*On leave of absence from Khartoum University, Khartoum, Sudan.

[1]J. L. Anderson, *Principles of Relativity Physics* (Academic, New York, 1968), p. 414.
[2]A. Peres, Am. J. Phys. 40, 398 (1972).
[3]R. H. Dicke, in *Proceedings of the International School of Physics, Enrico Fermi Course XX*, edited by C. Møller (Academic, New York, 1961), p. 31.
[4]R. H. Dicke, *The Theoretical Significance of Experimental Relativity* (Gordon and Breach, New York, 1964), p. 8.
[5]See, for instance, C. Fronsdal and R. W. Huff, Phys. Rev. D 6, 2760 (1972).
[6]A. R. Prasanna, Nuovo Cimento Lett. 6, 420 (1973).
[7]C. Brans and R. H. Dicke, Phys. Rev. 124, 925 (1961).
[8]R. V. Pound and A. Rebka, Phys. Rev. Lett. 4, 337 (1960).
[9]G. Gamow, Phys. Rev. Lett. 19, 759 (1967).
[10]R. H. Dicke, Science 129, 621 (1959).
[11]J. Oppenheimer and H. Snyder, Phys. Rev. 56, 455 (1939).
[12]G. D. Birkhoff and R. E. Langer, *Relativity and Modern Physics* (Harvard Univ. Press, Cambridge, Mass., 1923).

PHYSICAL REVIEW D VOLUME 8, NUMBER 6 15 SEPTEMBER 1973

Unaccelerated-Returning-Twin Paradox in Flat Space-Time*

Carl H. Brans
Physics Department, Loyola University, New Orleans, Louisiana 70118

Dennis Ronald Stewart†
Physics Department, Tulane University, New Orleans, Louisiana 70118

(Received 14 May 1973)

The twin paradox in a flat space-time which is spatially closed on itself is considered. In such a universe, twin B can move with constant velocity away from twin A and yet return younger than A. This paradox cannot be resolved in the usual way since neither twin is accelerated or locally subject to other than flat Minkowski geometry. Thus there are no obvious kinematic, dynamic, or geometric distinctions between the two and yet one experimentally verifies that moving clocks are slowed while the other does not. A global analysis leads to the conclusion that the description of the topology of this universe has imposed a preferred state of rest so that the principle of special relativity, although locally valid, is not globally applicable.

INTRODUCTION

The twin or clock paradox arising from the relativity of time-interval measurements in Einstein's special theory of relativity has had a long and controversial career. Einstein himself raised the issue in its basic form in his original 1905 paper, but without extensive comment. The conventional interpretation of the problem as presented in most textbooks[1] is satisfactory for the majority of workers in the field. In its simplest form the paradox can be stated as follows. One twin, or observer with a clock, A, is at rest in a certain inertial reference frame. A second, identical, twin, B, starts from A's position registering the same time as A, but travels along a different world line and returns younger than A. The standard explanation is based on the assumption of special relativity that any standard clock will record the proper-time lengths of its world line as determined by the Minkowski metric,

$$d\tau^2 = dt^2 - dx^2, \tag{1}$$

where we use units in which $c=1$. Since A and B have different world lines, it is understandable that they can be of different lengths, i.e., elapsed proper times. Further, in the geometry of an indefinite metric such as (1), a straight timelike world line is the *longest* distance between its end points.

After mechanics is introduced, a further, dynamical, asymmetry arises. If the reference frame of A is inertial, then it seems clear that B must accelerate with respect to this frame during part of his journey in order to return to A. Thus, B is distinguished from A by not being force-free throughout his path so that his comoving reference frame is not inertial. The development of general relativity permits other versions of the problem to be discussed in which B also travels a geodesic, and is thus always in free fall. However, it is necessary here for B to travel through regions of space-time having different metric properties from those in A's neighborhood, thus providing for a geometric distinction between the two. While these interpretations seem to be generally satisfactory to most physicists, there have been objections voiced by some others.[2]

The purpose of this paper is to consider the twin paradox in special relativity in a closed, but still flat, space-time. For simplicity, we will restrict most of the discussion to one space dimension. In such a universe, twin B can travel *without acceleration or experiencing any metric dissimilarities relative to A* and yet return younger than A. In this case the dynamical or geometric distinctions cannot be made. Both A and B are inertial observers and see a flat space-time, yet the statement "The moving twin ages more slowly than the one at rest" is true for one but not the other. The source of this distinction will have to be found in the relationship of the kinematics to the topology. In fact, we will find that such a universe possesses an absolute rest reference frame so that the usual principle of special relativity is not applicable.

