Model of Pion Excitation

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We study the vector and axial-vector excitation of a pion to an arbitrary single-particle state. The main inputs to our calculation are a hard-pion representation for the relevant off-shell matrix elements, and use of partial conservation of axial-vector current, the Bjorken-Johnson-Low theorem, and current algebra as constraints. We find that for a model incorporating a finite number of poles, the pion cannot be excited to arbitrarily high-spin states. In a particular model we find a cutoff of $J = 4$, and also obtain predictions involving branching ratios for the decay of $f(1268)$ and $g(1680)$ mesons into $A_1(1070)\pi$ and $\pi\pi$ composites. The results of our model are checked by comparing with sum rules based on local commutation relations.

I. INTRODUCTION

In this paper, we shall construct and analyze a model of the vector and axial-vector excitation of the pion to an arbitrary single-particle state. In principle, this phenomenon is observable in processes where the pion interacts by means of a current with external probes such as photons or lepton pairs. Although pion targets are not available, the type of reaction described above has been observed for the nucleon in electroproduction experiments. If both the invariant momentum transfer and the energy transfer are not too large [say, $-k^2 \lesssim 1$ $(\text{GeV}/c)^2$, $\Delta E \lesssim 2$ GeV], then excitation of the nucleon to resonances can dominate the cross section. In such situations, the magnitude and momentum-transfer dependence of transition-matrix elements can be read off from the data.

The model of pion excitation used in this paper is an extension of the method inaugurated by Schnitzer and Weinberg,[1] and modified later by Brown and West.[2] The original hard-pion calculation of Schnitzer and Weinberg involved taking the $A_1(1070)$, $\rho(765)$, and π mesons simultaneously off their mass shells by relying upon the well-known field-theoretic association of these particles with vector and axial-vector currents and the axial-vector divergence. Certain smoothness assumptions were then imposed upon vertex functions related by Ward identities. In the calculation of Brown and West, equivalent results were derived for subsystems of the full $A_1\rho\pi$ system by keeping one of the particles on its mass shell. For example, a typical quantity to be studied would be

$$\int d^4x\, e^{-iq\cdot x} \langle A_{1c}(p,\lambda)|\, T(V_b^\mu(0)\,\partial_\sigma A_a^\sigma(x))\,|0\rangle . \tag{1}$$

In addition, it was shown by Brown and West that application of the Bjorken-Johnson-Low (BJL) limit[3] to (1) yields constraints on the model. The results thus obtained turn out to be consistent with the original work of Schnitzer and Weinberg.

We shall be studying matrix elements of currents which connect a pion to an arbitrary-spin meson. Unfortunately, there is, in general, no known relation between an arbitrary-high-spin meson and a local operator which is, in some way, related to measurable phenomena. Thus, when we perform a calculation involving off-shell dynamics, we must keep the high-spin meson on its mass shell. For this reason, our calculation will resemble that of Brown and West more than that of Schnitzer and Weinberg.

To be more precise, we shall analyze off-shell amplitudes of the type

$$\int d^4x\, e^{-iq\cdot x} \langle M^J(p,\lambda)|\, T(J^\mu(0)\,\partial_\lambda A^\lambda(x))\,|0\rangle , \tag{2}$$

where M^J is a spin-J meson of momentum p and helicity λ, J^μ is either a vector or axial-vector current, and isospin indices are temporarily suppressed. Throughout, the usual off-shell relation between the pion and the divergence of the isospin axial-vector current [partial conservation of axial-vector current (PCAC)] will be adhered to.[4] Our dynamical model for (2) will consist of a collection of pion and $A_1(1070)$ [or $\rho(765)$, depending on the current] poles as well as constant terms.[5] When taken on the mass shell, such an amplitude contains particle poles plus constants. This approach is clearly meant to reproduce the correct behavior of the current matrix elements at reasonably small values of momentum transfer. The constants simulate contributions from states of higher mass than the π, A_1, ρ particles, perhaps even subtraction constants. As we shall show in Sec. III, for $J=1$ our model is equivalent to that of Schnitzer and Weinberg. Our parametrization automatically reproduces their smoothness assumptions. We shall use PCAC and current algebra to provide constraints on the model. Moreover, encouraged by the results of Brown and West, we shall apply the BJL theorem to our amplitudes. It will become evident for matrix elements involving excitation of the pion to a high-spin meson that this places rather powerful restrictions on the model. The key point is that imposition of the BJL theorem to an amplitude such as the one defined in (2) implies a magnitude for the amplitude of at most $O((q^0)^{-1})$. However, one can also express the off-shell amplitude in terms of form factors and spin wave functions. When the BJL limit is taken for this latter representation, the form factors become multiplied by large powers of q^0 arising from the orbital angular momentum needed by the pion + current composite to couple to the high-spin meson M^J. Thus the form factors are tightly constrained. In fact, we shall show that in a model incorporating π and A_1 (or ρ) poles for the current matrix element, the $\pi\pi$ and πA_1 (or $\pi\rho$) systems do not couple to resonances with $J \geq 4$.

Some discussion of our outlook is in order at this point. The model just described has both virtues and defects. Its chief limitations are probably its lack of unitarity, and the simultaneous imposition of PCAC and the BJL theorem as calculational procedures. The former limitation has been overcome in certain specific hard-pion models, where it has been shown that analyticity and unitarity can form the basis for a hard-pion calculation.[6] However, we cannot see how to do this generally, particularly for amplitudes containing high-spin mesons. Although, in principle, one could write a collection of unitarity relations, these would be too complicated, due to the presence of the many contributing intermediate states, to be useful. The latter of the above limitations is evident. When we employ PCAC in an amplitude constrained by the BJL theorem, we are evidently taking seriously the idea that the divergence of the axial-vector current properly describes the pion far off its mass shell. Clearly, any justification of this assumption must necessarily follow from the success of our results. Because it gave good results for a particular hard-pion calculation,[2] we are motivated to use it here.

To sum up, we believe that the model to be analyzed has enough truth in it to yield some interesting qualitative insights. It has the major virtue of being a soluble problem for what is a large class of amplitudes. It should be kept in mind that we are treating a rather general problem—pion excitation—to an *arbitrary* spin state. As we shall show in the ensuing sections, the quantitative predictions of the calculation turn out to be surprisingly good.

We now summarize the contents of this paper. In Sec. II, we define the relevant amplitudes, introduce the hard-pion representation which forms the basis of our approach, and then analyze the content of the PCAC, current algebra, and BJL theorem assumptions. Certain consequences of the analysis of Sec. II are explored in Sec. III. Decay width predictions emerge from an analysis of specific transition matrix elements. The meaning of our results is clarified in Sec. IV, where various current-algebra sum rules are studied. Our conclusions are discussed in Sec. V.

II. GENERAL ANALYSIS OF THE MODEL

Consider the excitation of an on-shell pion of momentum q, isospin a by the axial-vector current $A_b^\mu(0)$ to an isoscalar meson M^J of momentum p, spin J, and helicity λ. Form factors for this transition are defined by

$$\langle M^J(p,\lambda)|A_b^\mu(0)|\pi_a(q)\rangle = i\delta_{ab}h^*_{\sigma_1\cdots\sigma_J}(p,\lambda)q^{\sigma_2}\cdots q^{\sigma_J}[g^{\sigma_1\mu}F_A^J(t)+q^{\sigma_1}k^\mu G_A^J(t)+q^{\sigma_1}Q^\mu H_A^J(t)]. \quad (3)$$

The polarization vector of the spin-J meson is $h^*_{\sigma_1\cdots\sigma_J}(p,\lambda)$, and we use the definitions $k=q-p$, $Q=p+q$, $t=k^2$. The assumption that M^J is isoscalar is purely for convenience—if M^J carries isospin c, then the factor $i\delta_{ab}$ is replaced by ϵ_{abc}. The same is true of our use of the axial-vector current $A_b^\mu(0)$—results obtained will hold for vector-current matrix elements as well. The class of mesons M^J, which can be reached via excitation of the pion as described in (3), obey the spin-parity relation $P=(-)^J$. For the class of mesons with $P=(-)^{J+1}$, we must write

$$\langle M^J(p,\lambda)|A_b^\mu(0)|\pi_a(q)\rangle = i\delta_{ab}h^*_{\sigma_1\cdots\sigma_J}(p,\lambda)\epsilon^{\sigma_1\mu\beta\gamma}q^{\sigma_2}\cdots q^{\sigma_J}q_\beta p_\gamma E_A^J(t). \quad (4)$$

For definiteness, the analysis to follow will treat the situation described by (3) rather than (4), although the qualitative results are similar.

