Classical Electromagnetic Interaction of a Charged Particle with a Constant-Current Solenoid

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The classical electromagnetic interaction for a charged particle with a long, constant-current solenoid is investigated in the context of classical electromagnetism. The conservation laws connected with energy, linear momentum, and angular momentum are first verified. Then the changes in the electromagnetic field quantities are evaluated explicitly, and the values are seen to have interesting connections with the solenoid vector potential. The calculations are of interest in connection with the Aharonov-Bohm effect involving the passage of electrons past a solenoid, and the classical expressions for energy and momentum are reminiscent of terms in the quantum-mechanical calculations.

I. INTRODUCTION

It is a well-known result of classical electromagnetism that the magnetic induction field $\vec{B}$ outside a uniform solenoid may be made negligible by making a solenoid sufficiently long. However, working from the point of view of quantum mechanics, Ehrenberg and Siday,[1] and Aharonov and Bohm[2] predict an effect upon electrons passing outside a long solenoid in a region presumably free of classical electromagnetic fields; and this Aharanov-Bohm effect has been verified in experiments by Chambers[3] and by Möllenstedt and Bayh.[4] It is this contrast between the classical and quantum electromagentic points of view which prompted the present detailed analysis of the classical electromagnetic interaction between a solenoid and a passing charged particle.

Precise calculations in classical electromagnetism can be obtained only in very special circumstances which usually do not represent any actual physical arrangement in detail, but only in approximation. The present paper reflects this situation by performing exact calculations for a charged particle moving with constant velocity past an infinitely long solenoid carrying constant surface currents. This cannot represent the precise conditions of the experiment of Möllenstedt and Bayh, and it may or may not be a valid approximation to these experimental conditions. However, the classical analysis here seems of interest in any case because it corresponds to the conditions usually assumed in the quantum theoretical calculations following Aharonov and Bohm. In this paper we will treat a whole range of questions which are often raised in connection with the Aharonov-Bohm effect when considered in terms of the classical electromagnetic interaction of a charged particle with a constant-current solenoid.

The analysis begins with preliminaries regarding the field and flux of a solenoid and treats with care the question involved in the limits of physical quantities as the length of the solenoid becomes infinite. The paper then turns to the detailed verification of the general conservation theorems in electromagnetism when applied to the case of a charged particle passing a long solenoid with constant currents. The currents are maintained constant by external forces on the individual particles of the solenoid. The analysis is similar to the explicit verification of these same theorems, carried out despite singularities, for charged particles moving with constant velocities.[5] This mundane checking may seem superfluous to the reader until he finds that some of the published secondary literature[6] on the Aharonov-Bohm effect casts into question the conservation laws in classical electromagnetism. The explicit expressions obtained for the energy and momentum contained in the electromagnetic field bear striking resemblances to terms in the quantum analysis of the Aharonov-Bohm effect. The vector potential appears, however, not in a gauge-independent role, but rather restricted to the Coulomb gauge. The troublesome question of a connection with the quantum calculation will be treated in a subsequent publication.

II. PRELIMINARIES

A. The Fields of a Solenoid

We are interested in the electromagnetic interactions involving a long solenoid, and accordingly as a first step we will review the expressions for the fields and flux of a finite solenoid, noting the limits of these quantities as the solenoid length becomes infinite.

We think of a solenoid as a parallelopiped with uniform surface currents circulating perpendicular to its axis. The right-circular solenoid used in our calculations is indicated in Fig. 1 for a solenoid of length L, radius R, carrying a surface current i per unit length. The solenoid produces magnetic induction fields $\vec{B}$ which may be obtained by integrating the Biot-Savart law over the surface currents. Thus the field at the center of the solenoid is

$$\vec{B}_0 = \frac{i}{c}\int_S \frac{d\Sigma(\hat{k}\times\hat{n})\times\vec{r}}{r^3} \cong \frac{4\pi i}{c}\hat{k}, \quad (1)$$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is the radius vector from a current element to the center and $\hat{n}$ is the outward pointing normal associated with the surface element $d\Sigma$. Here $\hat{k}\times\hat{n} = \hat{\theta}$ is in the direction of the surface current, and the integral is over the surface S of the solenoid. The approximate expression for $\vec{B}_0$ holds when the radius R of the solenoid is much smaller than the length L. In the actual calculation,[7] it is convenient to break the surface up into cylindrical rings of differential height. The same field $\vec{B}_0$ will hold[8] across the median cross section of the solenoid in the same approximation that $R \ll L$.

The fields outside the solenoid may again be obtained from the Biot-Savart law, or alternatively[9] be regarding the solenoid as composed of two circular ends of magnetic charge of opposite sign at the top t and bottom b of the solenoid in Fig. 1. For a point in the equitorial plane of the solenoid and outside it a distance d from the center, the magnetic field $\vec{B}_s$ of the solenoid follows an electrostatic calculation as

$$\vec{B}_s = \frac{i}{c}\int_t \frac{dxdy\,\vec{r}}{[(x-d)^2+y^2+z^2]^{3/2}} - \frac{i}{c}\int_b \frac{dxdy\,\vec{r}}{[(x-d)^2+y^2+z^2]^{3/2}}, \quad (2)$$

where again $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is the radius vector from the source point to the field point. For $R \ll d$ and $R \ll L$, this reduces to

$$\vec{B}_s \cong \frac{-i\mathcal{A}L}{c[d^2+(L/2)^2]^{3/2}}\hat{k}, \quad (3)$$

where $\mathcal{A} = \pi R^2$ is the cross-sectional area of the solenoid. The equal contributions from the two terms add to cancel the 2 in the denominator $z = \pm L/2$.

In the limit that the length of the solenoid becomes infinite, the magnetic field inside the solenoid becomes the constant value

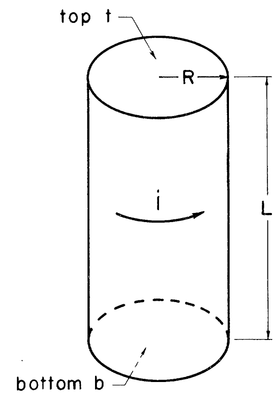

FIG. 1. A right-circular solenoid of length L, radius R, and surface current i per unit length.

$$\vec{B}_0 = \frac{4\pi i\hat{k}}{c}. \quad (4)$$

The fields outside the solenoid correspond to those of two discs of finite amounts of magnetic charge $\pm i\mathcal{A}/c$ which recede to spatial infinity. Hence the $\vec{B}$ field outside an infinitely long solenoid vanishes.

B. Flux Calculations

Although the field at any point outside an infinitely long solenoid vanishes, it is not clear and indeed not true that all integrals of the fields outside the solenoid also vanish. A specific example of the need for care regarding the limits involved for an infinite solenoid is provided by considerations of the magnetic flux passing through the equatorial plane of the solenoid.

