PHYSICAL REVIEW D VOLUME 8, NUMBER 5 1 SEPTEMBER 1973

Koba-Nielsen-Olesen Scaling at Finite Energies*

Alan Chodos and Morton H. Rubin
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19104

R. L. Sugar
Department of Physics, University of California, Santa Barbara, California 93106
(Received 15 January 1973; revised manuscript received 9 April 1973)

Since Slattery's analysis of the Serpukhov and National Accelerator Laboratory (NAL) multiplicity data shows Koba-Nielsen-Olesen (KNO) scaling behavior to remarkable accuracy, it is of interest to examine KNO scaling at finite energies. We discuss the fact that KNO's argument in favor of asymptotic scaling fails at any finite energy, and the corrections exceed the experimental limits at NAL energies. We show that energy conservation forbids exact moment scaling in any finite energy range, and we establish a sufficient condition for a distribution to obey scaling within the observed accuracy. We further point out that only the first two moments calculated by Slattery are independent. Finally, we produce a simple model showing how unitarity can generate the long-range correlations in the rapidity variable that KNO scaling implies.

Recently, Slattery[1] has analyzed the charged multiplicity data obtained in proton-proton collisions in the energy range $50 \leq E_{\text{lab}} \leq 300$ GeV, and has pointed out that, to a remarkable degree of accuracy, they satisfy a scaling law first proposed by Koba, Nielsen, and Olesen.[2]

Let σ be the total inelastic cross section at a given energy; let σ_n be the partial cross section for proton + proton $\rightarrow n$ charged particles. (For $n = 2$ we exclude the elastic contribution.) Define

$$\langle n^q \rangle = \sum_{n=2}^{\infty} n^q \frac{\sigma_n}{n}, \quad q = 1, 2, \dots \; . \tag{1}$$

Then the Koba-Nielsen-Olesen (KNO) scaling law can be stated in either of two ways:

$$\langle n^q \rangle = C_q \langle n \rangle^q \tag{2}$$

or

$$\frac{\sigma_n}{\sigma} = \frac{1}{\langle n \rangle} \psi\left(\frac{n}{\langle n \rangle}\right) . \tag{3}$$

Here the C_q's are independent of energy, and the function ψ depends on energy only through the variable $n/\langle n \rangle$, as indicated. KNO show that (2) and (3) are equivalent, provided that one is allowed to approximate the sum in (1) by an integral. (This will be permissible if $\langle n \rangle$ is large and the function ψ is reasonably smooth. In the range of Slattery's analysis, $5.4 < \langle n \rangle < 8.9$ and the empirically obtained scaling function is indeed quite smooth.)

The problem with this nice agreement between theory and experiment is, as Slattery pointed out, that KNO's result was based on arguments that are expected to hold only asymptotically, whereas the data obey Eqs. (2) and (3) in the energy range 50 to 300 GeV. In fact, KNO's argument explicitly fails in this energy range, for the following reason: They begin with the identity

$$A_q \equiv \langle n(n-1) \cdots (n-q+1) \rangle = \int dp_1 \cdots dp_q f^{(q)}(p_1, \dots, p_q) , \tag{4}$$

where $f^{(q)}$ is the q-particle inclusive cross section normalized to the total cross section. They assume Feynman scaling for $f^{(q)}$ so that

$$A_q = b_q (\ln s)^q + O((\ln s)^{q-1}) \; . \tag{5}$$

They then write that

$$A_q = \langle n^q \rangle + O((\ln s)^{q-1}) . \tag{6}$$

One thus has Eq. (2) with $C_q = b_q / b_1{}^q$, up to corrections of order $(\ln s)^{q-1}$. However, inspection of (4) shows us that

$$A_q = \langle n^q \rangle - \tfrac{1}{2} q(q-1) \langle n^{q-1} \rangle + \cdots$$

so that the corrections to (6) become comparable to the first term whenever $q \gtrsim \langle n \rangle^{1/2}$. In the range 50 to 300 GeV this means whenever $q \gtrsim 2$ or 3; and yet as Slattery and Weisberg[3] have pointed out, the data in this range obey Eq. (2) up to $q = 9$ or 10.