CLOSED SPACE

The model for the spatially closed space-time we will be using is the following. Consider a reference frame which assigns pairs of numbers, (x,t), to point events using standard procedures, including the Einstein synchronization of clocks. Now, however, we add the assumption that the space is closed by topologically identifying $x=0$ with $x=1$, for any fixed time, t. Thus, each spatial section t = constant has the topology of the circle and the full space-time, which we will denote by C^2, that of the cylinder (Fig. 1). In general, the pairs (x,t) are not unique representations of events, since (x,t) and $(x+n,t)$ represent the same event for any positive or negative integer n. It is also possible to use a complex representation which is unique, defining $z=e^{ix}$. In this notation the family of pairs (z,t), $|z|^2=1$, provides a one-to-one representation of the events in such a space-time.

The standard model for space-time is a manifold, which is covered by one or more coordinate patches. These are open sets, whose union covers the space-time and each of which provides a unique real numerical representation of the events within it. Further, the coordinate transformations in the intersection of two such patches must be regular functions in some sense.[3] It is clear then that the assignment of numbers (x,t) as described above for C^2 does not constitute a coordinate patch. In fact, C^2 requires at least two coordinate patches to cover it, as indicated, for example, in **Fig. 2.**

The generalization to a full three space dimensions is obvious, if not so easy to visualize. Numbers (x,y,z,t) are assigned to events in such a way that

$$\text{Event}(x,y,z,t)=\text{Event}(x+l,y+m,z+n,t), \qquad (2)$$

for any three integers l,m,n. In this case, the space-time structure is that of the topological product of a three-dimensional torus with the straight line. In this paper the discussion will be limited to the case of one space dimension although

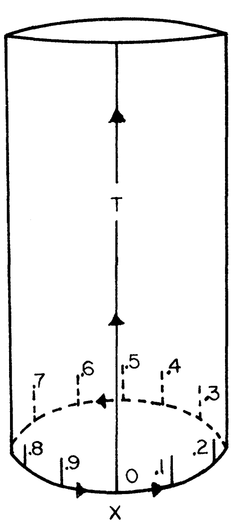

FIG. 1. The cylindrical model of our closed, flat space-time as seen by A. The unit-length stationary ruler has markings $x=0$, $x=0.1, \ldots,$ $x=1$, with the last point coinciding with the first. It should be noted that the ruler is straight, not curved, as might be suggested by this three-dimensional visualization.

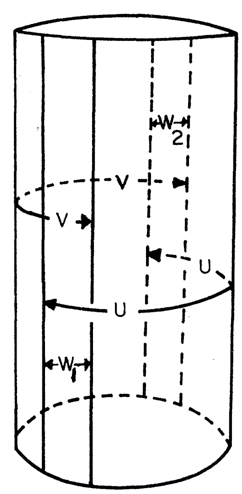

FIG. 2. This figure illustrates the domains U and V of two coordinate patches at rest with respect to A. The clocks associated with the two systems can agree with each other in each of the two overlap regions W_1 and W_2. This is possible because of the global synchronization transitivity in A's state of rest.

the three-dimensional one displays additional kinematic characteristics.

To fill out space-time with a Minkowski geometry, we assume that the (x,t) reference frame is inertial. Thus the metric is given by

$$\begin{aligned} ds^2 &= dx^2 - dt^2 , \\ d\tau^2 &= dt^2 - dx^2 . \end{aligned} \tag{3}$$

It should be noted that the nonuniqueness of the (x,t) representation does not affect the equations (3) since they involve only the interval dx, which is unique.