An off-shell amplitude, related to the matrix element defined in (3), is given by

$$M^\mu_{J,ab}(q,p)=i\int d^4x\, e^{-iq\cdot x}\langle M^J(p,\lambda)|T(\partial_\sigma A_a^\sigma(x)A_b^\mu(0))|0\rangle. \quad (5)$$

The prescription for returning to the pion mass shell is simply

$$\langle M^J(p,\lambda)|A_b^\mu(0)|\pi_a(q)\rangle = \lim_{q^2\to m_\pi^2} \frac{m_\pi^2 - q^2}{F_\pi m_\pi^2} M^\mu_{J,ab}(q,p), \tag{6}$$

where $F_\pi \cong 94$ MeV is the pion decay constant. An appropriate decomposition for (5) in terms of off-shell form factors is

$$M^\mu_{J,ab}(q,p) = i\delta_{ab} h^*_{\sigma_1\cdots\sigma_J}(p,\lambda) q^{\sigma_2}\cdots q^{\sigma_J}[g^{\sigma_1\mu}F_1^J(q^2,k^2) + q^{\sigma_1}k^\mu F_2^J(q^2,k^2) + q^{\sigma_1}Q^\mu F_3^J(q^2,k^2)]. \tag{7}$$

The off-shell quantities $F_i^J(q^2,k^2)$, $i=1,2,3$ are related to the on-shell form factors $F_A(t)$, $G_A(t)$, $H_A(t)$ by using the prescription (6).

Before adopting a specific model for $M^\mu_{J,ab}(q,p)$, let us examine the constraints that the BJL and soft-pion theorems, and current algebra place on it. The Bjorken limit involves taking $q^0\to\infty$, with $\vec{q}$ fixed. In the following we shall let $\mu=0$ and for convenience, choose $q^1=q^2=0$, $q^3\neq 0$. The BJL theorem states

$$M^0_{J,ab}(q,p) \underset{\substack{q^0\to\infty \\ \vec{q}\text{ fixed}}}{\sim} \frac{1}{q^0}\int d^3x\, e^{i\vec{q}\cdot\vec{x}}\langle M^J(p,\lambda)|[A_b^0(0), \partial_\sigma A_a^\sigma(0,\vec{x})]|0\rangle. \tag{8}$$

Alternatively, the right-hand side of (7) becomes asymptotically

$$i\delta_{ab}\left\{\left[(q^0)^{J-1}h_0 + (q^0)^{J-2}(q^3)\binom{J-1}{1}h_1 + \cdots + (q^3)^{J-1}h_{J-1}\right]F_1^J(q^2,k^2)\right.$$
$$\left. + \left[(q^0)^J h_0 + (q^0)^{J-1}(q^3)\binom{J}{1}h_1 + \cdots + (q^3)^J h_J\right]\{q^0[F_3^J(q^2,k^2)+F_2^J(q^2,k^2)] + p^0[F_3^J(q^2,k^2) - F_2^J(q^2,k^2)]\}\right\}, \tag{9}$$

where $h_0 \equiv h^*_{0\cdots 0}(p,\lambda)$, $h_1\equiv h^*_{300\cdots 0}(p,\lambda)$, etc. [we remind the reader that $h_{\sigma_1\cdots\sigma_J}(p,\lambda)$ is totally symmetric in its J indices]. Because p^μ is arbitrary, the quantities $h_0, h_1, \ldots$ are generally nonzero. Finally we have

$$\frac{1}{q^0}\int d^3x\, e^{+i\vec{q}\cdot\vec{x}}\langle M^J(p,\lambda)|[A_b^0(0), \partial_\sigma A_a^\sigma(0,\vec{x})]|0\rangle$$
$$= i\delta_{ab}\left\{(q^0)^{J+1}h_0F_+^J(q^2,k^2) + (q^0)^J\left[p^0h_0F_-^J(q^2,k^2) + (q^3)\binom{J}{1}h_1F_+^J(q^2,k^2)\right]\right.$$
$$\left. + (q^0)^{J-1}\left[h_0F_1^J(q^2,k^2) + h_1(q^3)p^0\binom{J}{1}F_-^J(q^2,k^2) + h_2\binom{J}{2}(q^3)^2F_+^J(q^2,k^2)\right] + \cdots\right\}, \tag{10}$$

where $F_\pm^J = F_3^J \pm F_2^J$. It is apparent from (10) that the BJL theorem places rather stringent conditions upon the high-energy behavior of the amplitude describing the excitation of a pion to a high-spin meson.

To facilitate our study of the current-algebra constraints on $M^\mu_{J,ab}(q,p)$, we define an auxiliary amplitude,

$$N_{J,ab}(q,p) = i\int d^4x\, e^{-iq\cdot x}\langle M^J(p,\lambda)|T(\partial_\sigma A_a^\sigma(x)\partial_\mu A_b^\mu(0))|0\rangle. \tag{11}$$

Multiplication of $M^\mu_{J,ab}(q,p)$ by ik_μ yields

$$ik_\mu M^\mu_{J,ab}(q,p) = N_{J,ab}(q,p) + i\langle M^J(p,\lambda)|[F_b^5(0), \partial_\sigma A_a^\sigma(0)]|0\rangle. \tag{12}$$

We shall see that the current-algebra relation (12) is quite effective in reducing the number of arbitrary coefficients in a hard-pion amplitude. Finally, we consider the soft-pion theorems,

$$\lim_{q\to 0} M^\mu_{J,ab}(q,p) = -i\langle M^J(p,\lambda)|[F_a^5(0), A_b^\mu(0)]|0\rangle \tag{13a}$$

and

$$\lim_{q\to 0} N_{J,ab}(q,p) = -i\langle M^J(p,\lambda)|[F_a^5(0), \partial_\sigma A_b^\sigma(0)]|0\rangle. \tag{13b}$$

It is clear that useful information is contained in the soft-pion theorems only if meson M^J has spin 0 or 1. For $J>1$, the content of the soft-pion theorem is trivial, implying $0=0$.

In order to exploit these strictures, we adopt a dynamical model for the form factors $F_i^J(q^2,k^2)$—inclusion of $\pi, A_1(1070)$ poles where appropriate—together with constant behavior for the remainder. Thus we write for $i=1,2,3$,

$$F_i^J(q^2,k^2)=\frac{a_i^J}{(m_\pi^2-q^2)(m_\pi^2-k^2)}+\frac{b_i^J}{m_\pi^2-q^2}+\frac{c_i^J}{(m_\pi^2-q^2)(m_A^2-k^2)}+\frac{d_i^J}{m_\pi^2-k^2}+\frac{e_i^J}{m_A^2-k^2}+f_i^J\,, \tag{14}$$

where $a_i^J, b_i^J, \ldots, f_i^J$ are constants. Note that $a_1^{J=0}, b_1^{J=0}, \ldots, f_1^{J=0}=0$ from kinematical considerations. Also observe that $F_1^J(q^2,k^2)$ and $F_3^J(q^2,k^2)$ cannot contain pion pole terms in the variable k^2. This is deduced by examining intermediate states allowed in (5) and comparing them with (7). Thus we have $a_1^J=a_3^J=d_1^J=d_3^J=0$.

We now demand the validity of the BJL theorem. Upon inserting the representation (14) into Eq. (10), we find the following:

If $J\geq 0$:

$$f_2^J+f_3^J=0\,,\quad f_2^J-f_3^J=0\,,$$
$$i\delta_{ab}(b_2^J+d_2^J+e_2^J+b_3^J+e_3^J)=\langle M^J(p,\lambda)|[F_b^5(0),\partial_\mu A_a^\mu(0)]|0\rangle\delta_{J,0}\,.$$

If $J\geq 1$:

$$f_1^J=0\,,\qquad b_2^J+d_2^J+e_2^J=0\,,\quad b_3^J+e_3^J=0\,. \tag{15}$$

If $J\geq 2$:

$$a_2^J+c_2^J+c_3^J-b_1^J-e_1^J=0\,,\quad b_1^J+e_1^J=0\,.$$

If $J\geq 3$:

$$a_2^J+c_2^J=0\,,\quad c_3^J=0\,.$$

If $J\geq 4$:

$$c_1^J=0\,.$$

Before writing down the relations which follow from the current-algebra constraint (12), we must define a hard-pion parametrization for $N_{J,ab}(q,p)$:

$$N_{J,ab}(q,p)=i\delta_{ab}h^*_{\sigma_1\cdots\sigma_J}(p,\lambda)q^{\sigma_1}\cdots q^{\sigma_J}\times\left[\frac{\alpha^J}{(m_\pi^2-q^2)(m_\pi^2-k^2)}+\frac{\beta^J}{m_\pi^2-q^2}+\frac{\beta^J}{m_\pi^2-k^2}+\gamma^J\right]. \tag{16}$$