The lines of magnetic induction $\vec{B}$ may be regarded as continuous because of the apparent absence of magnetic monopoles,

$$\int \vec{B}\cdot d\vec{\sigma} = 0, \quad (5)$$

where the integration is over any closed surface. Consider a circular disc in the equatorial plane of a finite solenoid, and complete a closed surface with a hemispherical cap as in Fig. 2(a). The flux through this closed surface vanishes. For a large radius r for the hemisphere, the solenoid appears as a dipole of moment $M = i\mathcal{A}L/c$. Thus the contribution to the magnetic flux over the hemispherical portion goes as

$$\Phi_{\text{hemisphere}} \sim \frac{i\mathcal{A}L}{cr^3}\,2\pi r^2 \propto \frac{1}{r} \to 0 \text{ as } r\to\infty. \quad (6)$$

Thus we conclude that the flux through the equatorial plane of the solenoid is also zero. The lines of $\vec{B}$ run upwards through the plane inside the solenoid, and then downwards through the plane outside the solenoid.

The calculation may be performed explicitly. Choosing an upwards pointing normal for the plane, the flux inside is $\cong B_0\mathcal{A}$, and that outside is

$$\int_{\text{outside}} \vec{B}\cdot d\vec{\sigma} \cong \int_{r=R}^{\infty} \frac{-i\mathcal{A}L}{c[r^2+(L/2)^2]^{3/2}}\,2\pi r\,dr = \frac{-2\pi i\mathcal{A}L}{c[R^2+(L/2)^2]^{1/2}}\,. \tag{7}$$

The expressions for the fields are accurate only for a long solenoid $L \gg R$, in which case the flux outside becomes

$$\lim_{L\to\infty} \frac{-2\pi i\mathcal{A}L}{c[R^2+(L/2)^2]^{1/2}} = \frac{-4\pi i}{c}\mathcal{A} = -B_0\mathcal{A} \tag{8}$$

while the flux inside is $= +B_0\mathcal{A}$. Indeed the flux outside is opposite in sign to that inside the solenoid. Thus we have verified that although the magnetic field $\vec{B}$ outside a long solenoid vanishes as $L\to\infty$, the flux through the median plane outside the solenoid becomes a nonvanishing constant.

The order of taking the limits is crucial for understanding the behavior of a long solenoid. Above we took the limit of a large hemispherical cap $r\to\infty$ before the limit of a long solenoid $L\to\infty$. If we had reversed the order of the limits, then our diagram becomes that of Fig. 2(b). Now there is a nonvanishing contribution to the flux from the hemispherical surface. The flux through the equatorial plane outside the solenoid vanishes for this case, and the contribution from inside the solenoid is balanced by the contribution of opposite sign from the hemispherical cap.

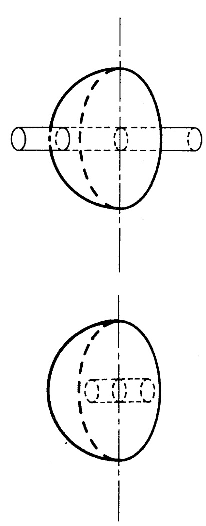

FIG. 2. Solenoid and closed surfaces for flux calculations.

C. Lorentz Force on a Passing Particle

The question which this paper has in mind is just what does happen to a charged particle which passes a long solenoid. Having seen that the flux outside an infinitely long solenoid is nonvanishing although the fields vanish, one must enquire as to whether a passing charged particle is not deflected by the magnetic field of a long solenoid with constant currents. This question arises because here again a double limit occurs, the limit that the particle starting point is at spatial infinity, and the limit that the solenoid has infinite length.

If the limit that the solenoid is infinite is taken first and the solenoid currents are held constant, then clearly the magnetic field outside the solenoid vanishes, and a passing particle is undeflected by any Lorentz force. The other limit involves a particle starting from spatial infinity and passing a solenoid of finite length L. We are assuming that the solenoid currents are constant and that the external particle passes with constant velocity in the equatorial plane of the solenoid. The impulse due to the magnetic field of the solenoid acting on the passing particle of charge e is

$$\vec{\mathcal{J}} = \int_{-\infty}^{\infty} \vec{F}\,dt = \int_{-\infty}^{\infty} e\frac{\vec{v}}{c}\times\vec{B}\,dt = \frac{e}{c}\int_{-\infty}^{\infty} d\vec{r}\times\vec{B}\,. \tag{9}$$

Taking $\vec{B}$ in (3), valid for the approximation that $R \ll L$, $R \ll d$, and using the coordinate system of Fig. 3, this becomes an elementary integral

$$\vec{\mathcal{J}} \cong \frac{e}{c}\int_{-\infty}^{\infty} dy\,\hat{j}\times\hat{k}\,\frac{(-i\mathcal{A}L)}{c[(x-d)^2+y^2+(L/2)^2]^{3/2}} = \frac{-2ei\mathcal{A}L}{c^2[d^2+(L/2)^2]}\,\hat{i}\,. \tag{10}$$

But then for large L,

$$\vec{\mathcal{J}} \cong \frac{-8ei\mathcal{A}}{c^2L}\,\hat{i} \to 0 \quad \text{as} \quad L\to\infty\,. \tag{11}$$

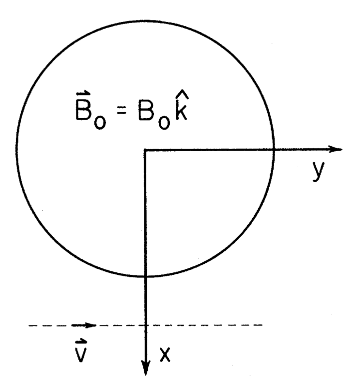

FIG. 3. Coordinate system for a particle passing in the median plane of the solenoid.

Thus for a long solenoid, this impulse is negligible.

The deflection angle θ of the passing particle from the forward direction may be approximated as

$$\theta \sim \frac{\Delta p}{p}, \tag{12}$$

where Δp is the transverse momentum acquired due to the impulse $\vec{\mathcal{I}}$. Here we find the impulse and hence the deflection angle is finite for passage of the particle past a finite solenoid, but the deflection decreases to zero as the solenoid length becomes infinite. This effect will be omitted from our further investigations.

III. ENERGY, LINEAR MOMENTUM, AND ANGULAR MOMENTUM FOR A POINT PARTICLE AND A SOLENOID WITH CONSTANT CURRENTS

A. Conservation Theorems: General Theorems vs Specific Examples

Classical electromagnetism is a mixture of Newtonian mechanics and classical field theory. The point charges in the theory experience forces due to the electromagnetic fields and move according to Newton's second law. The electromagnetic fields carry linear momentum yet are not viewed in terms of Newtonian forces, but rather are calculated in terms of their sources at the charged particles. Virtually every advanced text book[10] on classical electromagnetism gives a general proof that the energy, linear momentum, and angular momentum of a system are all correctly supplied by the external forces on the particles of the system provided the system energy and momentum includes contributions from both the particles and the electromagnetic fields. However, it is in general difficult to give exact calculations for the energy and momentum of the fields, and hence illustrations of the general theorem are rare in the literature. Perhaps it is this lack of examples which has led some researchers considering the Aharonov-Bohm effect to suggest rather surprising mechanisms for conservation,[11] or to conclude erroneously that the general theorems are not valid after all.[6]

The general conservation theorems have been illustrated exactly recently with some striking examples involving charged particles moving with constant velocities.[5] The external forces needed to balance the interparticle Lorentz forces are found to account exactly for the changes in the energy and momentum of the fields. In the present paper, we present an analogous series of calculations for the case of a charged particle moving with constant velocity passing a solenoid having constant currents. We will verify the conservation laws for the specific electromagnetic system, and also calculate explicitly the changes in the system energy, linear momentum, and angular momentum.