THE UNACCELERATED ROUND TRIP

Now consider the twin paradox in a space such as C^2. Let A be the twin who remains at rest at the (x,t) spatial origin while B moves in the positive x direction with speed v. Thus, the world line of A is the set of points $(0,t)$ while that of B is (vt,t). These are sketched in Fig. 3. Because of the closed nature of space, B returns to A after time intervals of $t_A = 1/v$ as measured by A's clock. On the other hand, if the analysis of special relativity is valid *locally* throughout B's trip, then his clocks are constantly registering intervals $dt_B = (1 - v^2)^{1/2} dt$. Hence, upon his first return B's clock will read $t_B = (1 - v^2)^{1/2} t_A = (1 - v^2)^{1/2}/v$. In other words, B returns to A younger than A by the factor $(1 - v^2)^{1/2}$ even though B has been in constant, uniform motion relative to A throughout the entire period. Since this is so, an attempt to apply Einstein's principle of special relativity leads to the following dilemma. Since A is inertial and B is unaccelerated relative to him, B also should be an inertial observer. Thus, all the laws of physics should be valid in the same form for both. However, A observes: "A clock moving with respect to me runs slower than my clocks and returns to me showing less elapsed time than mine." However, if we look at the same round trip from B's viewpoint, he has remained at rest and A has circled the universe and returned. Thus B sees A's clock as moving but nevertheless returning showing *more* elapsed time than B's. As a consequence, B, who apparently should also be an inertial observer, disagrees with A about the "law" that moving clocks run slow.

ANALYSIS OF THE PROBLEM

It is therefore apparent that something is wrong with Einstein's principle of special relativity in such a universe, but precisely where does it break down? Let us return to the two coordinate patch picture of the universe, Fig. 2. It is clear that in each such region of space-time the physics of special relativity should be valid and Lorentz transformations could be made to link the local rest system of A to one associated with B. However, a global extension of this to the entire universe uncovers some basic problems.

First, let us try to define a global reference frame attached to B similar to (x,t) for A. Such a system would assign coordinates (x',t') to events, with the world line of B being $(0,t')$. The corresponding Lorentz transformation is

$$\begin{aligned} x' &= \gamma(x - vt), \\ t' &= \gamma(t - vx), \end{aligned} \tag{4}$$

where $\gamma = (1 - v^2)^{1/2}$. Now the identification (2) must be translated into (x',t') system to give

$$\text{Event}(x',t') = \text{Event}(x' + \gamma n, t' - \gamma v n) \tag{5}$$

for arbitrary integer, n. It becomes clear very soon however that the identification (5) leads to insurmountable obstacles to considering (x',t') in any physically acceptable sense. Even though it is not unique, the (x,t) reference frame of A could be visualized in terms a ruler spanning the universe with the markings $x=0$, $x=1$ overlapping and with Einstein synchronous clocks attached. Why, now, can the (x',t') not be thought of as just such a ruler set in motion along with B? The first objection is that the line $t'=0$ is not closed under the identification (5). This means that a global family of clocks moving with B cannot be synchro-

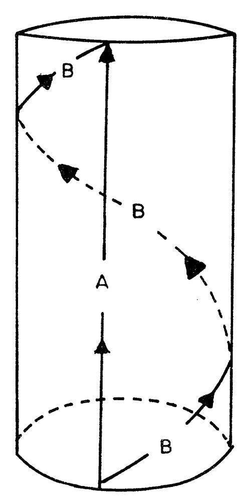

FIG. 3. The paths of the two twins A and B are shown here. In order to show both of their meetings within a diagram of reasonable size, the slope of B's line has been lowered. Note again that both A's path and that of B are straight lines in flat geometry, with the apparent curvature of B's path arising only from our attempt to visualize C^2 as imbedded in a flat three dimensional space. These drawings are tracings from photographs of straight lines drawn on flat transparent plastic which was then wrapped around a cylinder.