Clearly then, some other explanation than an early onset of Feynman scaling must be sought for the data, and several such explanations have appeared in the literature in the last few months. These models fall into two categories: (i) those in which scaling begins at, roughly, 50 GeV and persists to all energies[4]; and (ii) those in which scaling in the 50–300-GeV range is an "accident" due to the competing effects of two separate mechanisms, and is violated at higher energies.[5] We feel that there is at present insufficient experimental evidence to distinguish between these two categories, and *a fortiori* among the various models in each of the categories. Rather, we present three model-independent remarks, which, although they have less physical content than a specific model, must necessarily apply to any successful explanation of the data:

(A) No energy-conserving model can obey Eq. (2) exactly over a finite range of energy. *Proof:* Introduce the generating function

$$G(h)=\sum_{n=1}^{\infty} f(n)h^n \ , \tag{7}$$

where $f(n)\equiv \sigma_n/\sigma$. By definition $G(1)=1$. Since the $f(n)$ are all positive, $G(h)$ is an analytic function of h inside the unit circle. In fact at any finite energy $f(n)$ vanishes for n sufficiently large, so $G(h)$ is really a polynomial in h. We observe that

$$\begin{aligned}\langle n^q\rangle &= \sum_{q=0}^{\infty} n^q f(n)\\ &=\left[\left(\frac{d}{d\ln h}\right)^q G(h)\right]_{\ln h=0} ,\end{aligned} \tag{8}$$

so

$$G(h)=\sum_{q=0}^{\infty}\langle n^q\rangle(\ln h)^q/q\,! \ . \tag{9}$$

Now energy conservation required that $f(n)$ vanish for $n>n_{\max}$, where $n_{\max}$ is the largest number of particles that can be produced at center-of-mass energy $s^{1/2}$; $n_{\max}=\text{const}\times s^{1/2}$. As a result, $\langle n^q\rangle \leqslant (n_{\max})^q$, and the series in Eq. (9) converges for all values of $\ln h$. If we now assume that the $\langle n^q\rangle$ are given by Eq. (1) we find that

$$G(h)=\sum_{q=0}^{\infty}\frac{C_q}{q\,!}\,[\langle n\rangle \ln h]^q \ . \tag{10}$$

Thus $G(h)$ is a function of $h^{\langle n\rangle}$, which means that $G(h)$ has a branch point at $h=0$ except when $\langle n\rangle$ is an integer and possibly at some other isolated values of $\langle n\rangle$. (Recall that the C_q are independent of energy and therefore of $\langle n\rangle$.) Since $G(h)$ must be analytic within the unit circle, we have a contradiction, which shows that there must be deviations from Eq. (2) even at very high energies.

We note that the continuous variability of $\langle n\rangle$ is necessary for our result. Thus it is not strictly true that KNO moment scaling must break down for large enough q by energy conservation alone. For example, the distribution

$$\frac{\sigma_n}{\sigma}=\delta_{n,N/2}\ ,$$

where N is the maximum number of particles that can be produced at a given energy yields exact KNO moment scaling.

(B) Although Eq. (2) cannot hold exactly in the range 50–300 GeV, the data seem to indicate that it holds to about 1%, i.e., that the corrections are of order $(1/\langle n\rangle)^2$ and not $1/\langle n\rangle$. We show that this can be achieved by a distribution of the form

$$\frac{\sigma_n}{\sigma}=\frac{1}{b}\psi(an)\ , \tag{11}$$

where all the energy dependence is contained in the factors a and b, provided $\psi(0)=0$. We assume that a is a small parameter, and that ψ is a differentiable function of its argument. Then

$$b=\sum_n \psi(an)$$

so

$$\frac{\partial b}{\partial a}=\sum_n n\,\psi'(an)\ .$$

We now use the relation between the derivative and the finite difference:

$$\psi'(an)=\frac{1}{a}[\ln(1+\Delta)]\,\psi(an)\ , \tag{12}$$

where $\Delta\psi(an)=\psi(a(n+1))-\psi(an)$ and keep enough terms on the right-hand side of (12) to give us the desired accuracy. For example, if we write

$$\psi'(an)\approx\frac{1}{a}(\Delta-\tfrac12\Delta^2)\psi(an)\ ,$$

we obtain

$$\frac{\partial b}{\partial a}=-\frac{b}{a}-\frac{1}{2a}\psi(a)\ ,$$

and therefore[6]

$$b^{-1}=[\,a(1+\kappa a)]+O(a^3)\ ,$$

where $\kappa=\frac12\psi(0)$.

