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Physical Interpretation of the Multiparticle Generating Functional and Partition Function*

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It is shown that the multiparticle generating functional and the multiplicity generating function (or partition function) are experimental observables and can be measured directly for finite ranges of their parameters. Their derivatives can also be measured directly. Since these functions are observables, they are subject to statistical fluctuations due to the necessarily finite number of measurable events used in their evaluation. The expected rms fluctuations of the partition function are shown to be simply expressed in terms of the total number of events and the partition function itself. Examples are given to clarify these results and their physical interpretation.

I. INTRODUCTION

The large number of secondary particles produced in high-energy collisions, and the surfeit of possible experimental variables, has led to an inclusive approach to the description of such events. Consequently, a set of multiparticle correlation functions are introduced which allow a succinct characterization of the data. The inclusive differential cross sections are conveniently related to the exclusive cross sections by means of a formal generating functional.[1-4] Particular exclusive and inclusive cross sections can be found by taking functional derivatives with respect to its parametric function. The generating function of the multiplicity distribution, which was originally introduced by Mueller,[5] is achieved when the parametric function is replaced by a constant fugacity.

The analogy between the distribution of produced secondaries and the ensemble distribution of a gas or liquid system in statistical mechanics has been rather thoroughly discussed.[6,7] The multiplicity generating function (partition function) has been used to conveniently derive properties of certain models of the production processes.[8,9] The introduction of long-range correlations into the multiperipheral model has been achieved using this approach,[10] and the possibility of "phase transitions" has been discussed.[7,11]

In the above works, the generating functional and the multiplicity generating function are introduced as purely formal devices to simplify the mathematical discussion. In this paper, we wish to point out that the generating functional and the partition function are observables and can be directly measured by experiment, at least for a finite range of their variables (i.e., fugacity less than one).

The fugacity of a particular type of particle turns out to be the probability that such a particle, once produced, will not be detected. The partition function is then found to be the difference between the true total cross section, which can be measured by an absorption experiment, for example, and the one calculated by summing over detected events. The derivatives of the partition function can also be measured by varying the detector efficiency. Even the functional derivatives of the generating functional can be measured by the same technique.[12]

Since the partition function $Q(z)$ is an observable, it is subject to statistical fluctuations due to the finite number of observed events used to compute it. This rather unusual aspect of $Q(z)$ will be discussed and a simple formula for the expected fluctuations will be derived. This result is clarified by an explicit calculation of the fluctuations for two different multiplicity distributions. Let us turn now to a brief review of the generating functional approach.

II. PHYSICAL GENERATING FUNCTIONALS

In order to develop our formalism and interpretations in a convenient and simple form, it will be assumed that only one particle type is involved in the collision processes. The generalization to several particle species is straightforward. Following the formulation and notation of Brown, Ref. 1, the exclusive differential cross section for the production of n particles in the momentum interval $(d^3q_1)\cdots(d^3q_n)$ is written as $(n\geq 2)$

$$d\sigma_n^{\text{exc}}=\prod_{a=1}^{n}\frac{d^3q_a}{q_a^0}\,\delta^4\left(\sum_{b=1}^{n}q_b-P\right)|T_n|^2\,, \qquad (1)$$

where P^μ is the initial total four-momentum and T_n is the transition amplitude (including an incident-flux factor). This is a symmetric function of the n momenta $q_1, \ldots, q_n$. The exclusive generating functional $E[\phi]$ is now introduced in the form

$$E[\phi]=\sum_{n=2}^{\infty}\frac{1}{n!}\int d\sigma_n^{\text{exc}}\phi(q_1)\cdots\phi(q_n). \tag{2}$$