nized by the Einstein process which is basic to the definition of physical reference frames.[4] This process involves the stationing of an observer midway between pairs of clocks. If this observer sees light signals from each clock arriving simultaneously to him indicating that each clock is registering the same time, then the two clocks are synchronous. In the definition of the (x,t) reference frame we have assumed that the clocks used to measure t for different x's have been synchronized by such a procedure. However, in C^2 we must look at two aspects of this synchronization procedure that are taken for granted when it is applied in the usual, topologically Euclidean, space-time. The first concerns the question of whether there is a unique midpoint between two clocks in space and the second is whether or not this procedure is transitive, i.e., if clock 1 is synchronous with 2 and 2 with 3, will 1 necessarily be synchronous with 3? In C^2, there are obviously two midpoints between two given spatial points, so we must check to see whether or not different results are obtained when the observer is placed at one and then the other. Further, as the synchronization process is continued for clocks at x and $x+dx$ from $x=0$ around to the original clock at $x=1$, the question of transitivity is critical. For the system (x,t) at rest with A, these questions are easily resolved: Synchronization with light going one way around the universe yields the same result as for light going the other way, and the process is transitive. In fact, a brief analysis of the intersection of light rays, $x=\pm t$, with A's rest clocks, having paths $x=$constant, subject to the identification (2) verifies these statements. The global transitivity of synchronization becomes equivalent to the fact that for A's rest observers light circles the universe with the same elapsed time in both directions.

However when an attempt is made to build a global rest system for B these problems become crucial. In the first place, the single clock at rest with B reads different times for the paths of light rays going around the universe in different directions. To see this, express B's path and that of the two light rays in A's system as

$$x=vt, \qquad x=\pm t. \tag{6}$$

Using the identification, (2), these paths intersect regularly with elapsed intervals $t_1=1/(1-v)$ and $t_2=1/(1+v)$ or, as read by B's clock, $t_{B1}=\gamma/(1-v)$ and $t_{B2}=\gamma/(1+v)$. Thus B must say that the universe has a *longer circumference in one direction than in the opposite one*. Another anomaly would be the periodic reoccurrence of B's space-time origin, $x'=0$, $t'=0$, at the separated events whose (x,t) coordinates are $(v^2n\gamma^2, vn\gamma^2)$ for integer n. This result follows immediately from (4) and (5). Other difficulties also become apparent in looking at the (x',t') lines induced by (4) as sketched in Fig. 4.

Because of these problems, it is clear that no physically acceptable system with globally synchronized clocks can be set up moving along with B. This provides the basis for the distinction between A and B. The topological structure of C^2 contains within it the choice of an absolute rest system. That is, for A the lines of constant time are closed, but this is not true for B or any other moving observer. Hence, the principle of special relativity is not valid in such a system. Clocks moving with respect to A are slowed, but the reverse statement is not true, because of A's distinguished state of rest in C^2.

Of course, the principle of relativity can be *locally* valid in C^2. That is, C^2 can be covered by two coordinate patches corresponding to reference frames at rest with respect to B (Fig. 5). Just as the coordinates used in the patches U and V can differ from (x,t) by only constants in their respective domains the same is true of the "candy striping" patches, U' and V', in which the coordinates differ from (x',t') of (4) by only constants. In this case, however, *both the x' and the t' variables must disagree in at least one overlap*. For example, let B be at rest in the U' region with x' given by (4), but with x' restricted to the interval $(0,\frac{1}{2})$

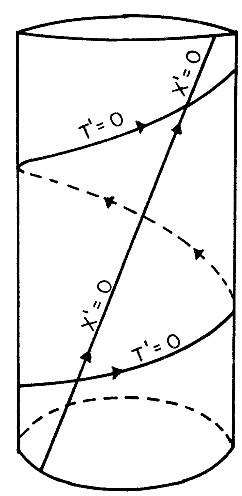

FIG. 4. Two of the coordinate lines of the (x', t') system as defined by (4) are indicated. Note that the origin, $x'=0$, $t'=0$, occurs periodically at separated space-time events. Further, the line $t'=0$ is not closed, indicating the impossibility of global synchronization of moving clocks.

and t' always increasing. This can be done by proper choice of integer n and replacing x by $x+n$. The same thing can be done in V' which will overlap U' in two regions, W_1' and W_2', near $x'=0$ and $x'=\frac{1}{2}$, respectively. The coordinates can even be made to agree in one of the overlap regions say W_1'. If so, they cannot agree in the other, W_2'.