Similarly, we have

$$\begin{aligned}G(h,a)&=\sum_{n=1}^{\infty} h^n\frac{1}{b}\psi(an)\\ &\equiv h\left[g_0\left(\frac{h-1}{a}\right)+a\,g_1\left(\frac{h-1}{a}\right)+\cdots\right] .\end{aligned} \tag{13}$$

Evaluating $\partial G/\partial a$ in the same way we calculated $\partial b/\partial a$, we obtain the equations, with $x=(h-1)/a$:

$$\begin{aligned}&g_0(x)\ \text{arbitrary},\\ &g_1(x)=(\kappa-x)g_0-\tfrac12 x^2 g_0'-\kappa\ ,\\ &g_2(x)=\tfrac56\,\kappa x-\rho+(\rho-\kappa x+x^2)g_0\\ &\qquad+\tfrac12 x^2(\tfrac53 x-\kappa)g_0'+\tfrac18 x^4 g_0''\ ,\end{aligned} \tag{14}$$

with $\rho=\kappa^2+\frac{1}{12}\psi'(0)$. We can now calculate

$$\begin{aligned}\langle n^q\rangle &=\left(h\frac{\partial}{\partial h}\right)^q G(h,a)\bigg|_{h=1}\\ &=\tilde C_q\,(1/a)^q(1+\kappa a+\rho a^2)\\ &\quad-\tfrac16\,\delta_{q1}\kappa a+O((1/a)^{q-3})\,.\end{aligned} \tag{15}$$

Here

$$\tilde{C}_q = \left(\frac{d}{dx}\right)^q g_0(x)\bigg|_{x=0} .$$

Thus if $\kappa = 0$ [i.e., $\psi(0) = 0$], we have

$$\langle n\rangle = \tilde{C}_1/a, \quad C_q = \tilde{C}_q/(\tilde{C}_1)^q ,$$

and

$$\langle n^q\rangle = C_q\langle n\rangle^q + O(\langle n\rangle^{q-2}). \tag{16}$$

It is interesting to observe that the data are consistent with $\kappa = 0$. Note also that this result does not rely on approximating the discrete sum (1) by an integral.

(C) In view of the various models attempting to fit the data, and in particular the values of C_q, $q = 2, \ldots, 10$ which have been extracted from the data, one may ask how significant it is to fit all ten moments once the first two or three have been fitted. A partial answer to this may be obtained by looking at the cumulants of the distribution rather than at the moments themselves. The cumulants κ_q are given by equating coefficients of like powers of t in the equation

$$\exp\left(\sum_{q=1}^{\infty} \kappa_q \frac{t^q}{q!}\right) = 1 + \sum_{q=1}^{\infty} C_q \frac{t^q}{q!} . \tag{17}$$

The κ_q have had the effects of the lower moments subtracted out in much the same way that, for example, one defines q-particle correlation functions by subtracting out the q-body and lower inclusive cross sections. The experimental values of the C_q, $q = 1, \ldots, 9$, and the corresponding κ_q are shown in Table I. Since the equation relating the C_q and κ_q are of the form

$$C_q = \kappa_q + F_q(\kappa_1, \ldots, \kappa_{q-1}) , \tag{18}$$

it is clear from the errors quoted in Table I that all the κ_q, $q > 3$, are statistically consistent with zero. Thus one might expect that a one or two parameter fit to the data is indicated. Also, since $\kappa_q = 0$, $q > 2$ for a Gaussian distribution, it is not surprising that one of the best empirical fits to $\psi(n/\langle n\rangle)$ is of the form of a power times a Gaussian.[3]