The exclusive cross sections can be extracted from $E[\phi]$ by taking the appropriate number of functional derivatives with respect to $\phi(q)$ and setting $\phi=0$. The total cross section can be written as

$$\sigma_{\text{tot}}=E[\phi=1]. \tag{3}$$

The inclusive cross sections are defined as $(n+m\geqslant 2)$

$$d\sigma_n^{\text{inc}}=\prod_{a=1}^{n}\frac{d^3q_a}{q_a^0}\sum_{m=0}\frac{1}{m!}\int\prod_{b=1}^{m}\frac{d^3q_b}{q_b^0}\delta^4\left(\sum_{c=1}^{n+m}q_c-P\right)\times|T_{n+m}|^2, \tag{4}$$

and the corresponding inclusive generating functional $I[\phi]$ is found to be related to the exclusive generating functional by

$$I[\phi]=E[1+\phi]. \tag{5}$$

Let us now turn our attention to a superficially quite different problem, namely, the effect of an imperfect experimental detection efficiency on the measured cross sections.[13] The experimental detection probability of a particle of momentum $\vec{q}$ is denoted by $d(\vec{q})$ and will be assumed to be known. The raw experimental exclusive cross section $d\Sigma_n^{\text{exc}}$, which is deduced directly from the data without correcting for the detection efficiency, depends on the probability of detecting n particles and on the probability of missing the rest $(n+m\geqslant 2)$:

$$d\Sigma_n^{\text{exc}}=\prod_{a=1}^{n}\frac{d^3q_a}{q_a^0}d(\vec{q}_a)\sum_{m=0}\frac{1}{m!}\int\prod_{b=1}^{m}\frac{d^3q_b}{q_b^0}[1-d(\vec{q}_b)]\times\delta^4\left(\sum_{c=1}^{n+m}q_c-P\right)|T_{n+m}|^2. \tag{6}$$

The experimental exclusive cross section is therefore a type of inclusive cross section. Note that there is a finite probability that only one particle will be detected in the final state, hence $d\Sigma_1^{\text{exc}}$ is not zero. The measured elastic scattering cross section is $d\Sigma_2^{\text{exc}}$.

The experimental total cross section Σ_{tot} is found by integrating the above over the n-particle phase space and summing from $n=1$ to $n=\infty$. The result is

$$\begin{aligned}\Sigma_{\text{tot}}&\equiv\sum_{n=1}^{\infty}\frac{1}{n!}\int d\Sigma_n^{\text{exc}}\\&=\sum_{l=2}^{\infty}\frac{1}{l!}\int\prod_{a=1}^{l}\frac{d^3q_a}{q_a^0}\delta^4\left(\sum_{b=1}^{l}q_b-P\right)|T_l|^2\\&\quad\times\left[1-\prod_{c=1}^{l}[1-d(\vec{q}_c)]\right],\end{aligned} \tag{7}$$

or in other words,

$$\begin{aligned}E[1-d]&=I[-d]\\&=\sigma_{\text{tot}}-\Sigma_{\text{tot}}[d].\end{aligned} \tag{8}$$

This relation allows one to directly measure the functional dependence of $E[\phi]$ for arbitrary values of ϕ such that $0<\phi(\vec{q})<1$. The physical interpretation of this equation is clear. The difference between the true total cross section and the experimental total cross section is $E(1-d)$, which simply measures the total number of events in which no particles at all are detected.

The experimental multiplicity is a quantity of considerable interest and it is given by

$$\begin{aligned}\langle n\rangle_{\text{exp}}\Sigma_{\text{tot}}&\equiv\sum_{n=1}\frac{n}{n!}\int d\Sigma_n^{\text{exc}}\\&=\sum_{l=2}\frac{l}{l!}\int\prod_{a=1}^{l}\frac{d^3q_a}{q_a^0}\delta^4\left(\sum_{b=1}^{l}q_b-P\right)|T_l|^2d(\vec{q}_1)\\&=\int d\sigma_1^{\text{inc}}d(\vec{q}_1),\end{aligned} \tag{9}$$

which is the average value of $d(q)$ over the true inclusive cross section.

