $\pi(q^2)$, the proper self-energy of the photon. The manner in which the exact photon propagator

$$D'_{F\mu\nu}(q)=-\frac{g_{\mu\nu}}{q^2}[1+e_0^2\pi(q^2)]^{-1}$$

develops a pole at $q^2=e_0^2/\pi$ as the fermion mass m tends to zero also deserves comment. The second-order contribution becomes

$$\pi_2(q^2)=\frac{1}{\pi}\int_0^1 dv\, v(1-v)/[m^2-v(1-v)q^2-i\epsilon] \qquad (m\neq 0),$$

instead of

$$\pi_2(q^2)=-1/\pi(q^2+i\epsilon) \quad (m=0),$$

and the photon propagator has a pole at $q^2=0$:

$$D'_{F\mu\nu}(q)\sim -g_{\mu\nu}Z_3/q^2 \quad (q^2\to 0),$$

$$Z_3=[1+e_0^2/6\pi m^2+O(e_0^4/m^4)]^{-1} \quad (m\neq 0).$$

Infrared divergences change the position of the pole from $q^2=0$ for $m\neq 0$ to $q^2=e_0^2/\pi$ ($m=0$), and $\lim_{m\to 0}Z_3$ vanishes. Thus infrared effects are more severe than in four dimensions, where the $q^2=0$ pole changes into a branch cut at $q^2=0$ as m vanishes.

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[18]For $n>2$, Eq. (5.3) also holds in Schwinger's model (i.e., away from the eigenvalue).

[19]Eigenvalue conditions have been derived for the ϵ and $1/N$ expansions discussed in Ref. 8. However, an axial-vector current cannot be introduced because the dimension of space-time is not an even integer.