In conclusion we wish to present a simple model which displays KNO scaling to leading order in $\langle n\rangle^{-1}$. As was pointed out by KNO such models must have significant long-range correlations in the rapidity variables. The model we present shows how unitarity can generate such long-range correlations. A particularly simple example can be obtained by assuming that the protons interact via the exchange of an arbitrary number of chains with each chain being allowed to emit or absorb only one secondary particle.[7,8] This model is very different in spirit from the multiperipheral model. Following the notations of Ref. 8 we write the amplitude for the exchange of a single chain in the form[9]

$$W(Y, B; y, b) = s\Lambda e^{-B^2/2R^2} e^{-b^2/2r^2} . \tag{19}$$

Here $Y = \ln(s/m^2)$ is the rapidity difference of the protons and B their relative impact parameter; y and b are the rapidity and impact parameter of the secondary. Λ, R, and r are constants. The cross section for the production of n secondaries is[8]

$$\sigma_n = \int d^2B e^{-C(Y,B)}[C(Y,B)]^n/n! , \tag{20}$$

where

$$C(Y,B) = \lambda Y e^{-B^2/R^2} \tag{21}$$

and λ is a constant. It should be noted that the n in Eq. (8) is not the n in the KNO scaling formula, since the latter is larger by two because of the protons. One easily finds that

$$f(n) = \frac{\pi R^2}{\sigma}\frac{1}{\langle n\rangle}\left(\frac{n}{\langle n\rangle}\right)^{-1}\theta\left(\frac{\pi R^2}{\sigma} - \frac{n}{\langle n\rangle}\right) + O\left(e^{-\lambda Y}\frac{(\lambda Y)^{n-1}}{n!}\right) , \tag{22}$$

with $\langle n\rangle = \lambda Y \pi R^2/\sigma$. Thus Eq. (3) is approximately satisfied at high energies.[10] On the other hand,

$$\langle n^q\rangle = \left(\frac{\sigma}{\pi R^2}\right)^{q-1} q^{-1}\langle n\rangle^q [1 + O(q/\langle n\rangle)], \tag{23}$$

so Eq. (2) breaks down for $q \gtrsim \langle n\rangle$. This should be contrasted with the case of a Poisson distribution where Eq. (2) breaks down for $q \gtrsim \langle n\rangle^{1/2}$.

TABLE I. The C_q are as calculated by Slattery from the data (except $C_1 = \kappa_1 = 1$ by definition), and as reported by Weisberg, Ref. 3. The κ_q are calculated using Eq. (17).

q	$C_q = \langle n_q\rangle/\langle n\rangle^q$	κ_q
1	1	1
2	1.2438 ± 0.0056	0.2438
3	1.813 ± 0.020	0.081
4	2.973 ± 0.057	0.0064
5	5.36 ± 0.15	−0.013
6	10.43 ± 0.39	−0.042
7	21.6 ± 1.1	−0.034
8	47.0 ± 2.8	−0.064
9	107.4 ± 7.8	1.24

The above model is obviously unrealistic. It merely illustrates the fact that it is possible to construct simple models with approximate KNO scaling. The challenging problem is to construct a realistic model which explains Weisberg's remarkable fit to the data.[11]

*Work supported in part by the Atomic Energy Commission and the National Science Foundation.

[1]P. Slattery, Phys. Rev. D 7, 2073 (1973); Phys. Rev. Lett. 29, 1624 (1972).

[2]Z. Koba, H. B. Nielsen, and P. Olesen, Nucl. Phys. **B40**, 317 (1972); Phys. Lett. **38B**, 25 (1972). Equation (2) will hereafter be referred to as KNO scaling.

[3]H. Weisberg, Phys. Rev. D 8, 331 (1973).

[4]S. Barshay, Phys. Lett. **42B**, 457 (1972); H. B. Nielsen and P. Olesen, Phys. Lett. **43B**, 37 (1973); H. Moreno, Phys. Rev. D 8, 268 (1973); D. Levy, Saclay Report No. DPh-T/73/9 (unpublished).