The mathematical functional $E[\phi]$ is introduced purely formally in order to reproduce the exclusive cross sections by taking functional derivatives with respect to ϕ and then setting ϕ to zero. However, these purely formal manipulations can actually be carried out and have a definite operational meaning. Consider two experiments, one run at a detection efficiency of $d(q)$ and the other at a slightly different value of $d(q)+\delta d(q)$, where $\delta d(q)$ is an arbitrary change in the detection efficiency $d(q)$. The difference of the experimental total cross sections is

$$\begin{aligned}\delta\Sigma_{\text{tot}}&=\Sigma_{\text{tot}}[d+\delta d]-\Sigma_{\text{tot}}[d]\\&=\int\frac{\delta E[1-d]}{\delta[1-d]}\delta d\\&=\sum_n\frac{n}{n!}\int d\sigma_n^{\text{exc}}\prod_{a=2}^{n}[1-d(\vec{q}_a)]\delta d(\vec{q}_1)\\&=\int\delta d(\vec{q})d\Sigma_1^{\text{exc}}/d(\vec{q}).\end{aligned} \tag{10}$$

By choosing $\delta d(q)$ to be zero except for a narrow range of values of $\vec{q}$, the exclusive cross section in this range is determined by this difference in the measured total cross sections. Higher-order differences determine the higher exclusive cross sections. For example, the double difference yields

$$\delta^2\Sigma_{\rm tot} = \Sigma_{\rm tot}[d+\delta d] - 2\Sigma_{\rm tot}[d] + \Sigma_{\rm tot}[d-\delta d]$$
$$= \int \frac{\delta^2 E[1-d]}{\delta(1-d)^2}\delta d\delta d$$
$$= -\int \delta d(\vec{q}_1)\delta d(\vec{q}_2) d\Sigma_2^{\rm exc}/d(\vec{q}_1)d(\vec{q}_2)\,, \qquad (11)$$

where $d\Sigma_2^{\rm exc}$ is the experimental elastic cross section. Clearly this is not a very practical way to extract information from the data, but it does illustrate that the functional derivatives of the generating functional are experimentally realizable.

III. PARTITION FUNCTIONS

The partition function can be introduced as a special case of the generating functionals. Setting the function $\phi(q)=z$, the partition function $Q(z)$ is defined as

$$Q(z)=E[z]$$
$$=\sum_{n=2} z^n \sigma_n^{\rm exc}\,.$$

Using this definition, the relation between the experimental and the true total cross sections is [see Eq. (8)]

$$Q(1-D) = \sigma_{\rm tot} - \Sigma_{\rm tot}\,, \qquad (12)$$

where D is the (constant) detection probability. Therefore, one sees that the partition function is an experimental observable, at least for values of its argument between zero and one, since $\Sigma_{\rm tot}$ is measured directly by counting events (and depends on D), and $\sigma_{\rm tot}$ can be measured independently (for example, by an absorption experiment). Since D can be varied experimentally, the derivatives of Q are also directly measurable. Following the discussion in Sec. II, the derivative of Q is related to the difference in two measurements of $\Sigma_{\rm tot}$ at slightly different detection efficiencies:

$$\delta\Sigma_{\rm tot}[D] = \Sigma_{\rm tot}[D+\delta D] - \Sigma_{\rm tot}[D]$$
$$=\delta D\, Q'(1-D)\,.$$

The multiplicity is therefore given by

$$\sigma_{\rm tot}\langle n\rangle = \left.\frac{d\Sigma_{\rm tot}}{dD}\right|_{D=0}$$
$$=Q'(1)\,. \qquad (13)$$

Thus a differential measurement of the total cross section at very poor detection efficiency ($D\sim 0$) measures the derivative of the partition function directly at one.

The experimental partition function $Q_x(z)$ is calculated by taking the measured values of $\Sigma_n^{\rm exc}$ and forming the sum

$$Q_x(z) = \sum_{n=1} z^n \Sigma_n^{\rm exc}\,. \qquad (14)$$

For the case of a general $d(\vec{q})$, the experimental partition function can be shown to be

$$Q_x(z) = E[1-(1-z)d] - E[1-d]\,, \qquad (15)$$

which at $z=1$ is equivalent to (8).