It then follows from Eqs. (7), (12), (14), (16) that

$$e_3^J=b_2^J=0\,,$$
$$c_1^J+c_2^J m_A^2+c_3^J(m_\pi^2-m_J^2)=0\,,$$
$$e_1^J+e_2^J m_A^2-c_3^J=0\,,$$
$$a_2^J m_\pi^2=-i\alpha^J\,, \tag{17}$$
$$b_3^J(m_\pi^2-m_J^2)-a_2^J-c_2^J+b_1^J=-i\beta^J\,,$$
$$d_2^J m_\pi^2=-i\beta^J\,,$$
$$i\delta_{ab}(f_1^J-b_3^J-d_2^J-e_2^J)=i\delta_{ab}\gamma^J+i\langle M^J(p,\lambda)|[F_a^5(0),\partial_\mu A_b^\mu]|0\rangle\delta_{J,0}\,.$$

We can obtain more information by considering the BJL theorem obeyed by $N_{J,ab}(q,p)$,

$$N_{J,ab}(q,p)\rightarrow-\frac{1}{q^0}\int d^3x\,e^{i\vec{q}\cdot\vec{x}}\times\langle M^J(p,\lambda)|[\partial_\sigma A_a^\sigma(0,\vec{x}),\partial_\mu A_b^\mu(0)]|0\rangle\,, \tag{18}$$

which together with (16) implies

$$\gamma^J=0\,,\quad \text{all } J$$
$$\beta^J=0\,,\quad J\geq 2 \tag{19}$$
$$\alpha^J=0\,,\quad J\geq 4\,.$$

We have proceeded deeply enough into the analysis of the hard-pion amplitudes (14) to reach a conclusion of real interest. The following is a compilation of our results so far:

For $J=0$,

$$a_3^{J=0}=d_3^{J=0}=e_3^{J=0}=f_3^{J=0}=f_2^{J=0}=b_2^{J=0}=0\,,$$
$$d_2^{J=0}=-\frac{i\beta^{J=0}}{m_\pi^2}\,,$$
$$a_2^{J=0}=-\frac{i\alpha^{J=0}}{m_\pi^2}\,, \tag{20a}$$
$$c_2^{J=0}=e_2^{J=0}(m_\epsilon^2-m_\pi^2)=\frac{c_3^{J=0}(m_\epsilon^2-m_\pi^2)}{m_A^2}\,,$$
$$b_3^{J=0}=-\frac{c_3^{J=0}}{m_A^2}-\frac{i\alpha^{J=0}}{m_\pi^2(m_\epsilon^2-m_\pi^2)}-\frac{i\beta^{J=0}}{m_\epsilon^2-m_\pi^2}\,,$$

where we denote the mass of the spin-zero meson by m_ϵ.

For $J\geq 1$,

$$a_1^J=d_1^J=f_1^J=b_2^J=f_2^J=a_3^J=d_3^J=e_3^J=f_3^J=0\,,$$
$$d_2^J=-\frac{i\beta^J}{m_\pi^2}\,,\quad a_2^J=-\frac{i\alpha^J}{m_\pi^2}\,,$$
$$c_3^J=e_1^J+m_A^2e_2^J\,, \tag{20b}$$
$$c_1^J=(m_J^2-m_\pi^2)(e_1^J+m_A^2e_2^J)-m_A^2c_2^J\,,$$
$$b_3^J=\frac{i\beta^J}{m_\pi^2}-e_2^J\,,$$
$$b_1^J=c_2^J-\frac{i\alpha^J}{m_\pi^2}-i\beta^J+(m_J^2-m_\pi^2)\left(\frac{i\beta^J}{m_\pi^2}-e_2^J\right).$$

In addition, for $J \geq 2$,

$$b_3^J = \beta^J = 0\,, \tag{20c}$$

for $J \geq 3$,

$$e_1^J = \frac{i\alpha^J}{m_\pi^2} - c_2^J\,, \tag{20d}$$

and finally, for $J \geq 4$,

$$b_1^J = a_2^J = c_2^J = \alpha^J = 0\,. \tag{20e}$$

All coefficients vanish if $J \geq 4$. Therefore, given the dynamical model described in this section, excitation of the pion by means of currents to single-particle states with $J \geq 4$ is forbidden. We check the consequences of this result in Sec. IV, where we analyze sum rules and present a more detailed discussion of it in the summary in Sec. VI.

We conclude this section by writing down constraints gleaned from the soft-pion limit. For $J = 0$, defining an operator σ as the isoscalar part of the commutator $i[F_a^5, \partial_\mu A_b^\mu]$, and ϵ as any $T = J = 0$ meson, we find[7]

$$\frac{i\alpha^{J=0}}{m_\pi^2(m_\epsilon^2 - m_\pi^2)} + \frac{(2m_\pi^2 - m_\epsilon^2)i\beta^{J=0}}{m_\pi^2(m_\epsilon^2 - m_\pi^2)} = \langle \epsilon(p)|\sigma(0)|0\rangle\,,$$

and (21a)

$$\frac{-a_2^{J=0}}{m_\pi^2(m_\pi^2 - m_\epsilon^2)} + \frac{b_2^{J=0}}{m_\pi^2} + \frac{c_3^{J=0} - c_2^{J=0}}{m_\pi^2(m_A^2 - m_\epsilon^2)} - \frac{d_2^{J=0}}{m_\pi^2 - m_\epsilon^2} - \frac{e_2^{J=0}}{m_A^2 - m_\epsilon^2} = 0\,.$$

For $J = 1$, letting ρ represent any $T = J = 1$ meson, we obtain

$$\frac{b_1^{J=1}}{m_\pi^2} + \frac{c_1^{J=1}}{m_\pi^2(m_A^2 - m_\rho^2)} + \frac{e_1^{J=1}}{m_A^2 - m_\rho^2} = -g_\rho\,, \tag{21b}$$

where

$$\langle 0|V_c^\mu(0)|\rho_a(k,\lambda)\rangle = g_\rho\,\delta_{ac}\,\epsilon^\mu(k,\lambda)\,. \tag{22}$$

III. SOME EXPERIMENTAL CONSEQUENCES OF THE MODEL

In this section we analyze specific examples of the results obtained above. First, however, let us verify that the Schnitzer-Weinberg (SW) calculation can be reproduced for $J = 1$. Taking the spin-one meson as the ρ we define:

$$\begin{aligned}\alpha^{J=1} &= -i2F_\pi^2 m_\pi^4 g_{\rho\pi\pi}\,,\\ e_1^{J=1} &= -\tfrac{1}{4}\delta g_A m_\pi^2\,,\\ c_2^{J=1} &= \frac{g_A}{m_A^2} F_\pi m_\pi^2\left[g_{A\rho\pi} - \tfrac{1}{2}(m_\rho^2 - m_\pi^2)\frac{\delta}{2F_\pi}\right].\end{aligned} \tag{23}$$

Here, g_A is given by

$$\langle 0|A_a^\mu(0)|A_{1b}(p,\lambda)\rangle = g_A\delta_{ab}\,\epsilon^\mu(p,\lambda)\,, \tag{24}$$

and $g_{A\rho\pi}$, $g_{\rho\pi\pi}$, and δ (the anomalous moment of the A_1 meson[8]) are defined by

$$g_{\rho\pi\pi}\epsilon_{abc}\rho_a^\mu\pi_b\partial_\mu\pi_c + g_{A\rho\pi}\epsilon_{abc}A_{1a}^\mu\rho_{b\mu}\pi_c - \frac{\delta}{2F_\pi}\epsilon_{abc}A_{1a}^\mu\rho_b^\nu\partial_\mu\partial_\nu\pi_c\,. \tag{25}$$

In order to obtain the SW results we note the following:

(i) By study of

$$\int d^4x\, e^{-iq\cdot x}\langle \pi_a(p)|T(\partial^\mu A_\mu^b(x)V_\nu^c(0))|0\rangle$$

via the techniques discussed in Sec. II, one finds that $\langle \pi_a(p)|\partial^\mu A_\mu^b(0)|\rho_c(k)\rangle$ satisfies an unsubtracted dispersion relation in k^2.[9] Hence $\beta^{J=1} = 0$.