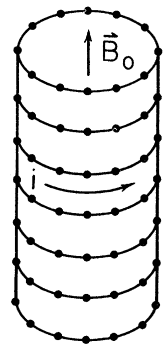

FIG. 4. Classical electron theory model for a solenoid. Charges of one sign move in circles about the surface of the solenoid. Charges of the opposite sign and of equal surface density hold fixed surface positions giving vanishing surface charge density for the solenoid.

B. Model for a Solenoid in Classical Electron Theory

A solenoid involves currents leading to magnetic fields. However, in the mathematical calculations, there is no mention of a physical mechanism for the currents. The model of classical electron theory describes the currents as due to point charges which move around the surface of the solenoid as in Fig. 4. The electrostatic charge of these moving particles is balanced by a set of stationary charges of equal density and opposite charge also on the surface of the solenoid.

This classical electron theory model accounts for the currents and neutrality of the solenoid. It is a crucial observation that the point charges moving in a circle do not radiate,[12] and hence once started, the currents will continue to flow at a constant rate for all time. This absence of radiation for a continuous progression of charges, whereas a single point charge moving in a circle does indeed radiate energy, is due to the coherent interference of the fields of the many point charges.

C. Energy in the Electromagnetic Fields

The energy $\mathcal{E}$ in the electromagnetic fields according to classical theory is

$$\mathcal{E} = \frac{1}{8\pi} \int d^3r (\vec{E}^2 + \vec{B}^2). \tag{13}$$

If several different sources contribute to the fields, then the total fields are $\vec{E}_1+\vec{E}_2+\vec{E}_3+\cdots$ and $\vec{B}_1+\vec{B}_2+\vec{B}_3+\cdots$, where $\vec{E}_i$ and $\vec{B}_i$ are the contributions from the ith system. The expressions $\vec{E}^2$ and $\vec{B}^2$ become

$$\begin{aligned}\vec{E}^2&=\vec{E}_1{}^2+\vec{E}_2{}^2+\cdots+2\vec{E}_1\cdot\vec{E}_2+\cdots,\\ \vec{B}^2&=\vec{B}_1{}^2+\vec{B}_2{}^2+\cdots+2\vec{B}_1\cdot\vec{B}_2+\cdots.\end{aligned} \tag{14}$$

The contributions $\vec{E}_i{}^2$ and $\vec{B}_i{}^2$ also hold when the ith system is isolated from other systems, and hence, if the sources in the ith system are held constant, these energies act as system self-energies. The interaction energies for constant sources appear in the terms $2\vec{E}_1\cdot\vec{E}_2+\cdots$ and $2\vec{B}_1\cdot\vec{B}_2+\cdots$.

D. Energy in the Particle-Solenoid System

In the present case, we consider a solenoid with constant currents interacting with a passing charged particle moving with constant velocity. Since the particles of the solenoid move with constant speed and the external particle moves with constant velocity, there is no change in the mechanical kinetic energy of the system. Thus all changes in system energy occur in the energy of the electromagnetic fields. Moreover, since the sources are constant, all energy changes appear in the form

$$\Delta\mathcal{E}=\frac{1}{8\pi}\int_V(2\vec{E}_s\cdot\vec{E}'+2\vec{B}_s\cdot\vec{B}')d^3r, \tag{15}$$

where the subscript s refers to the solenoid, the prime refers to the passing particle and the integration volume V is all space. The electric field $\vec{E}_s$ of the solenoid vanishes everywhere and hence the expression reduces to

$$\Delta\mathcal{E}=\frac{1}{4\pi}\int_V\vec{B}_s\cdot\vec{B}'\,d^3r.$$

We remark that the field $\vec{B}'$ of the passing particle depends specifically on the velocity $\vec{v}$

$$\vec{B}'=\frac{\vec{v}}{c}\times\vec{E}' \tag{16}$$

and vanishes when the particle is not moving. If the velocity of the external particle is zero, then there is no electromagnetic energy of interaction with a solenoid of constant currents.

E. The Interaction Energy Outside an Infinite Solenoid Vanishes

For the ease of calculating the interaction energy, we wish to go to the limit of a solenoid of infinite length. However, it is conceivable that even though the magnetic fields outside a long solenoid vanish in the limit of an infinite length, the interaction energy between a solenoid and a passing charged particle might not vanish. We will verify that indeed this interaction energy outside a long solenoid vanishes.

We first remark that the interaction energy between a finite solenoid and a charged particle is finite. The fields are finite except at the position of the passing particle which makes a contribution to the interaction energy of the form

$$\Delta\mathcal{E}\sim\int_{r=0}\text{const}\frac{1}{r^2}4\pi r^2dr<\infty.$$

At large distances, the fields of the solenoid are those of a dipole so that the energy at spatial infinity is

$$\Delta\mathcal{E}\sim\int^{\infty}\text{const}\frac{1}{r^3}\frac{1}{r^2}4\pi r^2dr<\infty.$$

The fields outside the solenoid may be obtained from two discs of magnetic charge $\pm i\mathcal{A}/c$ at the top and bottom of the solenoid. Thus for a very long solenoid, the dependence of the magnetic interaction energy outside the solenoid has the spatial dependence of the interaction energy of three widely separated electric point charges. This energy vanishes as the solenoid length becomes infinite. Thus for a point charge and a long solenoid, we may ignore the interaction energy outside the solenoid.

F. Verification of Energy Conservation for the System

When a charged particle moving with constant velocity passes outside a long solenoid, the electric and magnetic fields of the passing particle exert forces on the particles of the solenoid. Since the solenoid currents are assumed constant, there must be external forces on the particles of the solenoid which keep them moving with uniform speed in circles. These external forces do work which will appear as energy in the electromagnetic fields of the system.

It is only the components of the external forces on the solenoid particles in the direction of motion which do work on the particles. The forces of constraint which maintain the motion in a circle are perpendicular to the direction of motion and so do no work. In particular, no work is done by the external forces required to balance the forces $q_i(\vec{v}_i/c)\times\vec{B}'$. due to the external particle magnetic field $\vec{B}$. Here q_i and $\vec{v}_i$ are the charge and instantaneous velocity of the ith solenoid particle. Thus the only external forces doing work are those balancing the tangential components of forces on the solenoid particles due to the electric field $\vec{E}'$ of the passing particle,

$$\text{work done on system per unit time} = \sum_i \vec{F}_{i\,\text{external}} \cdot \vec{v}_i = \sum_i (-q_i \vec{E}') \cdot \vec{v}_i = -\int_V \vec{j} \cdot \vec{E}' d^3 r , \tag{17}$$

where

$$\vec{j}(\vec{r}) = \sum_i q_i \vec{v}_i \delta^3(\vec{r} - \vec{r}_i) \tag{18}$$

represents the currents of the solenoid and $\vec{r}_i$ gives the position of the ith solenoid particle. Notice that it is only the circulating particles and not the stationary particles of the solenoid which are involved in the work calculation.

Although the passing particle exerts forces on the solenoid, it is not true that the solenoid exerts a force on the passing particle. We have assumed that the currents in the solenoid are constant and that the solenoid is infinitely long, and hence it follows that there is no magnetic field $\vec{B}$ outside the solenoid.[13] We may say that the passing particle causes confusion behind the scenes but the external forces counteract the confusion so that the passing particle is uninfluenced.