[5]F. Hayot and A. Morel, Saclay Reports No. DPh-T/73/4 and No. Dph-T/73/19 (unpublished); H. Harari and E. Rabinovici, Phys. Lett. **43B**, 49 (1973); K. Fialkowski and H. T. Miettinen, Phys. Lett. **43B**, 61 (1973); L. Van Hove, Phys. Lett. **43B**, 65 (1973); W. R. Frazer, R. D. Peccei, S. S. Pinsky, and Chung-I Tan, Phys. Rev. D **7**, 2647 (1973); C. Quigg and J. D. Jackson, NAL Report No. THY-93 (unpublished); R. C. Arnold and G. H. Thomas, Argonne Report No. ANL/HEP 7257 (unpublished). Not all the above authors treat KNO scaling specifically, but all are concerned with fitting the same data which Slattery has used to demonstrate the KNO scaling behavior.

[6]An arbitrary constant has been fixed by demanding that $1/b = a + O(a^2)$.

[7]G. Calucci, R. Jengo, and C. Rebbi, Nuovo Cimento **4A**, 330 (1971).

[8]R. Aviv, R. Blankenbecler, and R. Sugar, Phys. Rev. D **5**, 3250 (1972).

[9]For simplicity we neglect the quantum numbers of the secondaries.

[10]Actually $\sigma = \pi R^2 \ln(\lambda Y)$, but at the energy range of present accelerators variations in $\ln Y$ can be neglected.

[11]One approach would be to consider the more sophisticated models in which production occurs from more than one chain [S. J. Chang and T.-M. Yan, Phys. Rev. Lett. **25**, 1586 (1970); S. Auerbach, R. Aviv, R. Blankenbecler, and R. Sugar, Phys. Rev. D **6**, 2216 (1972)]. All such models will have long-range correlations.

PHYSICAL REVIEW D VOLUME 8, NUMBER 5 1 SEPTEMBER 1973

Effects of the Neutral Weak Current on the Process $\nu + (Z) \to \nu + \mu + \bar{\mu} + (Z)$

Kazuo Fujikawa*

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, England

(Received 26 December 1972)

The effects of the neutral weak current on the spectra of charged leptons in the process $\nu + (Z) \to \nu + \mu + \bar{\mu} + (Z)$ are discussed. We show that the effects of the neutral current can be expressed in terms of various distributions based on the purely $V-A$ interaction.

Following an interesting remark by Gell-Mann *et al.*[1] on the diagonal coupling of weak lepton currents, the neutrino production of the muon pair in a nuclear Coulomb field (see Fig. 1)

$$\nu_\mu + (Z) \to \nu_\mu + \mu + \bar{\mu} + (Z) \tag{1}$$

has attracted much attention.[2] Among those authors who studied the process (1) in the past,[3,4] Czyż, Sheppy, and Walecka[4] were the first to perform a detailed numerical analysis of the total cross section. With the advent of experimental feasibility of the process at NAL,[5] detailed numerical studies of various distributions in the process (1) have been performed.[6-9] All of these calculations are based on the "conventional" $V-A$ effective coupling

$$\begin{aligned}\mathcal{L} &= \frac{G}{\sqrt{2}} \bar{\psi}_\mu \gamma_\alpha (1-\gamma_5) \psi_{\nu_\mu} \bar{\psi}_{\nu_\mu} \gamma^\alpha (1-\gamma_5) \psi_\mu \\ &= \frac{G}{\sqrt{2}} \bar{\psi}_\mu \gamma_\alpha (1-\gamma_5) \psi_\mu \bar{\psi}_{\nu_\mu} \gamma^\alpha (1-\gamma_5) \psi_{\nu_\mu} .\end{aligned} \tag{2}$$

The effective Lagrangian (2) predicts several characteristic features of $V-A$ coupling. A marked asymmetry between μ and $\bar{\mu}$ is one of these characteristics.[6-9]

The coupling in (2), however, is not the only possible coupling among leptons. The recent study of unified models of leptonic interactions based on the spontaneously broken gauge symmetry (Higgs mechanism) indicates various other coupling schemes.[10] Those models of lepton interactions are particularly attractive because they are