(ii) By study of

$$\int d^4x\, e^{-iq\cdot x}\langle \pi_a(p)|T(A_\mu^b(x)V_\nu^c(0))|0\rangle$$

by the same techniques, one finds that $H_A^{J=1}(k^2)$ satisfies an unsubtracted dispersion relation in k^2.[10] Hence $b_3^{J=1} = 0$. These conditions are obtained automatically for $J > 1$. From Eq. (20b), $b_3^{J=1} = i\beta^{J=1}/m_\pi^2 - e_2^{J=1}$, so that if $b_3^{J=1} = \beta^{J=1} = 0$, we must have $e_2^{J=1} = 0$. We then obtain

$$\begin{aligned}F_1^{J=1}(q^2,k^2) &= \frac{(g_A/m_A^2)F_\pi m_\pi^2[g_{A\rho\pi} - \frac{1}{2}(m_\rho^2 - m_\pi^2)\delta/2F_\pi] - F_\pi^2 m_\pi^2 g_{\rho\pi\pi}}{m_\pi^2 - q^2}\\ &\quad - \frac{g_A F_\pi m_\pi^2 g_{A\rho\pi}}{(m_\pi^2 - q^2)(m_A^2 - k^2)} - \frac{\frac{1}{4}\delta g_A m_\pi^2}{m_A^2 - k^2}\,,\\ F_2^{J=1}(q^2,k^2) &= -\frac{2F_\pi^2 m_\pi^4 g_{\rho\pi\pi}}{(m_\pi^2 - q^2)(m_\pi^2 - k^2)} + \frac{g_A}{m_A^2}F_\pi m_\pi^2\,\frac{g_{A\rho\pi} - \frac{1}{2}(m_\rho^2 - m_\pi^2)\delta/2F_\pi}{(m_\pi^2 - q^2)(m_A^2 - k^2)}\,,\\ F_3^{J=1}(q^2,k^2) &= -\frac{\frac{1}{4}\delta g_A m_\pi^2}{(m_\pi^2 - q^2)(m_A^2 - k^2)}\,.\end{aligned} \tag{26}$$

There are only two free parameters here since δ, $g_{\rho\pi\pi}$, $g_{A\rho\pi}$ are constrained by the soft-pion equation:

$$g_\rho = g_A \frac{m_\rho^2}{m_A^2} \frac{1}{m_A^2 - m_\rho^2} F_\pi \left[g_{A\rho\pi} + \frac{\delta}{4F_\pi}(m_A^2 - m_\rho^2 + m_\pi^2)\right] + F_\pi^2 g_{\rho\pi\pi}\,. \tag{27}$$

These are exactly the results of Brown and West and of Schnitzer and Weinberg.

Now that we have verified the hard-pion results for $J=1$, we can move on to other values of J. In particular, the model can be used to relate $A_1\pi$ and $\pi\pi$ decay modes of arbitrary spin mesons. As a first step, we show how the parameter α^J can be determined experimentally. Consider the on-shell transition amplitude:

$$\begin{aligned}\langle \pi_a(q)\pi_b(k)\,|\,M^J(p,\lambda)\rangle &= \lim_{q^2,k^2\to m_\pi^2} i\,\frac{(m_\pi^2-q^2)(m_\pi^2-k^2)}{F_\pi^2 m_\pi^4} N_{J,ab}(q,p)\\ &= \delta_{ab}\frac{-\alpha^J}{F_\pi^2 m_\pi^4} h_{\sigma_1\cdots\sigma_J}(p,\lambda)q^{\sigma_1}\cdots q^{\sigma_J}\,,\end{aligned} \tag{28}$$

for the decay of an $I=0$, spin-J meson. If $I=1$, simply take $\delta_{ab}\to -i\epsilon_{abc}$. Squaring the amplitude and summing over available phase space, we obtain

$$\Gamma(M^J\to\pi\pi) = C^I\left(\frac{\alpha^J}{F_\pi^2 m_\pi^4}\right)^2 \frac{(m_J^2-4m_\pi^2)^{J+1/2}}{32\pi m_J^2}\frac{(J!)^2}{(2J)!}\frac{1}{2^J(2J+1)}\,, \tag{29}$$

where C^I is an isospin factor given by

$$C^I = \begin{cases} 2 \text{ if } I=1\,,\\ 3 \text{ if } I=0\,.\end{cases} \tag{30}$$

It is also possible to write a formula relating the parameters c_1^J and c_3^J to $M^J\to A_1\pi$ decay widths. The appropriate transition amplitude is given by

$$\begin{aligned}\langle \pi_a(q)A_{1b}(k,\lambda')\,|\,M^J(p,\lambda)\rangle &= \lim_{\substack{q^2\to m_\pi^2\\ k^2\to m_A^2}} (-i)\frac{m_\pi^2-q^2}{F_\pi m_\pi^2}\frac{m_A^2-k^2}{g_A}\epsilon_\mu^*(k,\lambda')M^\mu_{J,ab}(p,\lambda)\\ &= -\frac{i\delta_{ab}}{F_\pi m_\pi^2 g_A} h_{\sigma_1\cdots\sigma_J}(p,\lambda)q^{\sigma_1}\cdots q^{\sigma_{J-1}}\epsilon_\mu^*(k,\lambda')(c_1 g^{\mu\sigma_J}+c_3 q^\mu q^{\sigma_J}).\end{aligned} \tag{31}$$

After a lengthy exercise in kinematics, we find

$$\begin{aligned}\Gamma(M^J\to A_1\pi) = C^I \frac{|\vec q\,|^{2J-1}}{(m_J m_A m_\pi^2 F_\pi g_A)^2}\frac{[(J-1)!]^2 2^{J-2}}{(2J-1)!}\bigg[&\left(m_A^2+\frac{J}{2J+1}\vec q^{\,2}\right)\frac{c_1^2}{4\pi}+\frac{J}{2J+1}m_J^2\vec q^{\,4}\frac{c_3^2}{4\pi}\\ &+\frac{2J}{2J+1}m_J\omega_A\vec q^{\,2}\frac{c_1c_3}{4\pi}\bigg].\end{aligned} \tag{32}$$

We may use existing data on the decay of $\epsilon(700)$, $\rho(765)$, $f(1269)$, $f'(1514)$, and $g(1650)$ (Ref. 11) along with Eqs. (29) and (32) to constrain α^J, c_1^J, and c_3^J for $J=0,1,2,3$. In view of the $J\geq 4$ cutoff which occurs in our model, we shall concentrate in the following upon relating $\pi\pi$ and $A_1\pi$ branching ratios for the $J=2,3$ states. In order to have definite *predictions*, we are forced to make two additional assumptions:

(a) $F_A^{J=2}(k^2)$ obeys an unsubtracted dispersion relation in k^2. This is essentially an A_1-dominance assumption. It gives $b_1^{J=2}=0$. Note that this assumption does not hold for $J=1$.

(b) The decay of a high-spin meson into a two-body final state is dominated by that channel having the lowest available value of orbital angular momentum. Some details pertinent to this assumption are presented in Appendix B. Here we merely note it implies $c_3^{J=2}\cong 0$.

The main consequence of (a) and (b) is that $c_1^{J=2}=-m_A^2\alpha^{J=2}/m_\pi^2$. This latter condition allows us to relate the $\pi\pi$ and $A_1\pi$ branching ratios for the decay of a $J=2$ meson. Moreover, when we consider the $J=3$ problem, it is necessary just to assume either (a) or (b) (one automatically implies the other). We then find $c_1^{J=3}=-m_A^2\alpha^{J=3}/m_\pi^2$, again relating the $\pi\pi$ and $A_1\pi$ decay modes. Inserting the above relations between c_1^J, c_3^J, and α^J into Eqs. (29) and (32), and substituting in the appropriate numerical values, we predict

$$\frac{\Gamma(f\to A_1\pi)}{\Gamma(f\to\pi\pi)}=0.15\,,\quad \frac{\Gamma(f'\to A_1\pi)}{\Gamma(f'\to\pi\pi)}=1.2\,,\quad \frac{\Gamma(g\to A_1\pi)}{\Gamma(g\to\pi\pi)}=0.59\,. \tag{33}$$

Current experiments give

$$\frac{\Gamma(f(1269)\to 2\pi^+2\pi^-)}{\Gamma(f(1269)\to \pi\pi)} \cong \tfrac{1}{13} \quad \text{(Ref. 12)}.$$

Noting that the $A_1\pi$ mode may contribute to the $2\pi^+2\pi^-$ final state via the chain $f(1269)\to A_1\pi\to \rho\pi\pi\to 4\pi$, we predict that

$$\frac{\Gamma(f(1269)\to 2\pi^+2\pi^-)_{A_1\pi}}{\Gamma(f(1269)\to \pi\pi)} \cong \tfrac{1}{20},$$

consistent with the data. The implication of our $f'(1514)$ prediction is that since the $\pi\pi$ mode is highly suppressed, so also then is the $A_1\pi$ mode. Finally, using the current value, $\Gamma(g(1650)\to\pi\pi)\cong 64$ MeV, we predict

$$\Gamma(g(1650)\to 4\pi)_{A_1\pi}\cong 38 \text{ MeV},$$

whereas experimentally, one finds

$$\Gamma(g(1650)\to 4\pi)\cong 80 \text{ MeV} \quad \text{(see Ref. 11)}.$$

To summarize, the above predictions each fall within existing experimental limits. We hope that the numerical values for decay widths given above stimulate further experimental study of the $A_1\pi$ decay mode. It is worth pointing out here that if our model is qualitatively correct, $\pi\pi$ and $A_1\pi$ decay modes of a resonance with $J\geq 4$ should be suppressed. Presumably, such a higher-spin state M^J would decay, for example, via pion emission into another high-spin state M^{J-1}, and so on.[13] Future experiments will decide on this. Fortunately our $J=4$ cutoff has further implications which can be studied here; it proves to be quite important in the evaluation of various sum rules, as discussed in Sec. IV.