Despite this lack of symmetry in the forces, all the conservation laws hold. Here we will show that the change in energy of the system is given by work in (17) done by the external forces. The change in system energy is entirely due to that of the electromagnetic fields. This change in energy $\Delta\mathcal{E}$ is just the interaction energy of Sec. III D.

$$\Delta\mathcal{E} = \frac{1}{8\pi} \int_V 2\vec{B}_s \cdot \vec{B}' d^3 r = \frac{1}{4\pi} \int_{V_s} \vec{B}_0 \cdot \vec{B}' dx dy dz = \frac{B_0}{4\pi} \int_{-\infty}^{\infty} dz \int_s \vec{B}' \cdot d\vec{\sigma} , \tag{19}$$

where $\vec{B}_0 = B_0 \hat{k}$ is the magnetic field inside the solenoid and $d\vec{\sigma} = dx dy \hat{k}$. The first integral is over the volume V of all space, the second is over the volume V_s inside the solenoid, and the third involves the circular cross-sectional area s in Fig. 5. The time rate of change of the energy is

$$\frac{d\Delta\mathcal{E}}{dt} = \frac{B_0}{4\pi} \int_{-\infty}^{\infty} dz \frac{d}{dt} \int_s \vec{B}' \cdot d\vec{\sigma} . \tag{20}$$

However,

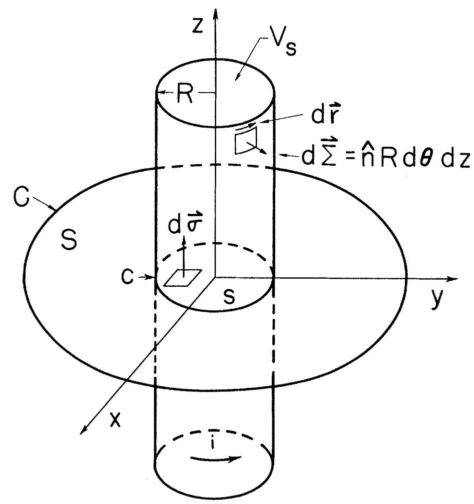

FIG. 5. Surface and volume elements of the solenoid used in the conservation theorems. V_s is the interior volume of the solenoid, R the solenoid radius. The curve c surrounds the cross-sectional area s, and curve C surrounds the area S. The surface of the solenoid is labelled $\mathcal{S}$. The surface elements indicated are $d\vec{\sigma} = \hat{k}\, dx dy$, and $d\vec{\Sigma} = \hat{n} R\, d\theta dz$. The line element $d\vec{r}$ is in the same direction as the surface current i.

$$\frac{d}{dt} \int_s \vec{B}' \cdot d\vec{\sigma}$$

is the rate of change of flux through the open surface s bounded by the walls of the solenoid as in Fig. 5. By Maxwell's equations,

$$\frac{d}{dt} \int_s \vec{B}' \cdot d\vec{\sigma} = -c \int_c \vec{E}' \cdot d\vec{r} , \tag{21}$$

where c is the circle enclosing the open surface s. We also know that $B_0 = 4\pi i / c$ where i is the surface current per unit length. Thus combining (20), (21), and (4),

$$\frac{d\Delta\mathcal{E}}{dt} = -\frac{cB_0}{4\pi} \int_{-\infty}^{\infty} dz \int_c \vec{E}' \cdot d\vec{r} = -\int_{-\infty}^{\infty} dz \int_c i \vec{E}' \cdot d\vec{r} = -\int_V \vec{j} \cdot \vec{E}' d^3 r , \tag{22}$$

where

$$\vec{j} = i\,\delta(r - R)\hat{\theta}$$

is the volume current density of the solenoid as in (18). The final expression in (22) is in agreement with the power calculation in (17).

The one further step required in this work is the check that all the integrals are indeed finite so that the manipulations make sense. This is easily verified. The interaction energy in (19) is bounded above for large z by

$$\Delta \mathcal{E} \sim \int^{\infty} \text{const} \frac{1}{z^2}\, dz < \infty .$$

This completes the explicity demonstration that the rate of change of system energy is indeed given by the power expended by external forces acting on the particles of the solenoid.

G. Calculation of the Energy Changes

The verification of energy conservation above was performed without ever actually evaluating the integrals for the changes of energy. It is of interest to carry out this evaluation and to notice that the energy change can be related approximately to a particular choice for the vector potential for the solenoid.

The fields $\vec{E}'$ and $\vec{B}'$ for a particle moving with constant velocity $\vec{v} = v\hat{j}$ may be written[14] as

$$\vec{E}' = \frac{\gamma e (x\hat{i} + y\hat{j} + z\hat{k})}{[x^2 + (\gamma y)^2 + z^2]^{3/2}} , \qquad \vec{B}' = \frac{\vec{v}}{c} \times \vec{E}' , \tag{23}$$

with

$$\gamma = (1 - v^2/c^2)^{-1/2}$$

or as

$$\vec{E}' = -\nabla\phi + \vec{\beta}\,\vec{\beta}\cdot\nabla\phi , \qquad \vec{B}' = -\vec{\beta}\times\nabla\phi , \tag{24}$$

where

$$\phi(\vec{r},\vec{r}\,') = \frac{e}{\{[(\vec{r}-\vec{r}\,')\cdot\vec{\beta}]^2 + (1-\beta^2)(\vec{r}-\vec{r}\,')^2\}^{1/2}} , \tag{25}$$

$\vec{\beta} = \vec{v}/c$ and $\vec{r}\,'$ is the particle position. Inserting the expression (24) into the particle-solenoid interaction energy (19), and using the same manipulations as in the Sec. III F,

$$\begin{aligned}
\Delta \mathcal{E} &= \frac{1}{4\pi}\int_{V_S} \vec{B}_0\cdot(-\vec{\beta}\times\nabla\phi)\, d^3r \\
&= -\frac{i}{c}\int_{-\infty}^{\infty} dz \int_s (\vec{\beta}\times\nabla\phi)\cdot d\vec{\sigma} \\
&= -\frac{i}{c}\int dz\, \vec{\beta}\cdot\int_s \nabla\phi\times d\vec{\sigma} \\
&= \frac{i}{c}\int dz\, \vec{\beta}\cdot\int_c \phi\, d\vec{r} \\
&= \frac{\vec{v}}{c^2}\cdot\int_V \vec{j}\,\phi\, d^3r ,
\end{aligned} \tag{26}$$

where again $\vec{j}$ stands for the density of the solenoid currents.