If d has the constant value D, then note that the experimental exclusive cross sections are

$$\Sigma_n^{\rm exc} = \sum_{m=0}^{\infty} \frac{(n+m)!}{n!m!} D^n (1-D)^m \sigma_{n+m}^{\rm exc}\,.$$

In this case, it is a simple matter to define $1-z=(1-x)/D$ and expand both sides of Eq. (15) to yield the inverse relation

$$\sigma_l^{\rm exc} = \sum_{n=l}^{\infty} \frac{n!}{l!(n-l)!}(D-1)^{n-l}D^{-n}\Sigma_n^{\rm exc}\,, \qquad (15')$$

which expresses a truly exclusive quantity in terms of inclusive ones. The fact that the left-hand side vanishes for $l=0$ and 1 allows one to express $\Sigma_1^{\rm exc}$ and $\Sigma_0^{\rm exc}$ [$=Q(1-D)$] in terms of Σ_n for $n>2$.

One can introduce an experimental exclusive generating function in analogy to Eq. (2) with $d\Sigma_n^{\rm exc}$ in place of $d\sigma_n^{\rm exc}$. Proceeding as above, and assuming that $d(\vec{q})$ does not vanish, one finds the more general inverse relation

$$\sigma_l = \sum_{n=l}^{\infty} \frac{n!}{l!(n-l)!}$$
$$\times \int d\Sigma_n^{\rm exc} \prod_{a=1}^{n-l}[d(\vec{q}_a)-1]\Big/ n! \prod_{b=1}^{n} d(\vec{q}_b)\,, \qquad (16)$$

which reduces to Eq. (15') above when d is constant. This is a generalized statement of the well-known relation expressing the exclusive cross sections in terms of the inclusive ones.

The experimental value of the average multiplicity is given by

$$\langle n\rangle_{\rm exp} = \left.\frac{d}{dz}\ln Q_x(z)\right|_{z=1}$$
$$= \Sigma_{\rm tot}^{-1}\left.\frac{d}{dz}E[1-(1-z)d]\right|_{z=1}\,,$$

which agrees with Eq. (9). Thus $Q_x(z)$ is the proper partition function for characterizing experimental quantities since it takes into account the detection efficiency. To extend the above discussion to several species of produced particles, one simply notes that each type has its own detection probability and hence its own fugacity, and proceeds accordingly by expanding in multinomial series.

IV. STATISTICAL ERRORS OF $Q_x(z)$

Let us now consider the effect of statistical errors on the partition function by examining a simplified but not unrealistic experiment. It will be assumed that in a particular experiment a total of N events are detected with N_n being the number of events with measured multiplicity n, where

$$N = \sum_{n=1} N_n .$$

Our fundamental statistical assumption will be that the expected fluctuation in counts satisfies the familiar relation

$$\langle N_n N_m \rangle - \langle N_n \rangle \langle N_m \rangle = \delta_{nm} \langle N_n \rangle , \tag{17}$$

where the angular bracket stands for an ensemble average.

The partition function, normalized to the total number of events at $z=1$, is

$$Q_N(z) = \sum_n z^n N_n .$$

The expected fluctuation in this quantity follows from the expected fluctuation in the N_n:

$$\begin{aligned} \langle Q_N{}^2(z) \rangle - \langle Q_N(z) \rangle^2 &= \langle Q_N(z^2) \rangle \\ &= \langle Q_N(z^2) \rangle \langle Q_N(1) \rangle / N . \end{aligned} \tag{18}$$

Therefore the expected statistical fluctuations in the conventionally normalized partition function are expected to be of the order

$$Q_x(z) \simeq \langle Q_x(z) \rangle \pm N^{-1/2} [\langle Q_x(1) \rangle \langle Q_x(z^2) \rangle]^{1/2} . \tag{19}$$

This is a convenient form for estimating the errors of Q_x since it involves only a knowledge of Q_x itself and the total number of events.