IV. SUM RULES

As an additional test of our model, we shall perform evaluations of various current-algebra sum rules. For our first example, we take the operator

$$\int d^3x \int d^3y \{[A_+^0(x), A_-^0(y)] - 2\delta^3(x-y)V_3^0(x)\}_{x^0=y^0} = 0 \tag{34}$$

between pion states (the Adler sum rule[14] for a pion target) in the limit of infinite pion momentum. In doing so, we are working in a kinematic domain, $q^2=m_\pi^2$ and $k^2=0$, where we expect our parametrization has its greatest validity. We shall attempt to saturate the sum rule by using only the spin-J resonances described previously. This gives

$$1=\frac{1}{F_\pi^2 m_\pi^8}\Big[\,|\alpha^{J=0}|^2(m_{J=0}^2-m_\pi^2)^{-2}+\tfrac{1}{4}|\alpha^{J=1}|^2 m_{J=1}^{-2}+\tfrac{1}{24}|\alpha^{J=2}|^2(m_{J=2}^2-m_\pi^2)^2 m_{J=2}^{-4} + \tfrac{1}{160}|\alpha^{J=3}|^2(m_{J=3}^2-m_\pi^2)^4 m_{J=3}^{-6}+\cdots\Big]. \tag{35}$$

The $|\alpha^J|^2$ are related to the $J\to\pi\pi$ decay rates by Eq. (29). As a consequence of the BJL theorem and current-algebra constraints, we neglect all contributions from states with $J\geq 4$. Inserting the numerical values[15]

$m_{J=0}=700$ MeV, $\Gamma_{J=0}=300$ MeV,

$m_{J=1}=765$ MeV, $\Gamma_{J=1}=128$ MeV,

$m_{J=2}=1269$ MeV, $\Gamma_{J=2}=125$ MeV,

$m_{J=3}=1680$ MeV, $\Gamma_{J=3}=64$ MeV

into Eq. (35), we find

$$\begin{aligned}1&=0.31+0.47+0.10+0.04+\cdots\\&=0.92+\cdots,\end{aligned} \tag{36}$$

which, given the uncertainties in data and the approximate nature of our model, is in rather good agreement with the theoretical prediction. In particular, it lends support to the $J=4$ cutoff predicted by our model.

We can probe the commutator slightly away from the forward direction by insertion of the operator

$$0=\int d^3x\int d^3y\, e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\{[A_+^0(x),A_-^0(y)]-2\delta^3(\vec{x}-\vec{y})V_3^0(x)\}_{x^0=y^0} \tag{37}$$

between pion states. The derivative with respect to $\vec{k}^2$ is taken in the infinite-momentum frame and the resulting sum rule is again evaluated at $\vec{k}^2=0$.[16] However, now we must include the contribution of the odd-parity mesons, $B(1233, I=1)$, $D(1286, I=0)$, $E(1422, I=0)$.[17] For these states, we parametrize the $I=0$ off-shell amplitude $M^{\mu}_{J,ab}(q,p)$ as

$$M^{\mu}_{J,ab}(q,p)=i\delta_{ab}h^*_{\sigma_1\cdots\sigma_J}\epsilon^{\sigma_1\mu\lambda\alpha}q^{\sigma_2}\cdots q^{\sigma_J}q_\lambda p_\alpha E^J(q^2,k^2), \tag{38}$$

where

$$E^J(q^2,k^2)=\frac{a^J}{(m_\pi^2-q^2)(m_A^2-k^2)}+\frac{b^J}{m_\pi^2-q^2}+\frac{c^J}{m_A^2-k^2}+d^J, \tag{39}$$

a^J, b^J, c^J, d^J being constants. For $I=1$, we take $i\delta_{ab}\to\epsilon_{abc}$. The off-shell form factors $E^J(q^2,k^2)$ are related to the on-shell form factors $E_A^J(t)$ of Eq. (4), as described in Eq. (6). From the BJL theorem, we conclude that

$$J\geqslant 1,\ d^J=0;\quad J\geqslant 2,\ b^J+c^J=0;\quad J\geqslant 3,\ b^J=c^J=0;\quad J\geqslant 4,\ a^J=0. \tag{40}$$

Thus again, there is a $J=4$ cutoff. In terms of the on-shell form factors of Eqs. (3) and (4), the sum rule reads

$$\begin{aligned}
0={}&2H_A^{J=0}(0)(H_A^{J=0})'(0)\\
&+\frac{1}{4m_{J=1}^2}\{2[F_A^{J=1}(0)-(m_{J=1}^2-m_\pi^2)H_A^{J=1}(0)][(F_A^{J=1})'(0)-(m_{J=1}^2-m_\pi^2)(H_A^{J=1})'(0)-H_A^{J=1}(0)]\\
&\qquad-4m_{J=1}^2[H_A^{J=1}(0)]^2\}\\
&+\frac{1}{24m_{J=2}^4}\{2[(m_{J=2}^2-m_\pi^2)F_A^{J=2}(0)-(m_{J=2}^2-m_\pi^2)^2H_A^{J=2}(0)]\\
&\qquad\times[(m_{J=2}^2-m_\pi^2)(F_A^{J=2})'(0)+F_A^{J=2}(0)-(m_{J=2}^2-m_\pi^2)^2(H_A^{J=2})'(0)\\
&\qquad-2(m_{J=2}^2-m_\pi^2)H_A^{J=2}(0)+4m_{J=2}^2H_A^{J=2}(0)]-3m_{J=2}^2[F_A^{J=2}(0)]^2\}\\
&+\frac{1}{160m_{J=3}^6}\{-2(m_{J=3}^2+m_\pi^2)[(m_{J=3}^2-m_\pi^2)F_A^{J=3}(0)-(m_{J=3}^2-m_\pi^2)^2H_A^{J=3}(0)]^2\\
&\qquad+(m_{J=3}^2-m_\pi^2)^2\,2[(m_{J=3}^2-m_\pi^2)F_A^{J=3}(0)-(m_{J=3}^2-m_\pi^2)^2H_A^{J=3}(0)]\\
&\qquad\times[(m_{J=3}^2-m_\pi^2)(F_A^{J=3})'(0)+F_A^{J=3}(0)-(m_{J=3}^2-m_\pi^2)^2(H_A^{J=3})'(0)\\
&\qquad+4m_{J=3}^2H_A^{J=3}(0)-2(m_{J=3}^2-m_\pi^2)H_A^{J=3}(0)]-\tfrac{8}{3}m_{J=3}^2[F_A^{J=3}(0)]^2\}\\
&-\tfrac{1}{4}[E_A^{J=1}(0)]^2.
\end{aligned} \tag{41}$$

The final term refers to the sum of the contributions from the $J^P=1^+$, B, D, and E mesons. This becomes much simpler when we note that for $J\geqslant 2$, with the assumptions made previously, $H_A^J(k^2)=0$ and $F_A^{J\prime}(0)=(1/m_A^2)F_A^J(0)$. If we demand A_1 dominance for $H_A^{J=0}(k^2)$ we find from Eq. (41), along with Eq. (35),[18]

$$1=0(\text{spin }1)+0.04(\text{spin }2)+0.07(\text{spin }3)+\tfrac{1}{8}m_A^2\{[E_A^B(0)]^2+[E_A^D(0)]^2+[E_A^E(0)]^2\}+\cdots, \tag{42}$$

with obvious notation. Thus, the odd-parity mesons must make up most of the remainder. Assuming A_1-dominance for each of the $E_A^{J=1}(k^2)$, and using the latest data, we find that the contribution of the D and E mesons is probably quite small. However, since the B meson lies so near the $A_1\pi$ threshold, a reasonably large amplitude could be masked by a small $B\to A_1\pi$ width. We deduce from the relation

$$\tfrac{1}{8}m_A^2[E_A^{J=1}(0)]^2\cong\Gamma_{B\to A\pi}\,\frac{3}{2\pi}|\vec{q}|^3\frac{g_A^2}{m_A^2}, \tag{43}$$

that if $\Gamma(B\to A_1\pi)\cong 7$ MeV, the sum rule (42) is saturated. Of course, there are other states which can contribute, so this should be considered as an upper bound. It is hoped that future experiments seek evidence for this mode.