The result for $\Delta\mathcal{E}$ in (26) is exact. However, if we go to the nonrelativistic approximation for ϕ, then the integral may be easily evaluated in closed form. In the nonrelativistic approximation $\beta \ll 1$, the electrostatic potential is

$$\phi(\vec{r},\vec{r}\,') \cong \frac{e}{|\vec{r}-\vec{r}\,'|} , \tag{27}$$

and the integral

$$\frac{1}{c}\int \vec{j}\,\frac{\phi}{e}\, d^3r = \frac{1}{c}\int \frac{\vec{j}}{|\vec{r}-\vec{r}\,'|}\, d^3r \tag{28}$$

may be recognized as just the magnetic vector potential $\vec{A}(\vec{r}\,')$ (in the Coulomb gauge,[15] $\nabla\cdot\vec{A} = 0$), back at the position of the passing particle. We remark that there is no room for any gauge transformation on $\vec{A}$ when we relate the energy to the vector potential

$$\Delta\mathcal{E} \cong e\frac{\vec{v}}{c}\cdot\vec{A}(\vec{r}\,') . \tag{29}$$

Although it is perfectly feasible to carry out the integrals in (26) or in (28) by direct evaluation, once the integral is recognized as the vector potential $\vec{A}$ of the solenoid, we may take advantage of the properties of $\vec{A}$. The integral in (28) has the cylindrical symmetry of the solenoid. Thus integrating in a circular path C around the solenoid as in Fig. 5,

$$\begin{aligned}
\int_C \vec{A}\cdot d\vec{r} &= A2\pi r \\
&= \int_S (\nabla\times\vec{A})\cdot d\vec{\sigma} \\
&= \int_s \vec{B}_0\cdot d\vec{\sigma} \\
&= B_0\mathcal{A} ,
\end{aligned} \tag{30}$$

where S is the surface enclosed by C, and s is again the solenoid cross-sectional area. Then since $\nabla\cdot\vec{A} = 0$,

$$\vec{A} = \frac{B_0\mathcal{A}}{2\pi r}\,\hat{\theta} \tag{31}$$

or in the Cartesian coordinates of Fig. 3,

$$\vec{A}(\vec{r}) = \frac{B_0\mathcal{A}(x\hat{j} - y\hat{i})}{2\pi(x^2+y^2)} . \tag{32}$$

Thus the interaction energy of the particle-solenoid system is

$$\Delta\mathcal{E} \cong \frac{evB_0\mathcal{A}x}{2\pi c(x^2+y^2)} \tag{33}$$

when the passing particle is outside the solenoid. When the particle passes through the solenoid, then the integration in (30) gives

$$\vec{A} = \frac{B_0 r}{2}\,\hat{\theta} \tag{34}$$

and

$$\Delta \mathcal{E} \cong \frac{evB_0 x}{2c}\,. \tag{35}$$

Figure 6(a) graphs the interaction energy (29) of the particle-solenoid system as a function of the particle's position along its trajectory. The case treated is that of a positive charge passing outside the solenoid on the right-hand side as in Fig. 3. Figure 6(b) indicates the interaction energy at the point $y=0$ of closest approach of the particle to the center of the solenoid.

H. Verification of Linear Momentum Conservation

Having verified the conservation of energy theorem for the particle-solenoid interactions, we next turn to conservation of linear momentum. We wish to show that the external forces on the particles of the solenoid correctly account for the changes in the system linear momentum. Again since the solenoid currents are constant and the external particle moves with constant velocity, any change of system momentum involves the momentum of the electromagnetic fields.

The momentum in the electromagnetic fields is given by

$$\vec{\mathcal{P}} = \frac{1}{4\pi c}\int_V \vec{E}\times\vec{B}\,d^3r\,, \tag{36}$$

where the volume V is all space. The momentum is separated into contributions independent of the relative displacement of the solenoid and the particle, and those depending on the relative positions. Thus

$$\vec{\mathcal{P}} = \frac{1}{4\pi c}\int_V (\vec{E}_s + \vec{E}')\times(\vec{B}_s + \vec{B}')\,d^3r\,, \tag{37}$$

and the position-dependent part is

$$\Delta\mathcal{P} = \frac{1}{4\pi c}\int_V (\vec{E}_s\times\vec{B}' + \vec{E}'\times\vec{B}_s)\,d^3r\,. \tag{38}$$

Since the electric field $\vec{E}_s$ due to the solenoid vanishes and the solenoid magnetic field is confined to its interior, this reduces to

$$\Delta\vec{\mathcal{P}} = \frac{1}{4\pi c}\int_V \vec{E}'\times\vec{B}_s\,d^3r$$

$$= \frac{1}{4\pi c}\int_{V_s} \vec{E}'\times\vec{B}_0\,d^3r\,. \tag{39}$$

Before the arrival of the passing particle, there

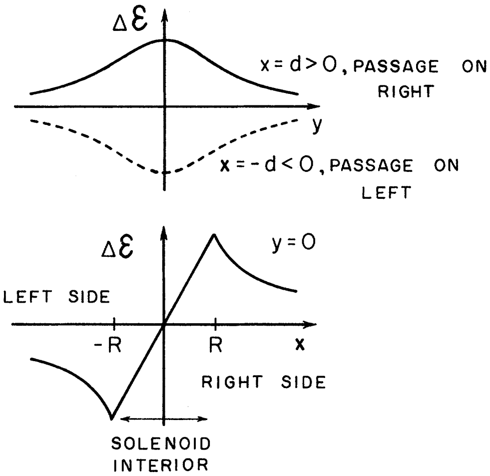

FIG. 6. Particle-solenoid interaction energy (33) following the coordinate system of Fig. 3.

are forces present upon the particles of the solenoid which confine the particles to the surface of the solenoid. However, these forces sum to zero around any circular curve c around the solenoid, and hence do not lead to a change in system linear momentum. The external forces which will lead to a change in momentum are those which balance the Lorentz forces on the solenoid particles due to the fields of the passing particle. These external forces are required to maintain the constancy of the mechanical momentum of the solenoid particles. The impulse delivered to the system by the external forces satisfies

$$\text{impulse on solenoid per unit time} = \sum_i \vec{F}_{i\ \text{external}}$$

$$= \sum_i -q_i\frac{\vec{v}_i}{c}\times\vec{B}'$$

$$= -\frac{1}{c}\int_V \vec{j}\times\vec{B}'\,d^3r\,, \tag{40}$$

where the symbols referring to the solenoid particles and current densities are just as in Sec. III F. The reader may wonder about the contributions from the external forces balancing the forces due to the electric fields of the passing particle. These forces are independent of the velocity of the solenoid particles and cancel between the circulating solenoid charges and the stationary charges of opposite sign. The velocity-dependent forces do not so cancel and so appear in Eq. (40).