In order to clarify this result and its implications, it is instructive to consider and example. Let us assume that the experimental values of N_n happen to have a Poisson distribution and, as is customary, we will set $\langle N_n \rangle$ equal to the experimental distribution and estimate the expected fluctuations. Since we have ($\langle N_0 \rangle = 0$)

$$\langle N_{n+1} \rangle = N \frac{c^n}{n!} e^{-c} ,$$

the associated partition function is

$$\langle Q_N(z) \rangle = \{ N z e^{(z-1)\langle n-1 \rangle} \} ,$$

where $\langle n-1 \rangle = c$ (note that this multiplicity is shifted compared to the ordinary one). The expected fractional fluctuation in $Q_x(z)$ is then

$$\frac{Q_x(z)}{\langle Q_x(z) \rangle} \simeq 1 \pm N^{-1/2} \exp[\tfrac{1}{2}(z-1)^2 \langle n-1 \rangle] . \tag{20}$$

The fractional error is a minimum at $z=1$ since the maximum amount of experimental information is used. This is true for any distribution function $\langle N_n \rangle$. For small values of z, only the lowest multiplicities matter, and since they involve only a small fraction of the total number of particles, the statistical errors are larger. A similar argument holds for the more interesting region of large z, which is sensitive to the decreasing number of events with multiplicities much above the average.

As a second example, consider the generalized distribution which has been discussed by Hoang,[14]

$$\langle N_{n+1} \rangle \propto \frac{(c+bn)^n}{n!} .$$

This becomes a Poisson distribution in the limit of zero b, but is otherwise quite different for large values of n. For this distribution the partition function is

$$\begin{aligned} \langle Q_x(z) \rangle &= \langle Q_x'(0) \rangle \sum_0^\infty z^{n+1} (c+bn)^n / n! \\ &= \langle Q_x'(0) \rangle y (1-by)^{-1} e^{(c-b)y} , \end{aligned} \tag{21}$$

where $y = y(z)$ is given by $z = y e^{-by}$. The sum converges if $|zbe| < 1$ $(0 < |y| < b^{-1})$. Defining $y_1 = y(z=1)$, the total cross section is given by

$$\Sigma_{\text{tot}} = \langle Q_x'(0) \rangle (1-by_1)^{-1} e^{cy_1} .$$

Many other interesting distributions of this type can be generated by differentiating with respect to the parameters of this distribution.

The average multiplicity $\langle n \rangle$ is easily found by differentiating with respect to z and then setting $z=1$, and by using the fact that dz/dy is known. The result is

$$\langle n-1 \rangle = y_1 (1-by_1)^{-2} [c(1-by_1)+b] .$$

It is also easily computed that

$$\begin{aligned} &\langle (n-1)(n-2) \rangle - \langle n-1 \rangle^2 \\ &\quad = b y_1{}^2 (1-by_1)^{-4} [c(1-by_1)(2-by_1) + b(3-by_1)] , \end{aligned}$$

which vanishes for a Poisson distribution ($b=0$).

The expected statistical errors are easily computed from Eq. (21). If terms of order b^2 are dropped, one finds that the fractional statistical error is

$$\frac{Q_x(z)}{\langle Q_x(z) \rangle} = 1 \pm N^{-1/2} \exp[\tfrac{1}{2}(z-1)^2 \langle n-1 \rangle A] , \tag{22}$$

where

$$A = 1 + b[(1+z)^2 - 2] + O(b^2)$$

and

$$\langle n-1\rangle = c(1+2b)+b+O(b^2)\,.$$

Thus we see that this distribution leads to roughly the same type of statistical error as the Poisson, at least for values of b small compared to 1. For both examples, one sees that for a given fractional error in Q, the value of z that can be reached increases only as the square root of the logarithm of the total number of events.

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Unitarity Constraints on Moments of Inclusive Cross Sections

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It is shown that various moments of inclusive cross sections, such as the average transverse momentum, must satisfy rigorous inequality constraints involving the slope of the diffraction peak.

Considerable progress has been made recently in deriving rigorous unitarity constraints on inclusive cross sections.[1-3] Most of the results obtained so far have taken the form of fixed-angle bounds in terms of elastic-scattering quantities. The purpose of this paper is to show that one can also derive, from unitarity alone, a class of relations between moments of inclusive cross sections and the diffraction slope parameter,

$$B(s)=\left[\frac{d}{dt}\ln A(s,t)\right]_{t=0}, \tag{1}$$