We find then that of the two axial-vector sum rules, the first is in satisfactory agreement with the theoretical prediction, while the second yields a prediction for $B\to A\pi$ which is not inconsistent with current experimental bounds. If, however, we try to extend the Adler sum rule to larger values of momentum transfer, things soon break down. There are two ways in which to take $k^2 \neq 0$. One method is to proceed as before but refrain from taking the infinite-momentum limit. For instance, if we evaluate the sum rule in the pion rest frame, only the $J=0$ intermediate states contribute and we have

$$1=\sum_n \left(\frac{\alpha_n^{J=0}}{F_\pi m_\pi^2}\right)^2 \frac{1}{4m_\pi^2 (m_n^{J=0})^3} \frac{[(m_n^{J=0})^2-4m_\pi^2]^{1/2}}{(m_n^{J=0}-2m_\pi)^2(m_n^{J=0}-m_\pi)^2}, \tag{44}$$

where the sum runs over all possible $J=0$ resonances. Upon taking into account just the $\epsilon(700)$ contribution, we find $1=0.10+\cdots$. Clearly, the agreement is quite poor. Another procedure is to use a covariant dispersion relation with k^2 fixed at some arbitrary value.[19] We then find

$$\begin{aligned}
1=&[H_A^{J=0}(k^2)]^2+\frac{1}{4m_{J=1}^2}\{[F_A^{J=1}(k^2)-(m_{J=1}^2-m_\pi^2+k^2)H_A^{J=1}(k^2)]^2-4k^2m_{J=1}^2[H_A^{J=1}(k^2)]^2\}\\
&+\frac{1}{24m_{J=2}^4}\{[(m_{J=2}^2-m_\pi^2+k^2)F_A^{J=2}(k^2)-((m_{J=2}^2-m_\pi^2+k^2)^2-4m_{J=2}^2k^2)H_A^{J=2}(k^2)]^2-3k^2m_{J=2}^2[F_A^{J=2}(k^2)]^2\}\\
&+\frac{1}{160m_{J=3}^6}[[(m_{J=3}^2-m_\pi^2)^2-2k^2(m_{J=3}^2+m_\pi^2)+k^4]\\
&\qquad\times(\{(m_{J=3}^2-m_\pi^2+k^2)F_A^{J=3}(k^2)-[(m_{J=3}^2-m_\pi^2+k^2)^2-4k^2m_{J=3}^2]H_A^{J=3}(k^2)\}^2\\
&\qquad-\tfrac{8}{3}m_{J=3}^2k^2[H_A^{J=3}(k^2)]^2)]\\
&-\frac{k^2}{4}[E_A^{J=1}(k^2)]^2+\cdots.
\end{aligned} \tag{45}$$

This reduces to the infinite-momentum result at $k^2=0$ but, since all form factors drop off no faster than $1/k^2$ in our model, we see that disagreement sets in before too long.[20] Thus only the results near $k^2=0$ appear to be valid, which is consistent with our feeling that the model works best when applied to situations where q^2, k^2 are relatively small. This also suggests that when the pion is excited by the axial-vector current, resonance production is dominant near $k^2=0$, but multiparticle states become important at larger momentum transfers.[21]

Finally, it is interesting to evaluate the small $-k^2$ sum rules for the case in which the axial-vector currents are replaced by vector currents. That is, we evaluate

$$\iint d^3x\,d^3y\,e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\{[V_+^0(x),V_-^0(y)]-2\delta^3(x-y)V_3^0(x)\}_{x^0=y^0}=0 \tag{46}$$

between pion states in the infinite-momentum frame. In the limit $\vec{k}^2=0$, conservation of vector current[22] (CVC) implies that only the pion intermediate state contributes. Thus, the $\vec{k}^2=0$ sum rule is a trivial identity. However, if we differentiate with respect to $\vec{k}^2$, and evaluate the sum rule at $\vec{k}^2=0$ and in the infinite-momentum frame as before, we obtain a nontrivial relation for the pion radius. In describing the vector-current matrix elements, we use the same formalism as that employed for the axial-vector current. We define

$$\begin{aligned}
(M_V)^\mu_{J,ab}(q,p)&=i\int d^4x\,e^{-iq\cdot x}\langle M^J(p,\lambda)|T(\partial_\sigma A_a^\sigma(x)V_b^\mu(0))|0\rangle\\
&=i\delta_{ab}h^*_{\sigma_1\cdots\sigma_J}(p,\lambda)q^{\sigma_2}\cdots q^{\sigma_J}[g^{\sigma_1\mu}G_1^J(q^2,k^2)+q^{\sigma_1}k^\mu G_2^J(q^2,k^2)+q^{\sigma_1}Q^\mu G_3^J(q^2,k^2)],
\end{aligned} \tag{47}$$

with the $G_i^J(q^2,k^2)$ having the same structure as before, except that pion poles in k^2 are not present and A_1 poles are replaced by ρ poles:

$$G_i^J(q^2,k^2)=\frac{b_i^J}{m_\pi^2-q^2}+\frac{c_i^J}{(m_\pi^2-q^2)(m_\rho^2-k^2)}+\frac{e_i^J}{m_\rho^2-k^2}+f_i^J. \tag{48}$$

The on-shell counterparts of G_1, G_2, G_3 are F_V, G_V, H_V, respectively. The BJL and current-algebra constraints are identical to those given previously if we set $a_i^J=d_i^J=\alpha^J=\beta^J=\gamma^J=0$, the latter three conditions being required by CVC.[23]

For the isospin-zero odd-parity mesons, we define

$$(M_V)^\mu_{J,ab}(q,p)=i\delta_{ab}h^*_{\sigma_1\cdots\sigma_J}\epsilon^{\sigma_1\mu\lambda\alpha}q^{\sigma_2}\cdots q^{\sigma_J}q_\lambda p_\alpha H^J(q^2,k^2), \tag{49}$$

with

$$H^J(q^2,k^2)=\frac{a^J}{(m_\pi{}^2-q^2)(m_\rho{}^2-k^2)}+\frac{b^J}{m_\pi{}^2-q^2}+\frac{c^J}{m_\rho{}^2-k^2}+d^J, \tag{50}$$

and analogously for the isospin-one odd-parity states except that $i\delta_{ab}\to\epsilon_{abc}$. The analysis of the BJL limit goes through as before.

The structure of the vector current sum rule is then identical to the axial-vector sum rule (41) if we make the replacements $F_A, H_A, E_A \to F_V, G_V, E_V$, respectively. We shall use the sum rule to solve for the pion form factor in terms of the other contributions. Because we are working in a pole model, the sum of these other contributions is to be compared with the ρ-dominance value for the pion form factor,[24]

$$\begin{aligned} m_\rho{}^2 r_\pi{}^2 &= 6 \quad (\rho\text{-dominance value}) \\ &= \tfrac{3}{4} m_\rho{}^2\left([H_V^{A_1}(0)]^2+[E_V^\omega(0)]^2+[E_V^\phi(0)]^2+\frac{(m_{A_2}{}^2-m_\pi{}^2)^2}{8m_{A_2}{}^2}[E_V^{A_2}(0)]^2+\cdots\right) \\ &= 5.72\, m_\rho{}^2+\cdots . \end{aligned} \tag{51}$$

Again, agreement is quite good, suggesting that contributions from higher spin states are relatively unimportant.

V. CONCLUSION

We conclude with a review of our results and a discussion of their physical significance. In this paper, we have undertaken an analysis of the class of transitions, "current + pion → spin-J meson." The dynamical content of our on-shell amplitudes involves π, A_1, or ρ poles plus constant terms. These constants simulate higher-mass contributions or even the nonvanishing of form factors at infinite energy. The intent of this kind of model is to give a reasonable description of the transition amplitudes at moderate values of momentum transfer. Because we restrict the calculation to pion targets, we are able to exploit the powerful off-shell constraints of current algebra, PCAC, and the BJL theorem. Several different types of results were obtained.

Of the most immediate experimental interest are the $A_1\pi$ branching ratio predictions. Data on these branching ratios are sparse at this time, so we are restricted to concluding simply that our predictions are consistent with those experimental results now available. This lends tentative support to our parametrization of the k^2 dependence of the axial transition amplitudes in terms π and A_1 poles. The pattern of predictions [$\Gamma(f, g\to A_1\pi)$ measurable, $\Gamma(f'\to A_1\pi)$ suppressed] is distinct enough to allow future experiments to make decisive statements about them.