The impulse due to the external forces should lead to a change in the linear momentum of the system. Thus we compute

$$\begin{aligned}\frac{d\Delta\vec{\mathcal{P}}}{dt}&=\frac{1}{4\pi c}\int_{V_s}\frac{\partial\vec{E}'}{\partial t}\times\vec{B}_0 d^3r\\&=-\frac{i\hat{k}}{c}\times\int_{V_s}\nabla\times\vec{B}'d^3r\\&=-\frac{i\hat{k}}{c}\times\int_S(\hat{n}\times\vec{B}')d\Sigma\\&=\frac{i}{c}\int_S\vec{B}'\times(\hat{k}\times\hat{n})d\Sigma\\&=\frac{i}{c}\int_{-\infty}^{\infty}dz\int_c\vec{B}'\times d\vec{r}\\&=-\frac{1}{c}\int_V\vec{j}\times\vec{B}'d^3r.\end{aligned}\tag{41}$$

In the manipulations, we have used Maxwell's equation

$$\nabla\times\vec{B}'=\frac{1}{c}\frac{\partial\vec{E}'}{\partial t}\tag{42}$$

for a source-free region, and have indicated by $\hat{n}\,d\Sigma=\hat{n}\,dz\,R\,d\theta$ an element of the solenoid surface with outward pointing normal $\hat{n}$ as in Fig. 5. In the expression of the integrand

$$\vec{B}'\times(\hat{k}\times\hat{n})=\hat{k}(\vec{B}'\cdot\hat{n})-\hat{n}(\vec{B}'\cdot\hat{k}),\tag{43}$$

we may omit the term $\hat{k}(\vec{B}'\cdot\hat{n})$ because the contributions to the integral at equal distances above and below the median plane cancel each other. The conversion from $-\hat{k}\times(\hat{n}\times\vec{B}')$ to $\vec{B}'\times(\hat{k}\times\hat{n})$ then follows since

$$\hat{n}(\vec{B}'\cdot\hat{k})=\hat{k}\times(\hat{n}\times\vec{B}').\tag{44}$$

Comparing Eqs. (40) and (41), we indeed conclude that the external forces on the particles of the solenoid (which balance the Lorentz forces) exactly supply the change of field momentum.

Inasmuch as we have seen that the changes in system energy and momentum are provided by external forces on the particles of the solenoid, it is of interest to see that the energy-momentum relations can be written in a somewhat misleading form. Using the relation between $\vec{E}'$ and $\vec{B}'$ in (23), the interaction energy can be rewritten

$$\begin{aligned}\Delta\mathcal{E}&=\frac{1}{8\pi}\int_V 2\vec{B}_s\cdot\vec{B}'d^3r\\&=\frac{1}{4\pi}\int\left(\frac{\vec{v}}{c}\times\vec{E}'\right)\cdot\vec{B}_s d^3r\\&=\vec{v}\cdot\frac{1}{4\pi c}\int_V\vec{E}'\times\vec{B}_s d^3r\\&=\vec{v}\cdot\Delta\vec{\mathcal{P}}.\end{aligned}\tag{45}$$

This expression looks appropriate for an external force acting on the external particle with velocity $\vec{v}$ to give a change in momentum and an accompanying change in energy (for differential changes). Actually the changes in energy and momentum refer not to the passing particle but to the electromagnetic fields.

I. Calculation of the System Linear Momentum

The change in system momentum due to the interaction of the particle and solenoid may be evaluated directly by integrating over the momentum density (39) using the expression (23) for $\vec{E}'$. However, as in the case of the energy, it is convenient to consider a nonrelativistic approximation in which the system momentum can be related to the value of the magnetic vector potential of the solenoid at the position of the passing particle.

Substituting (24) for the electric field $\vec{E}'$ of the passing particle,

$$\begin{aligned}\Delta\vec{\mathcal{P}}&=\frac{1}{4\pi c}\int_{V_s}\vec{E}'\times\vec{B}_0 d^3r\\&=\frac{1}{4\pi c}\int_{V_s}(-\nabla\phi+\vec{\beta}\,\vec{\beta}\cdot\nabla\phi)\times\vec{B}_0 d^3r\\&=\frac{i}{c^2}\int_{-\infty}^{\infty}dz\int_s(-\nabla\phi+\vec{\beta}\,\vec{\beta}\cdot\nabla\phi)\times d\vec{\sigma}.\end{aligned}\tag{46}$$

In the nonrelativistic approximation, we may drop the term $\vec{\beta}\vec{\beta}\cdot\nabla\phi$, and take $\phi(\vec{r},\vec{r}')$ as in (27). The linear momentum becomes

$$\begin{aligned}\Delta\vec{\mathcal{P}}&\cong-\frac{ie}{c^2}\int_{-\infty}^{\infty}dz\int_s\nabla\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\times d\vec{\sigma}\\&=\frac{ie}{c^2}\int_{-\infty}^{\infty}dz\int_c\frac{1}{|\vec{r}-\vec{r}'|}d\vec{r}\\&=\frac{e}{c^2}\int\frac{\vec{j}}{|\vec{r}-\vec{r}'|}d^3r\\&=\frac{e}{c}\vec{A}(\vec{r}'),\end{aligned}\tag{47}$$

where $\vec{A}(\vec{r}')$ is the vector potential (in the Coulomb gauge) of the solenoid at the position of the passing particle, given in Eq. (32) for $\vec{r}'$ outside the solenoid. Figure 7 shows the x and y components of the system linear momentum as a function of the passing particle's position on its trajectory.

In the case that the particle passes through the solenoid, the change in system linear momentum is again as in (47) with the expression for $\vec{A}(\vec{r}')$ as in (34). However, when the particle is in the solenoid interior, there is a Lorentz force on the particle due to the magnetic field $\vec{B}_0$, and a further external force must act on the particle in order to keep it moving with constant velocity.

This further external force does no work, but it contributes in the calculations for the rate of change of system linear momentum. In particular, Eq. (41) is no longer correct because the Maxwell equation (42) no longer involves a source-free region but rather has a contribution from the current of the particle through the integration volume V_S. The conservation theorem can be completed easily for this case by inserting the particle current

$$\frac{4\pi}{c}\vec{j}=\frac{4\pi}{c}e\vec{v}\delta^3(\vec{r}-\vec{r}\,')$$

in Eq. (42)

J. System Angular Momentum

Having considered the system energy and linear momentum, we turn to the angular momentum. For the situation involving constant solenoid currents, the changes in angular momentum are particularly simple being related directly to the changes in linear momentum.

As pointed out repeatedly, it is the external forces on the solenoid particles which keep the moving particles circulating with constant speed and keep the stationary particles stationary despite the forces exerted by the fields of the passing particle. Before the arrival of the passing particle, the external forces exert no torques and there is no change in the solenoid angular momentum about its center. Further, the external forces required to balance the forces due to the electric field of the passing particle cancel between the circulating and stationary solenoid particles which have opposite charges, and hence do not give any net torque on the system. The torques about the center of the solenoid required to balance those due to the forces from the magnetic field of the passing particle are of the form

$$\vec{\tau}_0=-\frac{1}{c}\int\vec{r}\times(\vec{j}\times\vec{B}')d^3r$$
$$=-\frac{1}{c}\int[\vec{j}(\vec{r}\cdot\vec{B}')-\vec{B}'(\vec{r}\cdot\vec{j})]d^3r\ . \tag{48}$$

Now the term $\vec{r}\cdot\vec{B}'$ is odd in z and hence $\vec{j}(\vec{r}\cdot\vec{B}')$ receives cancelling contributions from volumes equal distances above and below the equitorial plane of the solenoid. The term $\vec{B}'(\vec{r}\cdot\vec{j})$ vanishes because the currents $\vec{j}$ are always perpendicular to the radius vector $\vec{r}$ from the center of the solenoid. Hence the net torque about the center of the solenoid vanishes for the particle-solenoid system, $\vec{\tau}_0=0$.