The physical interpretation of these coordinate patches could then be as follows. B has a measuring rod, with attached clocks, moving with him, but covering only half of the universe. This is U'. Another rod and attached clocks, at rest with respect to the first, covers the other half of the universe. Within each patch, U' or V', the clocks are synchronized in a self-consistent, transitive way *as long as the light rays remain entirely within that patch*. At one of the meetings, say W_1' the markings on the rods and clocks agree. However, at their other meeting, W_2', not only will the spatial coordinates differ, but *so will the clocks*. As long as an observer such as B performs experiments entirely within one patch, the principle of relativity will hold and he will in no way be able to detect any effects of his motion relative to A. However, when he probes the other coordinate patch, he discovers that, although it is at rest with respect to him and some of its clocks are synchronous with his, not all of them are. Hence when he sends light rays travelling around the universe in different directions he is aware they will enter another system. As they do, they pass this system's clocks which run successively faster, or slower, with respect to B's depending on the direction of motion of the particular light ray. Since the light still has unit velocity in this other system, B will have to expect that rays traveling in different directions will have different round-trip times as measured by him.

None of these problems arise for A, who can have two coordinate patches at rest with respect to him, but with totally synchronized clocks. It is this global synchronization that A can perform, but moving observers cannot, that distinguishes A from them and provides the preferred state of rest.

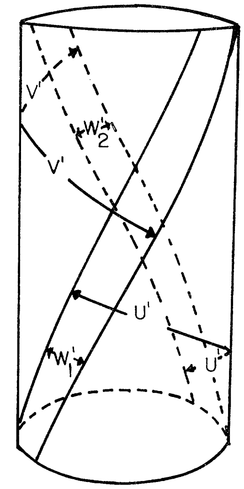

FIG. 5. Coordinate patches, U' and V', at rest with respect to B are shown. This is the analog for B of Fig. 2 for A. However, if the clocks of U' and V' agree in W_1' they will not agree in W_2'.

The implications of these ideas for the more realistic closed universes of general relativity are minimal for several reasons. In the first place, general relativity points out that special relativity need not be even approximately valid more than locally. Secondly, the cosmologies in general relativity make use of the matter in the universe to pick out a preferred state of rest. Hence, the fact that a global extension, even approximately, of a principle of special relativity breaks down is not surprising.

One of us (D.R.S.) would like to thank Les Fishbone and Vince Moncrief for helpful discussions with regard to the synchronization problems for moving clocks.

*Work based in part on a thesis presented by one of us (D.R.S.) to Tulane University for an Honors B.S. degree.

†Present address, Physics Department, University of Maryland, College Park, Maryland 20742.

[1]See for example, J. L. Anderson, *Principles of Relativity Physics* (Academic, New York, 1967), p. 174, or A. P. French, *Special Relativity* (W. W. Norton, New York, 1968), p. 154.

[2]A recent discussion was initiated by M. Sachs, Phys. Today 24 (No. 9) 23 (1971). This article was answered by numerous rebuttals in the Letters section of the January, 1972 issue of that journal.

[3]For a mathematical introduction to manifolds addressed to physicists see the article by R. Geroch, *General Relativity and Cosmology,* Proceedings of the International School of Physics "Enrico Fermi," Course XLVII (Academic, New York, 1971), p. 71.

[4]The related problem of trying to synchronize clocks on a rotating disk in conventional special relativity is discussed in H. P. Robertson and T. W. Noonan, *Relativity and Cosmology* (W. B. Saunders, Philadelphia, 1968), p. 58. Note however that these clocks are accelerated, whereas ours are not.