The result which proved most fruitful to explore, and which is probably the most important result of this paper, is that in our model, excitation of the pion by a current to a meson with $J\geq 4$ is forbidden. In order to show that this result is dynamic and not kinematic, and also to indicate how the cutoff value $J=4$ depends on the model, we shall study an extension of it in which $N_{J,ab}(q,p)$ [see Eq. (11)] is allowed to contain poles representing π', a hypothetical particle with the same quantum numbers as the pion, but with mass $M>m_\pi$. Thus, we temporarily replace Eq. (16) by

$$\begin{aligned} N_{J,ab}(q,p) = i\delta_{ab} h^*_{\sigma_1\cdots\sigma_J}(p,\lambda)q^{\sigma_1}\cdots q^{\sigma_J}\Big[&\frac{\alpha^J}{(m_\pi{}^2-q^2)(m_\pi{}^2-k^2)}+\frac{\beta^J}{m_\pi{}^2-q^2}+\frac{\beta^J}{m_\pi{}^2-k^2}+\gamma^J+\frac{\delta^J}{(M^2-q^2)(M^2-k^2)} \\ &+\frac{\eta^J}{M^2-q^2}+\frac{\eta^J}{M^2-k^2}+\frac{\phi^J}{(M^2-q^2)(m_\pi{}^2-k^2)}+\frac{\phi^J}{(m_\pi{}^2-q^2)(M^2-k^2)}\Big]. \end{aligned} \tag{52}$$

In Eq. (19), we showed how the BJL theorem constrains $\alpha^J, \beta^J, \gamma^J$. Upon evaluating Eq. (52) in the BJL limit and comparing to Eq. (18), we find

$$\begin{aligned} J\geq 0:&\quad \gamma^J=0; \\ J\geq 2:&\quad \beta^J+\eta^J=0; \\ J\geq 4,5:&\quad 2\phi^J+\delta^J+\alpha^J-2m_\pi{}^2\beta^J-2M^2\eta^J=0; \\ &\quad \cdots . \end{aligned} \tag{53}$$

Clearly the results of Eq. (19) are not insensitive to modifications in our dynamical model. It follows that the $J=4$ cutoff value obtained from the calculation performed in Sec. II depends on the specific model employed there. Hence, the significance of this result requires a certain amount of interpretation. We feel that Eq. (10), in which the BJL constraint upon the model-independent amplitude is exhibited, clearly shows that the amplitude for pion excitation to a high-spin meson must obey stringent conditions. In any reasonable model, this must lead to a suppression in producing the highest-spin mesons. The question is: For which value of J does this suppression start to become really effective? Our opinion is that, despite its apparent simplicity, the hard-pion model studied here has enough truth in it to give a reasonably correct estimate for the cutoff in spin. It is fortunate that this conclusion need not be taken on faith. There does exist a meaningful testing ground in the form of local current-algebra sum rules.

The connection between sum rules[25] and the calculation of pion transition amplitudes lies in our ability to evaluate contributions from the single-particle intermediate states. Due to the nature of the model, we analyzed first the sum rules at and near $k^2=0$. In the cases studied, the sum rules were not inconsistent with saturation by the states which our model allowed even though there was no reason to expect, solely on the basis of our hard-pion model, that multiparticle contributions could be ignored for $k^2 \cong 0$. This perhaps fortuitous event implied that the $J=4$ cutoff provided a reasonable, if approximate, indicator of when pion excitation becomes suppressed. This result is interesting because, although saturation by single-particle states has been a popular method in the evaluation of sum rules and spectral functions, there is generally no firm criterion for selecting the set of states to be included. We have mentioned in Sec. IV why our model is not appropriate for studying sum rules with fixed, but large, values of momentum transfer.

Finally, we want to transmit the hope that our conclusions about pion excitation might be substantiated in alternative theoretical approaches. Although for calculational reasons we have considered only pion targets in this paper, we conjecture that any hadronic target would behave analogously. That is, it is a property of hadronic matter that excitation with local currents at small values of momentum transfer leads predominantly to transitions to a limited number of states nearby in mass and spin. By associating a ρ meson with the appropriate vector current, it is not hard to see that we could reproduce the qualitative results discussed here for the case of a ρ target. However, in general, the relation between an arbitrary hadron and a quantum field is obscure, so the means by which our conjecture might be proved or disproved is not evident. Perhaps light-cone techniques can lead to further insights, although in one method wherein both external particles in a form factor are kept on the mass shell,[10] much information is lost due to the presence of unknown equal-time commutation relations.

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APPENDIX A

We discussed in Sec. III the evaluation of various of our parameters for $J=1,2,3$ by means of additional dynamical assumptions and use of experimental data. In this appendix, we show how to evaluate the $J=0$ parameters by employing Eq. (21a) in a simple model. As in Sec. IV, let us conjecture[15] that the ϵ meson with $m_\epsilon = 700$ MeV and $\Gamma(\epsilon\pi\pi)=300$ MeV may be used to characterize the $J=T=0$ $\pi\pi$ interaction for moderate energies. We assume that the matrix element of the σ operator between pion states, $\delta_{ab}T(t)=\langle \pi_a(q)|\sigma(0)|\pi_b(p)\rangle$, can be expressed off the mass shell in terms of the effective-range representation:

$$T(q^2,p^2;t)=\frac{A+B(q^2+p^2)+Ct}{m_\epsilon^{\,2}-t}\,, \qquad \text{(A1)}$$

where A, B, C are constants. Soft-pion theorems and the $(3,\bar{3})+(\bar{3},3)$ model of chiral symmetry breaking imply the constraints

$$q^2=p^2=t=0: \qquad A=m_\epsilon^{\,2}(m_\pi^{\,2}-X)\,, \qquad \text{(A2)}$$

$$q^2=p^2=m_\pi^{\,2},\ t=0:\ A+2m_\pi^{\,2}B=m_\pi^{\,2}m_\epsilon^{\,2}Y\,, \qquad \text{(A3)}$$

$$q^2=0,\ p^2=t=m_\pi^{\,2}:$$
$$A+m_\pi^{\,2}(B+C)=m_\pi^{\,2}(m_\epsilon^{\,2}-m_\pi^{\,2})\,, \qquad \text{(A4)}$$

where

$$X=\frac{i}{F_\pi^{\,2}}\int d^4x\,\langle 0|T(\sigma(x)\sigma(0))|0\rangle\,,\qquad Y=\frac{T(0)}{m_\pi^{\,2}}\,. \qquad \text{(A5)}$$

Defining

$$\langle \epsilon(\vec{k})|\sigma(0)|0\rangle = m_\pi^{\,2}F_\pi g_\epsilon\,, \qquad \text{(A6)}$$

we can use Eq. (21a) to obtain the additional constraint

$$q^2=0,\ p^2=t=m_\epsilon^{\,2}:\qquad A+m_\epsilon^{\,2}(B+C)=m_\pi^{\,2}(m_\epsilon^{\,2}-m_\pi^{\,2})g_\epsilon^{\,2}\,. \qquad \text{(A7)}$$

Equations (A2)–(A7) allow us to solve for A, B, C, and X in terms of g_ϵ, m_π, and m_ϵ. In particular, we find that X is forced to have its pole dominance value:

$$X=\frac{m_\pi^{\,4}g_\epsilon^{\,2}}{m_\epsilon^{\,2}}\,. \qquad \text{(A8)}$$

We can get a relation between g_ϵ and Y from $\epsilon \to \pi\pi$ data by using the equality

$$q^2 = p^2 = m_\pi^{\ 2},\ \ t = m_\epsilon^{\ 2}:$$

$$\frac{\alpha^{J=0} g_\epsilon}{m_\pi^{\ 2} F_\pi} = A + 2 m_\pi^{\ 2} B + m_\epsilon^{\ 2} C\,, \qquad \text{(A9)}$$

where $\alpha^{J=0}$ is given by Eq. (29). Finally, we obtain for the case $Y=1$

$$g_\epsilon = \frac{2F_\pi}{m_\epsilon}\left(\frac{32\pi\Gamma(\epsilon\pi\pi)}{2m_\epsilon}\right)^{1/2}\left(1-\frac{4m_\pi^{\ 2}}{m_\epsilon^{\ 2}}\right)^{-1/4} \simeq 1.06\,. \qquad \text{(A10)}$$

APPENDIX B

In this appendix, we discuss relations between various sets of coupling constants pertaining to the reaction $f(p,\lambda) \to A_1(k,\lambda')\pi(q)$. First, there are couplings G_P, G_F defined by the Lagrangian

$$\mathcal{L}_{fA\pi}(x) = G_P f^{\mu\nu}(x)\vec{A}_{1\mu}(x)\cdot\partial_\nu\vec{\pi}(x) + G_F f^{\mu\nu}(x)\vec{A}_1^{\lambda}(x)\cdot\partial_\mu\partial_\nu\partial_\lambda\vec{\pi}(x)\,. \qquad \text{(B1)}$$