If conservation of angular momentum is to hold, then it requires that there should be no changes

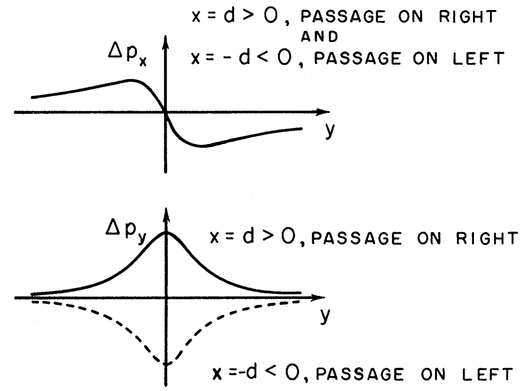

FIG. 7. The momentum (47) in the electromagnetic fields of the particle-solenoid system.

in the system angular momentum. Since the solenoid currents and the passing particle velocity are constant, there is no change in the mechanical angular momentum of the system, and all changes must come from the electromagnetic fields.

In order to obtain the changes as contrasted with the total field angular momentum, we subtract off the system angular momentum at spatial infinity and consider

$$\Delta\vec{\mathfrak{L}}_0=\frac{1}{4\pi c}\int_V\vec{r}\times(\vec{E}'\times\vec{B}_s)d^3r$$
$$=\frac{1}{4\pi c}\int_{V_s}\vec{r}\times(\vec{E}'\times\vec{B}_0)d^3r\ , \tag{49}$$

where the angular momentum is computed about the center of the solenoid. From symmetry of the system under reflection through the solenoid equatorial plane, it follows that only a z-component of angular momentum is possible. In evaluating the integral in (49), we choose spherical polar coordinates with origin at the instantaneous position of the passing particle. For each differential solid angle as shown in Fig. 8, we wish to consider two differential volume elements equal distances from the midpoint of the part of the solid angle cone lying inside the solenoid. The angular momentum contributions about the center of the solenoid cancel for these two volume elements. This follows since the contributions are

$$-aE'_\alpha B_0 r_\alpha{}^2\sin\theta d\theta\, d\phi\, dr\hat{k}\ +aE'_\beta B_0 r_\beta^2\sin\theta\, d\theta\, d\phi\, dr\hat{k}, \tag{50}$$

where r_α and r_β are distances from the origin of the polar coordinate system to the volume elements. The electric field of the passing particle is radially outward from the instantaneous position of the particle and has magnitude[16]

$$E' = \frac{e}{r^2}\,\frac{1-\beta^2}{(1-\beta^2\sin\Theta)^{3/2}} \quad , \tag{51}$$

where Θ is the angle between the radius vector from the particle and the velocity of the charge. Thus the factors of r_α^{-2} and r_β^{-2} from the field $\vec{E}'$ cancel the r_α^2 and r_β^2 from the volume elements, leaving contributions of equal magnitude and opposite sign. Pairing all contributions to the integral in this fashion, we see that the total system angular momentum about the center of the solenoid vanishes.

If some point other than the center of the solenoid is chosen for computations of the angular momentum, then the conservation theorem reduces to the linear momentum theorem confirmed above in Sec. III H. Thus

$$\begin{aligned}\vec{\tau}_Q &= -\frac{1}{c}\int_V (\vec{r}-\vec{r}_Q)\times(\vec{j}\times\vec{B}')d^3r \\ &= -\frac{1}{c}\int_V \vec{r}\times(\vec{j}\times\vec{B}')d^3r + \vec{r}_Q\times\frac{1}{c}\int_V \vec{j}\times\vec{B}'d^3r \\ &= \vec{\tau}_0 - \vec{r}_Q\times\vec{F}_{\text{external}} ,\end{aligned} \tag{52}$$

$$\begin{aligned}\Delta\vec{\mathfrak{L}}_Q &= \frac{1}{4\pi c}\int_V (\vec{r}-\vec{r}_Q)\times(\vec{E}'\times\vec{B}_s)d^3r \\ &= \frac{1}{4\pi c}\int_V \vec{r}\times(\vec{E}'\times\vec{B}_s)d^3r \\ &\quad -\vec{r}_Q\times\frac{1}{4\pi c}\int_V \vec{E}'\times\vec{B}_s d^3r \\ &= \Delta\vec{\mathfrak{L}}_0 - \vec{r}_Q\times\Delta\vec{\mathcal{P}} ,\end{aligned} \tag{53}$$

where $\vec{r}_Q$ is the radius vector from the center of the solenoid to the point Q. Above we showed that

$$0 = \vec{\tau}_0 = \frac{d\Delta\vec{\mathfrak{L}}_0}{dt} \tag{54}$$

and from Sec. III H, we have that

$$\vec{r}_Q\times\vec{F}_{\text{external}} = \vec{r}_Q\times\frac{d\Delta\vec{\mathcal{P}}}{dt} \quad . \tag{55}$$

Thus

$$\vec{\tau}_Q = \frac{d\Delta\vec{\mathfrak{L}}_Q}{dt} \tag{56}$$

and the conservation theorem for angular momentum is confirmed. The expression (52) for the torque about Q corresponds to the force $\vec{F}_{\text{external}}$ always acting at the center of the solenoid; the expression (53) for the system angular momentum about Q corresponds to a momentum which is always at the center of the solenoid, changing in magnitude but never position.[17] From (47), the nonrelativistic approximation for the angular momentum about Q is

$$\Delta\vec{\mathfrak{L}}_Q \cong \vec{r}^Q\times\frac{e}{c}\vec{A}(\vec{r}') , \tag{57}$$

where $\vec{r}^Q$ is the fixed displacement vector from Q to the center of the solenoid $\vec{r}^Q = -\vec{r}_Q$, and $\vec{A}(\vec{r}')$ is the magnetic vector potential (in the Coulomb gauge) of the solenoid at the position of the passing particle.

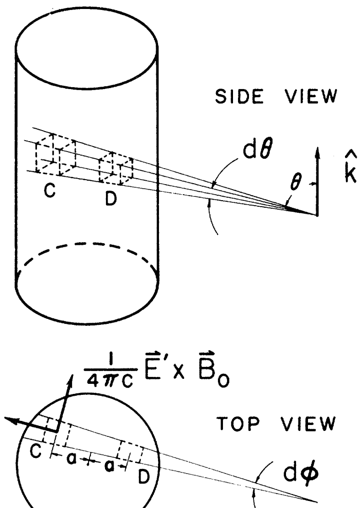

FIG. 8. Coordinate system for evaluating electromagnetic angular momentum about the center of the solenoid.

IV. INTERACTION OF A SOLENOID WITH A BEAM OF PARTICLES AND THE LIMIT OF A STEADY CURRENT

As an extension of the ideas presented above involving a solenoid and a point charge, it is natural to consider the classical electromagnetic interactions of a solenoid with a beam of point particles and, in the limiting case, with a line charge and steady current.