The matrix element in momentum space corresponding to this Lagrangian is

$$M_{\lambda\lambda'}(k) = h^{\mu\nu}(p,\lambda)\epsilon^{\sigma*}(k,\lambda')(G_P g_{\mu\sigma}k_\nu + G_F k_\mu k_\iota$$

where from this point on, we work in the rest frame ($\vec{p}=0$) of the parent particle. Alternatively there are helicity coupling constants defined by

$$M_{\lambda\lambda'}(\theta) = d^{(2)}_{\lambda\lambda'}(\theta)g^{(\lambda')}\,, \qquad \text{(B3)}$$

which are constrained by parity to obey $g^{(\lambda')} = g^{(-\lambda')}$. Hence there are just two independent helicity couplings, $g^{(1)}$ and $g^{(0)}$. By comparing (B2) and (B3) for particular values of initial and final helicity, we find

$$G_P = \frac{\sqrt{2}\,g^{(1)}}{k}\,,$$
$$G_F = \frac{1}{k^3 m_f}\left(\frac{3}{\sqrt{6}}\,m_A g^{(0)} - \sqrt{2}\,\omega_A g^{(1)}\right). \qquad \text{(B4)}$$

Finally, by using the relation between helicity eigenstates and states in the LS basis ($S=1$ here),

$$|J=2,\lambda;L1\rangle = \left(\frac{2L+1}{5}\right)^{1/2}\sum_{\lambda'} C(L12;0,\lambda')C(101;\lambda',0)|J=2,\lambda;\lambda'\rangle\,, \qquad \text{(B5)}$$

we can relate coupling constants g_p, g_f, which represent coupling to final states with a specific value of orbital angular momentum, to the helicity couplings $g^{(0)}, g^{(1)}$. Then from Eq. (B4), we deduce

$$G_P = \frac{1}{k}\left(\tfrac{2}{5}\right)^{1/2}\left[\left(\tfrac{3}{2}\right)^{1/2} g_p + g_f\right],$$
$$G_F = \frac{1}{\sqrt{5}\,k^3 m_f}\left[\sqrt{3}\,g_p(m_A-\omega_A) - \left(\frac{3m_A+2\omega_A}{\sqrt{2}}\right)g_f\right]. \qquad \text{(B6)}$$

Equation (B6) correctly exhibits a point about which there is sometimes confusion in the literature—that the Lorentz couplings G_P, G_F are not identical to the orbital couplings g_p, g_f.

The dynamical assumption made in Sec. III (see also Ref. 26) is that $|g_f/g_p| \ll 1$. From Eq. (B6) we conclude that this implies G_F may be neglected relative to G_P in the amplitudes of Sec. III (one must be careful in deducing this because G_P and G_F are of different dimension). As a brief indication of the validity of this statement, note that if $G_F=0$, we have $g_f/g_p = -\sqrt{6}/600 \ll 1$.

[1]H. J. Schnitzer and S. Weinberg, Phys. Rev. 164, 1828 (1967).

[2]S. G. Brown and G. B. West, Phys. Rev. 168, 1605 (1968); 174, 1777 (1968).

[3]J. D. Bjorken, Phys. Rev. 148, 1467 (1966); K. Johnson and F. E. Low, Progress Theoret. Phys. (Kyoto) Suppl. 37-38, 74 (1966).

[4]M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960).

[5]The technique we adopt here was used previously in a different application by B. R. Holstein, Nuovo Cimento 68A, 369 (1970).

[6]See, e.g., J. J. Brehm and E. Golowich, Phys. Rev. D 2, 1668 (1970).

[7]In Appendix A, we show how Eq. (21a) may be used to help solve a simple model of the matrix element $\langle\pi_a(q)|\sigma(0)|\pi_b(p)\rangle$.

[8]Comparison of this model with experimental results for $\rho\to\pi\pi$, $A_1\rho\pi$ yields $\delta \cong -\frac{1}{2}$. See Ref. 6.

[9]Such a matrix element is discussed in Sec. IV. The assumption that such a matrix element is unsubtracted is the usual basis for the version of PCAC founded on dispersion theory. See S. L. Adler and R. F. Dashen, *Current Algebras and Applications to Particle Physics* (Benjamin, New York, 1968).

[10]Also, using light-cone techniques [e.g., see R. Chandra and C. Ryan, Nucl. Phys. B40, 573 (1972)], one can argue that in most models $G(k^2)$ and $H(k^2)$ should have the same asymptotic behavior.

[11]We use the current experimental results as given by Particle Data Group, Phys. Letters 39B, 1 (1972).

[12]B. Y. Oh *et al.*, Phys. Rev. D 1, 2494 (1970).

[13]This sequence of decays is suggested by the statistical bootstrap model of S. Frautschi, Phys. Rev. D 3, 2821

(1971).

[14]S. L. Adler, Phys. Rev. 143, 1144 (1966).

[15]The $I=0$ $\pi\pi$ phase shift given recently by S. D. Protopopescu *et al*. [see *Experimental Meson Spectroscopy–1972*, edited by A. H. Rosenfeld and K. Lai (American Institute of Physics, New York, 1972)] is not easy to interpret in terms of resonance behavior. In this paper, we shall assume the existence of a broad resonance, $\epsilon(700)$, to characterize the significant pion-pion interaction for energies $0.4 \le W \le 1.0$ (GeV). See also P. Carruthers, Phys. Rev. D 3, 959 (1971), for previous work on this subject.

[16]This is just the axial-vector-current analog to the derivation of the Cabibbo-Radicati sum rule [N. Cabibbo and L. A. Radicati, Phys. Letters 19, 697 (1966)].

[17]The spin-parity of D, E are, of course, as yet not known with certainty. If they are found to be 1^+ they must be included in our analysis. Nevertheless their contribution to the sum rule can be shown to be small.

[18]The ρ contribution is that obtained for $\delta=0$, and is not changed significantly if $|\delta| \le 1$.

[19]S. Fubini, G. Furlan, and C. Rossetti, Nuovo Cimento 40, 1171 (1965).

[20]This is particularly true when $|k^2|$ exceeds the values of momentum transfer for which the model was designed to be accurate, $|k^2| \lesssim 1$ GeV^2.

[21]By dominant, we mean that the sum rule $\int W_2(\nu, k^2) d\nu = \textit{constant}$ is well estimated for $k^2 \cong 0$ in terms of the resonance contributions. In view of this, a question which naturally arises is: What can one infer from the sum rule about experimental studies of inclusive processes? The only unambiguous statement we can make is that resonance production is the major component in the "neutrino + pion → lepton + everything" cross section, $d^2\sigma/(d|k^2| d\nu)$, for $k^2 \cong 0$, $\nu \lesssim \frac{1}{2}\nu_{\max}$.

[22]R. P. Feynman and M. Gell-Mann, Phys. Rev. 109, 193 (1958).

[23]That is, the vector-current analog of $N_{ab}(q, p)$ vanishes.

[24]I.e.,

$$m_\rho^2 r_\pi^2 = 6 m_\rho^2 \frac{d}{dq^2} \ln F_\pi(q^2)\Big|_{q^2=0} = -6 m_\rho^2 \frac{d}{dq^2} \ln\left(1 - \frac{q^2}{m_\rho^2}\right)\Big|_{q^2=0} \cong 6.$$

In obtaining numerical estimates for certain of the terms in Eq. (51) we referred to the work of V. S. Mathur and L. K. Pandit, Phys. Letters 19, 523 (1965).

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[26]E. Golowich, Phys. Rev. 184, 1815 (1969).

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Multiperipheral Theory of Massive-Muon Pair Production in Hadron Collisions*

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Massive-lepton pair production in high-energy hadron collisions is studied in the ABFST (Amati-Bertocchi-Fubini-Stanghellini-Tonin) multiperipheral model. The cross section for point electromagnetic couplings is given by $d\sigma/dQ^2 = Q^{-4} f(s/Q^2)$ when $Q^2 \gg M^2$, where $\sqrt{s}$ is the center-of-mass energy of the colliding protons, Q the mass of the lepton pair, and M the nucleon mass. The scaling function f is expressed in terms of πN and NN off-shell forward absorptive amplitudes. When $s \gg Q^2$, the function f behaves like $a\ln(s/Q^2) + b$, Pomeron dominance being assumed. Gauge invariance of the model is discussed.

I. INTRODUCTION

The interesting SLAC-MIT experiments on deep-inelastic electron scattering probe the electromagnetic structure of hadrons when the current carries a spacelike momentum. The BNL-Columbia experiment[1] extends the probe to timelike momentum by studying the reaction

$$\text{proton} + \text{proton} \to \mu^+ + \mu^- + \text{anything}. \tag{1}$$

If unpolarized protons of momenta p_1 and p_2, energies E_1 and E_2, and mass M collide to produce a muon pair of momentum q in addition to anything else (Fig. 1), then, summing over muon polarizations and momentum variables except $q^2 \equiv Q^2$, the cross section neglecting muon mass is given by

$$\frac{d\sigma}{dQ^2} = \frac{4\alpha^2}{3\pi^3} \frac{1}{[s(s-4M^2)]^{1/2}} W(Q^2, s), \tag{2}$$

where $s = (p_1 + p_2)^2$ and $W(Q^2; s) = \tilde{W}_\mu{}^\mu(Q^2, s)$, with