The linearity of the equations of classical electromagnetism makes it easy to evaluate these more complicated situations from the results already obtained for a point charge. The motion of each charge in our system is maintained quite independently of the presence of any other charges, and hence the fields due to the charges are independent. Since the changes of energy, linear momentum, and angular momentum when passing the solenoid all involve the product of the solenoid magnetic field $\vec{B}$ with the fields of the point charge, these quantities may be evaluated simply as sums over the single point-charge contribution $\Delta\mathcal{E}_j$, $\Delta\vec{\mathcal{P}}_j$, $\Delta\vec{\mathfrak{L}}_j$ from each charge,

$$\Delta\mathcal{E}_{\text{Beam}} = \sum_j \Delta\mathcal{E}_j \;,$$

$$\Delta\vec{\mathcal{P}}_{\text{Beam}} = \sum_j \Delta\vec{\mathcal{P}}_j \;, \tag{58}$$

$$\Delta\vec{\mathfrak{L}}_{\text{Beam}} = \sum_j \Delta\vec{\mathfrak{L}}_j \;.$$

An understanding of the changes may be obtained simply from Figs. 6 and 7. The graph for the situation of a beam of particles separated by distances δ_y is obtained by superimposing the graphs for the contributions from the individual particles, making a translation for the graph of the jth particle relative to the $j-1$st particle. Thus the energy change $\Delta\mathcal{E}_{\text{Beam}}$ for a beam of particles interacting with a solenoid follows from superimposing copies of Fig. 6, and will have a positive value with a ripple corresponding to the passage of each particle. In the limit of a steady line of charge passing, the interaction energy due to the presence of the solenoid is a constant unchanged in time. Analogous comments follow from superimposing many relatively-translated copies of Fig. 7. Thus $\Delta\mathcal{P}_{x\text{Beam}}$ will have a ripple in time about a zero value and will vanish for a steady current, corresponding to the absence of a component E_y for a line charge along the y axis. The component $\Delta\mathcal{P}_{y\text{Beam}}$ behaves like the energy, going to a constant value for a line charge. This corresponds to the presence of the radial electric field which gives a component E_x which combines with $\vec{B}_0 = B_0\hat{k}$ to give a y component of momentum in the electromagnetic field.

CONCLUSION

Although a constant-current solenoid exerts no forces upon a charged particle passing outside the solenoid, the fields of the passing particle exert forces upon the particles which carry the solenoid currents. We have verified in detail the laws of conservation of energy, linear momentum, and angular momentum for this situation, and have evaluated the contributions to these quantities contained in the electromagnetic fields. In the nonrelativistic approximation the expressions for the field energy and momentum take simple forms involving the vector potential due to the solenoid at the position of the passing particle. Specifically the change in energy in the electromagnetic field is

$$\Delta\mathcal{E} \cong e\frac{\vec{v}}{c}\cdot\vec{A}(\vec{r}\,'),$$

the change in momentum in the electromagnetic field is

$$\Delta\vec{\mathcal{P}} \cong \frac{e}{c}\vec{A}(\vec{r}\,'),$$

and the change in angular momentum in the electromagnetic field is

$$\Delta\vec{\mathfrak{L}}_Q \cong \vec{r}^{\,Q}\times\frac{e}{c}\vec{A}(\vec{r}\,').$$

Here the particle of charge e is moving with velocity $\vec{v}$. The vector potential due to the solenoid must be given in the Coulomb gauge and is evaluated at the instantaneous location $\vec{r}\,'$ of the passing charge. The change in field angular momentum $\Delta\vec{\mathfrak{L}}_Q$ is computed about the point Q where $\vec{r}^{\,Q}$ is the displacement of the center of the solenoid relative to Q. Computed about the center of the solenoid, $\Delta\vec{\mathfrak{L}}_Q$ vanishes.

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[8]See Ref. 7, Ampere's law and symmetry arguments form a traditional method for obtaining the field $\vec{B}_0$ near the center of a long solenoid.
[9]See, for example, J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962), Chapter 5. The magnetic induction $\vec{B}$ is the same whether the currents causing it are true currents or magnetization currents.
[10]See, for example, Ref. 9, Secs. 6.8 and 6.9.
[11]G. T. Trammel [Phys. Rev. 134, B1183 (1964)] raises the possibility of a solenoid circling around a passing charged particle.
[12]See Ref. 9, problems 14.11 and 14.12, pp. 502, 503. The present author has carried out computer calculations which show explicitly the cancellations due to the coherent interference of the fields of the particles moving in a circle.
[13]See the considerations of Sec. IIC.

[14]See Ref. 8, Eqs. (27)–(31).
[15]See Ref. 7, p. 197.
[16]See Ref. 7, p. 160 Eq. (12).
[17]The idea of connecting the nonrelativistic approximation for the energy, linear momentum, and angular momentum with the solenoid vector potential is taken from the work of G. T. Trammel, Phys. Rev. 134B, 1183 (1964). However, Trammel's calculation of the angular momentum is in error.

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Classical Electromagnetic Deflections and Lag Effects Associated with Quantum Interference Pattern Shifts: Considerations Related to the Aharonov-Bohm Effect

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Classical electromagnetic lag effects can give rise to quantum interference pattern shifts such as that observed experimentally in the Aharonov-Bohm effect involving electrons passing a solenoid. This paper presents an extensive comparison between interference pattern shifts based upon classical electromagnetic fields, and based upon classical electromagnetic potentials as suggested by Aharonov and Bohm. Stress is placed upon the difference between two types of interference pattern shifts: those involving deflection of the entire interference pattern and those involving a deflection of only the double-slit pattern while leaving the single-slit envelope undisplaced. The first type of shift is produced by a classical deflecting force. The second type of shift can be produced by classical electromagnetic lag effects, and is also the type of shift associated with the Aharonov-Bohm effect. The two types are confused in the literature. A new experiment is proposed which shows the relationship between a classical lag effect, due to electrostatic fields on electrons passing along different paths, and the associated quantum interference pattern shifts. The experiment is analyzed in detail using the WKB approximation in the Schrödinger equation and also semiclassical ideas. The classical limit for the situation illustrates the Bohr correspondence principle, showing the relative lag between the electron wave packets becoming a measurable classical lag with a disappearance of the interference pattern as the lag becomes large compared to the wave-packet dimensions. For small shifts, the phase change predicted for the new experiment is identical with the scalar potential effect proposed by Aharonov and Bohm for a slightly different, time-varying experimental arrangement. The theoretical and experimental differences for large phase shifts are noted. The possibility is raised that a new classical electromagnetic lag effect may occur for electrons passing a small solenoid. Using a particular model for energy conservation, the predicted lag effect can be calculated and is associated with a quantum interference pattern shift of the same magnitude as predicted by Aharonov and Bohm based upon the electromagnetic vector potential. Thus the possibility exists that the experiments of Chambers and of Möllenstedt and Bayh may not confirm the ideas of Aharonov and Bohm on the vector potential in quantum theory. Several experiments are suggested which allow confirmation that the Aharonov-Bohm effect indeed involves local effects of the classical electromagnetic potential, rather than local electromagnetic fields leading to a new classical lag effect and hence to the observed quantum interference pattern shift.

I. INTRODUCTION

A. The Need for an Analysis of Interference Pattern Shifts

The diffraction patterns produced by electrons passing through slits have formed a phenomenon[1] familiar to physicists for forty years. However, the shifts in these patterns due to electromagnetic effects have formed a subject of interest[2] within the last decade because they seem to present evidence for a new break between classical and quantum electrodynamics with regard to the role of the electromagnetic potentials. In the present paper, we will provide some new ideas and an extensive commentary on interference pattern shifts caused by classical electromagnetic fields, or, following the ideas of Aharonov and Bohm, by classical electromagnetic potentials.

It seems no surprise to physicists that classical electromagnetic fields lead to shifts in electron interference patterns. This influence of the classical upon the quantum aspects should be expected because of the close ties between classical and quantum electrodynamics. What seems unantici-